FRAMENESS BOUND FOR FRAME OF SUBSPACES

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Abstract. In this paper, we show that in each finite dimensional Hilbert space, a frame of subspaces is an ultra Bessel sequence of subspaces. We also show that every frame of subspaces in a finite dimensional Hilbert space has frameness bound.

1. Introduction

Let \( \mathcal{H} \) be a separable Hilbert space. We say that a sequence \( \{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H} \) is a frame for \( \mathcal{H} \), if there exist constants \( 0 < A, B < \infty \) such that

\[
A \|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad f \in \mathcal{H}.
\]

If \( A = B \) then \( \{f_i\}_{i=1}^{\infty} \) is called a tight frame and if \( A = B = 1 \), it is called a Parseval frame. If the right hand inequality of (1.1) holds for all \( f \in \mathcal{H} \), then we say \( \{f_i\}_{i=1}^{\infty} \) is a Bessel sequence for \( \mathcal{H} \).

In 2008, the concept of ultra Bessel sequences in Hilbert spaces introduced and investigated by Faroughi and Najati [4].

Definition 1.1. Let \( \mathcal{H}_0 \) be an inner product space. Let \( \{f_i\}_{i=1}^{\infty} \) be a sequence of members of \( \mathcal{H}_0 \). Then \( \{f_i\}_{i=1}^{\infty} \) is called an ultra Bessel sequence in \( \mathcal{H}_0 \), if

\[
\sup_{\|f\|=1} \sum_{i=n}^{\infty} |\langle f, f_i \rangle|^2 \to 0,
\]

as \( n \to \infty \), i.e., the series \( \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \) converges uniformly in unit sphere of \( \mathcal{H}_0 \).

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As a generalization of ordinary frame, frame of subspaces introduced by Casazza and Kutyniok in [2].

**Definition 1.2.** Let \( \{v_i\}_{i=1}^{\infty} \) be a family of weights, i.e., \( v_i > 0 \), for all \( i \geq 1 \). A family of closed subspaces \( \{W_i\}_{i=1}^{\infty} \) of a Hilbert space \( \mathcal{H} \) is a frame of subspaces with respect to \( \{v_i\}_{i=1}^{\infty} \) for \( \mathcal{H} \), if there exist constants \( 0 < C \leq D < \infty \) such that

\[
C \|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D \|f\|^2, \quad f \in \mathcal{H}.
\]

If the right hand inequality in (1.2) holds for all \( f \in \mathcal{H} \), we call \( \{W_i\}_{i=1}^{\infty} \) a Bessel sequence of subspaces with respect to \( \{v_i\}_{i=1}^{\infty} \) with Bessel bound \( D \).

**Definition 1.3.** For each family of subspaces \( \{W_i\}_{i=1}^{\infty} \) of Hilbert space \( \mathcal{H} \), we define the set

\[
\left( \sum_{i=1}^{\infty} \oplus W_i \right) \ell^2 = \{ \{f_i\}_{i=1}^{\infty} | f_i \in W_i, \sum_{i=1}^{\infty} \|f_i\|^2 < \infty \}.
\]

It is clear that \( \left( \sum_{i=1}^{\infty} \oplus W_i \right) \ell^2 \) is a Hilbert space with the point wise operations and with the inner product given by

\[
\langle \{f_i\}_{i=1}^{\infty}, \{g_i\}_{i=1}^{\infty} \rangle = \sum_{i=1}^{\infty} \langle f_i, g_i \rangle.
\]

It is proved in [2], if \( \{W_i\}_{i=1}^{\infty} \) is a frame of subspaces with respect to \( \{v_i\}_{i=1}^{\infty} \) for \( \mathcal{H} \) then the operator

\[
T_{W,v} : \left( \sum_{i=1}^{\infty} \oplus W_i \right) \ell^2 \rightarrow \mathcal{H}, \quad T_{W,v}(\{f_i\}_{i=1}^{\infty}) = \sum_{i=1}^{\infty} v_i f_i,
\]

is bounded and onto and its adjoint is

\[
T_{W,v}^* : \mathcal{H} \rightarrow \left( \sum_{i=1}^{\infty} \oplus W_i \right) \ell^2, \quad T_{W,v}^*(f) = \{v_i \pi_{W_i}(f)\}_{i=1}^{\infty}.
\]

The operators \( T_{W,v} \) and \( T_{W,v}^* \) are called the synthesis and analysis operators for \( \{W_i\}_{i=1}^{\infty} \) and \( \{v_i\}_{i=1}^{\infty} \), respectively.

Also, it is proved in [2], if \( \{W_i\}_{i=1}^{\infty} \) is a frame of subspaces with respect to \( \{v_i\}_{i=1}^{\infty} \), the operator

\[
S_{W,v} : \mathcal{H} \rightarrow \mathcal{H}, \quad S_{W,v}(f) = TT^*(f)
\]
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is a positive, self-adjoint and invertible operator on $\mathcal{H}$ and we have the reconstruction formula

$$ f = \sum_{i=1}^{\infty} v_i^2 S_{W_i}^{-1} \pi_{W_i} (f), \quad f \in \mathcal{H}. $$

The operator $S_{W_i}^{-1}$ is called the frame operator for $\{W_i\}_{i=1}^{\infty}$ and $\{v_i\}_{i=1}^{\infty}$.

The ultra Bessel sequence of subspaces introduced in [1] by the authors of this paper.

Definition 1.4. Let $\mathcal{H}_0$ be an inner product space. Let $\{W_i\}_{i=1}^{\infty}$ be a family of closed subspaces of $\mathcal{H}_0$. Then $\{W_i\}_{i=1}^{\infty}$ is called an ultra Bessel sequence of subspaces in $\mathcal{H}_0$, if

$$ \sup_{\|f\|=1} \sum_{i=n}^{\infty} v_i^2 \|\pi_{W_i} (f)\|^2 \to 0, $$

as $n \to \infty$, i.e., the series $\sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i} (f)\|^2$ converges uniformly in the unit sphere of $\mathcal{H}_0$.

Following proposition has been proved in [1] and we use it in the rest of this paper.

Proposition 1.5. Let $\{W_i\}_{i=1}^{\infty}$ be a family of closed subspaces in Hilbert space $\mathcal{H}$ and $\{v_i\}_{i=1}^{\infty}$ be a family of weights such that $\sum_{i=1}^{\infty} v_i^2 < \infty$. Then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$.

2. FRAMENESS BOUND

In this section, we prove that in a finite dimensional Hilbert space, each frame of subspaces is an ultra Bessel sequence of subspaces. Also we can divide a frame of subspaces $\{W_i\}_{i=1}^{\infty}$ in two sets $\{W_i\}_{i=1}^{N-1}$ and $\{W_i\}_{i=N}^{\infty}$, for which $\{W_i\}_{i=1}^{N-1}$ is not a frame of subspaces, but $\{W_i\}_{i=N}^{\infty}$ is a frame of subspaces.

Theorem 2.1. Let $\{W_i\}_{i=1}^{\infty}$ be a frame of subspaces for Hilbert space $\mathcal{H}$ such that for all $i \geq 1$, $\dim W_i < \infty$. Then

(i) if $\mathcal{H}$ is an infinite dimensional Hilbert space, then $\{W_i\}_{i=1}^{\infty}$ is not an ultra Bessel sequence of subspaces.

(ii) if $\mathcal{H}$ is a finite dimensional Hilbert space, then $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces, and there exists $N_0 \geq 1$ such that for each $1 \leq n < N_0$, $\{W_i\}_{i=1}^{n}$ is not a frame of subspaces of $\mathcal{H}$, but $\{W_i\}_{i=1}^{n}$ is a frame of subspaces of $\mathcal{H}$ for each $n \geq N_0$. 
Proof. (i) Since \( \{W_i\}_{i=1}^{\infty} \) is a frame of subspaces of \( \mathcal{H} \), there exist constants \( C, D \) such that
\[
C \|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D \|f\|^2, \quad f \in \mathcal{H}.
\]
Let \( \{W_i\}_{i=1}^{\infty} \) be an ultra Bessel sequence of subspaces. Let \( 0 < \varepsilon < C \).
Then there exists \( N > 0 \) such that
\[
\sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 < \varepsilon
\]
for each \( f \in \mathcal{H} \) with \( \|f\| = 1 \). Since \( \dim(\mathcal{H}) = \infty \), there exists \( f_0 \in \mathcal{H} \) such that
\[
\|f_0\| = 1, \quad f_0 \perp \text{span}(\bigcup_{i=1}^{N} W_i).
\]
From other hand
\[
(2.1)
\]
If we replace \( f \) by \( f_0 \) in (2.1), then
\[
C \|f\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 = \sum_{i=1}^{N} v_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2.
\]
If we replace \( f \) by \( f_0 \) in (2.1), then
\[
C = C \|f_0\|^2 \leq \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(f_0)\|^2 = \sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f_0)\|^2 < \varepsilon
\]
and this contradiction proves (i).
(ii) Let \( \dim \mathcal{H} = m \) and \( \{e_1, \ldots, e_m\} \) be an orthonormal basis for \( \mathcal{H} \).
\[
\sum_{i=1}^{\infty} v_i^2 = \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}\|^2 = \sum_{i=1}^{\infty} \|v_i \pi_{W_i}\|^2
\]
\[
\leq \sum_{i=1}^{\infty} \|v_i \pi_{W_i}\|_2^2
\]
\[
= \sum_{i=1}^{\infty} \sum_{j=1}^{m} \|v_i \pi_{W_i}(e_j)\|^2
\]
\[
= \sum_{j=1}^{m} \sum_{i=1}^{\infty} v_i^2 \|\pi_{W_i}(e_j)\|^2 \leq mD,
\]
where \( \|v_i \pi_{W_i}\|_2 \) denotes the Hilbert-Schmidt norm of the operator \( v_i \pi_{W_i} \).
Thus by Proposition 1.5, \( \{W_i\}_{i=1}^{\infty} \) is an ultra Bessel sequence of subspaces with respect to \( \{v_i\}_{i=1}^{\infty} \).
Let $0 < \varepsilon < C$. Then there exists $N > 0$ such that

$$
(2.2) \quad \sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 < \varepsilon \|f\|^2, \quad f \in \mathcal{H}.
$$

From other hand, we have

$$
C \|f\|^2 \leq \sum_{i=1}^{N} v_i^2 \|\pi_{W_i}(f)\|^2 + \sum_{i=N+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2, \quad f \in \mathcal{H}.
$$

So (2.2) implies that

$$
(C - \varepsilon) \|f\|^2 \leq \sum_{i=1}^{N} v_i^2 \|\pi_{W_i}(f)\|^2 \leq D \|f\|^2, \quad f \in \mathcal{H}.
$$

So for each $n \geq N$, $\{W_i\}_{i=1}^{n}$ is a frame of subspaces of $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{n}$. Now, let $N_0$ be the minimum of the all such $N$. Then $\{W_i\}_{i=1}^{n}$ can not be a frame of subspaces for each $n < N$. □

**Definition 2.2.** Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\{W_i\}_{i=1}^{\infty}$ be a frame of subspaces of $\mathcal{H}$. Then we call the number $N_0$ in the Theorem 2.1, the frameness bound of the frame of subspaces $\{W_i\}_{i=1}^{\infty}$.

**Lemma 2.3.** [3] Let $\mathcal{H}$, $\mathcal{K}$ be Hilbert spaces, and suppose that $U : \mathcal{K} \to \mathcal{H}$ is a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger : \mathcal{H} \to \mathcal{K}$ for which

$$
UU^\dagger f = f, \quad \forall f \in \mathcal{R}_U.
$$

If $\{W_i\}_{i=1}^{\infty}$ is a frame of subspaces for $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$, then

$$
T_{W,v}(T_{W,v}^*S_{W,v}^{-1}f) = f, \quad f \in \mathcal{H},
$$

so $T_{W,v}^\dagger = T_{W,v}^*S_{W,v}^{-1}$.

**Theorem 2.4.** Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\{W_i\}_{i=1}^{\infty}$ be a frame of subspaces of $\mathcal{H}$ with respect to $\{v_i\}_{i=1}^{\infty}$ an let $N_0$ be frameness bound of $\{W_i\}_{i=1}^{\infty}$. Let $n \geq N_0$ and $S_n$ and $T_n$ be the frame operator and synthesis operator of $\{W_i\}_{i=1}^{n}$, respectively. Then

(i) $S_n \to S_{W,v}$ in $B(\mathcal{H})$,

(ii) $T_n \to T_{W,v}$ in $B\left(\bigoplus_{i=1}^{\infty} W_i, \mathcal{H}\right)$ and $T_n^\dagger \to T_{W,v}^\dagger$

in $B\left(\mathcal{H}, \bigoplus_{i=1}^{\infty} W_i\right)$. 
Proof. (i) Let $n \geq N_0$ and $f \in \mathcal{H}$ with $\|f\| = 1$. Then
\[
\|S_n(f) - S_{W,v}(f)\| = \left\| \sum_{i=n+1}^{\infty} v_i^2 \pi_{W_i}(f) \right\| = \sup_{\|g\| = 1} \left| \sum_{i=n+1}^{\infty} v_i^2 \pi_{W_i}(f), g \right| \leq \sup_{\|g\| = 1} \left( \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2 \right)^{\frac{1}{2}} \left( \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \right)^{\frac{1}{2}} \leq \sqrt{B} \left( \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(f)\|^2 \right)^{\frac{1}{2}}.
\]
Since $\{W_i\}_{i=1}^{\infty}$ is an ultra Bessel sequence of subspaces in $\mathcal{H}$, thus $S_n$ converges to $S_{W,v}$ in $B(\mathcal{H})$.

(ii) Let $n \geq N_0$, $\{f_i\}_{i=1}^{\infty} \in \left( \sum_{i=1}^{\infty} \oplus W_i \right)_{\ell^2}$ with $\|\{f_i\}_{i=1}^{\infty}\| \leq 1$. Then
\[
\|T_n(\{f_i\}_{i=1}^{\infty}) - T_{W,v}(\{f_i\}_{i=1}^{\infty})\|^2 = \left\| \sum_{i=n+1}^{\infty} v_i f_i \right\|^2 = \sum_{i=n+1}^{\infty} \|v_i\|^2 \|f_i\|^2 \leq \sum_{\|g\| = 1}^{\infty} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2 \|f_i\|^2 \leq \sup_{\|g\| = 1} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2.
\]
therefore $T_n \to T_{W,v}$ in $B\left( \left( \sum_{i=1}^{\infty} \oplus W_i \right)_{\ell^2}, \mathcal{H} \right)$.

Now let $n \geq N_0$ and $f \in \mathcal{H}$ with $\|f\| \leq 1$. Then we have
\[
\sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(S_{W,v}^{-1}(f))\|^2 = \|S_{W,v}^{-1}(f)\|^2 \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(S_{W,v}^{-1}(f))\|^2 \leq \|S_{W,v}^{-1}(f)\|^2 \sup_{\|g\| \leq 1} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2 \leq \|S_{W,v}^{-1}(f)\|^2 \sup_{\|g\| = 1} \sum_{i=n+1}^{\infty} v_i^2 \|\pi_{W_i}(g)\|^2.
\]
So by (i), for each $\varepsilon > 0$, there exists $M > N_0$ such that for all $f \in \mathcal{H}$ with $\|f\| \leq 1$ and all $n \geq M$,
\[ \| S_n^{-1}(f) - S_{W,v}^{-1}(f) \| < (\varepsilon/2B)^{\frac{1}{2}}, \] 

and
\[ \sum_{i=n+1}^{\infty} v_i^2 \| \pi_{W_i}(S_{W,v}^{-1}(f)) \|^2 < \varepsilon/2. \]

From other hand we have
\[ \sum_{i=1}^{n} v_i^2 \| \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)) \|^2 \]
\[ = \sum_{i=1}^{n} v_i^2 \left( \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)), \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)) \right) \]
\[ = \left( \sum_{i=1}^{n} v_i^2 \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)), S_n^{-1}(f) - S_{W,v}^{-1}(f) \right) \]
\[ \leq \| S_n^{-1}(f) - S_{W,v}^{-1}(f) \| \left\| \sum_{i=1}^{n} v_i^2 \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)) \right\|. \]

Also
\[ \left\| \sum_{i=1}^{n} v_i^2 \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)) \right\| \]
\[ = \sup_{\| g \| = 1} \left| \left( \sum_{i=1}^{n} v_i^2 \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)), g \right) \right| \]
\[ = \sup_{\| g \| = 1} \left| \sum_{i=1}^{n} \left( v_i^2 \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)), v_i \pi_{W_i}(g) \right) \right| \]
\[ \leq \sup_{\| g \| = 1} \sum_{i=1}^{n} \left| v_i \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)) \right| \| v_i \pi_{W_i}(g) \| \]
\[ \leq \left( \sum_{i=1}^{n} v_i^2 \right) \left( \sum_{i=1}^{n} \| \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)) \|^2 \right)^{1/2} \]
\[ \leq \sup_{\| g \| = 1} \left( \sum_{i=1}^{n} v_i^2 \| \pi_{W_i}(g) \|^2 \right)^{1/2} \]
\[ \leq \sqrt{B} \left( \sum_{i=1}^{n} v_i^2 \| \pi_{W_i}(S_n^{-1}(f) - S_{W,v}^{-1}(f)) \|^2 \right)^{1/2}. \]
Now (2.5) and (2.6) imply that
\[ \left( \sum_{i=1}^{n} v_i^2 \right) \| \pi_{W_i} \left( S_n^{-1}(f) - S_{W,v}^{-1}(f) \right) \|_2^{1/2} \leq \sqrt{B} \| S_n^{-1}(f) - S_{W,v}^{-1}(f) \|. \]
Consequently by (2.3) we have
\[ \sum_{i=1}^{n} v_i^2 \| \pi_{W_i} \left( S_n^{-1}(f) - S_{W,v}^{-1}(f) \right) \|^2 < \varepsilon/2. \]
Finally, according to (2.4) and (2.7)
\[ \| T_n^\dagger f - T_{W,v}^\dagger f \|^2 = \sum_{i=1}^{n} v_i^2 \| \pi_{W_i} \left( S_n^{-1}(f) - S_{W,v}^{-1}(f) \right) \|^2 + \sum_{i=n+1}^{\infty} v_i^2 \| \pi_{W_i} \left( S_{W,v}^{-1}(f) \right) \|^2 \]
yields that
\[ \| T_n^\dagger f - T_{W,v}^\dagger f \| < \varepsilon, \quad f \in \mathcal{H}, \| f \| = 1, \]
so \( T_n^\dagger \to T_{W,v}^\dagger \) in \( B \left( \mathcal{H}, (\sum_{i=1}^{\infty} \oplus W_i)_{\ell^2} \right) \).

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**References**


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