Sahand Communications in Mathematical Analysis (SCMA) Vol. 1 No. 1(2014), 41-51
http://journal.maragheh.ac.ir

INVARIANCE OF FRÉCHET FRAMES UNDER PERTURBATION

ASGHAR RAHIMI*

Abstract. Motivating the perturbations of frames in Hilbert and Banach spaces, in this paper we introduce the invariance of Fréchet frames under perturbation. Also we show that for any Fréchet spaces, there is a Fréchet frame and any element in these spaces has a series expansion.

1. Introduction

Historically, theory of frames appeared in the paper of Duffin and Schaeffer in 1952. Around 1986, Daubechies, Grossmann, Meyer and others reconsidered ideas of Duffin and Schaeffer [9], and started to develop the wavelet and frame theory. Frames for Banach spaces were introduced by K. Gröchenig [10] and subsequently many mathematicians have contributed to this theory [1, 4, 5, 17]. There are some complete spaces which are not Banach spaces (like Fréchet spaces). This was the main motivation for frames on Fréchet spaces. The concept of Fréchet frames investigated by Pilipović and Stoeva in [15, 16]. Like Hilbert and Banach spaces, we will show that for any Fréchet space we can find a Fréchet frame and every element in a Fréchet space has a series expansion [4].

In this manuscript we are interested in the problem of finding conditions under which the perturbation of a Fréchet frame is also a Fréchet frame. The following result [3], is one of the most general and also typical results about frame perturbations for the whole space $\mathcal{H}$ which generalizes the main results in [7, 8].

2010 *Mathematics Subject Classification.* 42C15, 46A45.
Key words and phrases. Frame, Banach frame, Fréchet frames, Fréchet spaces, Perturbation, F-bounded.
Received: 30 June 2013 Accepted: 8 December 2013
* Corresponding author.

Theorem 1.1. [3] Let $F := \{f_i\}_{i\in I}$ be a frame for $\mathcal{H}$ with bounds A and B, and $G := \{g_i\}_{i\in I}$ is a sequence in $\mathcal{H}$. Suppose that there exist non-negative λ_1, λ_2 and μ with $\lambda_2 < 1$ such that

$$\|\sum_i c_i(f_i - g_i)\| \leq \lambda_1\|\sum_i c_i f_i\| + \lambda_2\|\sum_i c_i g_i\| + \mu\|c\|$$

for each finitely supported $c \in \ell^2(\mathbb{N})$ and $\lambda_1 + \frac{\mu}{\sqrt{A}} < 1$. Then G is a frame for $\mathcal{H}$ with bounds

$$A(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1+\lambda_2})^2 \quad \textit{and} \quad B(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1-\lambda_2})^2.$$

2. Fréchet frames

Throughout this paper, $(X, \|.\|)$ is a Banach space and $(X^*, \|.\|^*)$ is its dual, $(\Theta, |||.|||)$ is a Banach sequence space and $(\Theta^*, |||.|||^*)$ is the dual of Θ.

Definition 2.1. A sequence space X_d is called a Banach space of scalar valued sequences or briefly a BK-space, if it is a Banach space and the coordinate functionals are continuous on X_d , i.e., the relations $x_n = \{\alpha_j^{(n)}\}, x = \{\alpha_j\} \in X_d, lim x_n = x$ imply $lim \alpha_j^{(n)} = \alpha_j$.

Definition 2.2. Let X be a Banach space and X_d be a BK-space. A countable family $\{g_i\}_{i\in I}$ in the dual X^* is called an X_d -frame for X if

(i) $\{g_i(f)\}_{i=1}^{\infty} \in X_d, \forall f \in X$;
(ii) the norms $\|f\|_X$ and $\|\{g_i(f)\}_{i=1}^{\infty}\|_{X_d}$ are equivalent, i.e., there exist constants $A, B > 0$ such that

$$A\|f\|_X \leq \|\{g_i(f)\}_{i=1}^{\infty}\|_{X_d} \leq B\|f\|_X, \quad \forall f \in X$$

A and B are called X_d -frame bounds. If at least (1) and the upper condition in (2) are satisfied, $\{g_i\}_{i=1}^{\infty}$ is called an X_d -Bessel sequence for X.

If X is a Hilbert space and $X_d = \ell^2$, (2) means that $\{g_i\}_{i=1}^{\infty}$ is a frame, and in this case it is well known that there exists a sequence $\{f_i\}_{i=1}^{\infty}$ in X such that $f = \sum_{i=1}^{\infty}\langle f, f_i\rangle g_i = \sum\langle f, g_i\rangle f_i$.

Similar reconstruction formulas are not always available in the Banach space setting. This is the reason behind the following definition:

Definition 2.3. Let X be a Banach space and X_d a sequence space. Given a bounded linear operator $S : X_d \to X$, and an X_d -frame $\{g_i\}_{i=1}^{\infty} \subseteq X^*$, we say that $(\{g_i\}_{i=1}^{\infty}, S)$ is a Banach frame for X with respect to X_d if

$$S(\{g_i(f)\} = f, \quad \forall f \in X. \tag{2.1}$$

Note that (2.1) can be considered as some kind of generalized reconstruction formula, in the sense that it tells how to come back to $f \in X$ based on the coefficients $\{g_i(f)\}_{i=1}^{\infty}$.

There is a relationship between these definitions, a Banach frame is an atomic decomposition if and only if the unit vectors form a basis for the space X_d. The following Proposition states this result.

Proposition 2.4. [5] Let X be a Banach space and X_d be a BK-space. Let $\{y_i\}_{i=1}^{\infty} \subseteq X^*$ and $S : X_d \to X$ be given. Let $\{e_i\}_{i=1}^{\infty}$ be the unit vectors in X_d. Then the following are equivalent:

(i) $(\{y_i\}_{i=1}^{\infty}, S)$ is a Banach frame for X with respect to X_d and $\{e_i\}_{i=1}^{\infty}$ is a Schauder basis for X_d.

(ii) $(\{y_i\}_{i=1}^{\infty}, \{S(e_i)\}_{i=1}^{\infty})$ is an atomic decomposition for X with respect to X_d.

It is known [5] that every separable Banach space has a Banach frame.

Theorem 2.5. Every separable Banach space has a Banach frame with bounds $A = B = 1$.

The main motivation of Fréchet frames comes from some sequences $\{g_i\}_{i=1}^{\infty}$ which are not Bessel sequences but they give rise to series expansions. For Banach space X, let $\{g_i\}_{i=1}^{\infty} \subseteq X^*$ be given and let there exist $\{f_i\}_{i=1}^{\infty} \subseteq X$ such that the following series expansion in X holds

$$f = \sum_{i=1}^{\infty} g_i(f) f_i, \quad \forall f \in X. \tag{2.2}$$

Validity of (2.2) does not imply that $\{g_i\}_{i=1}^{\infty}$ is a Banach frame for X with respect to the given sequence space. As one can see in the following examples.

Example 2.6. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. Consider the sequence $\{g_i\}_{i=1}^{\infty} = \{e_1, e_1, 2e_2, 3e_3, 4e_4, ...\}$. This sequence is not a Banach frame for $\mathcal{H}$ with respect to ℓ^2. However, the series expansion in $\mathcal{H}$ in the form (2.2) is $\{f_i\}_{i=1}^{\infty} = \{e_1, 0, \frac{1}{2}e_2, \frac{1}{3}e_3, \frac{1}{4}e_4, ...\}$.

Validity of (2.2) implies that $\{g_i\}_{i=1}^{\infty}$ is a Banach frame for X with respect to the sequence space $\big\{\{c_i\}_{i=1}^{\infty} : \sum_{i=1}^{\infty} c_i f_i \quad converges\big\}$.

Recall, a complete locally convex space which has a countable fundamental system of seminorms is called a Fréchet space.

Let $\{Y_s, \|.\|_s\}_{s\in\mathbb{N}}$ be a sequence of separable Banach spaces such that

$$\{0\} \neq \cap_{s\in\mathbb{N}} Y_s \subseteq ... \subseteq Y_2 \subseteq Y_1 \subseteq Y_0 \tag{2.3}$$

$$\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \cdots \tag{2.4}$$

$$Y_F := \cap_{s\in\mathbb{N}} Y_s \quad is \quad dense \quad in \quad Y_s \quad \forall s \in \mathbb{N}. \tag{2.5}$$

Then Y_F is a Fréchet space with the sequence of norms $\|.\|_s$, $s \in \mathbb{N}$. We use the above sequences in two cases: $Y_s = X_s$ and $Y_s = \Theta_s$. Let $\{X_s, \|.\|_s\}_{s\in\mathbb{N}}$ and $\{\Theta_s, |||.|||_s\}_{s\in\mathbb{N}}$ be sequences of Banach and Banach sequence spaces, which satisfy (2.3)-(2.5). For fixed $s \in \mathbb{N}$, an operator $V : \Theta_F \to X_F$ is s-bounded if there exist constants $K_s > 0$ such that $\|V\{c_i\}_{i=1}^{\infty}\|_s \leq K_s|||\{c_i\}_{i=1}^{\infty}|||_s$ for all $\{c_i\}_{i=1}^{\infty} \in \Theta_F$. If V is s-bounded for every $s \in \mathbb{N}$, then V is called F-bounded. Note that an F-bounded operator is continuous but the converse dose not hold in general. The book of R. Meise, D. Vogt is a very useful text book about Fréchet spaces [13].

The Banach sequence space Θ is called solid if the condition $\{c_i\}_{i=1}^{\infty} \in \Theta$ and $|d_i| \leq |c_i|$ for all $i \in \mathbb{N}$, imply that $\{d_i\}_{i=1}^{\infty} \in \Theta$ and $|||\{d_i\}_{i=1}^{\infty}|||_\Theta \leq |||\{c_i\}_{i=1}^{\infty}|||_\Theta$. A BK-space which contains all the canonical vectors e_i and for which there exists a constant $\lambda \geq 1$ such that

$$|||\{c_i\}_{i=1}^{n}|||_\Theta \leq \lambda|||\{c_i\}_{i=1}^{\infty}|||_\Theta, \quad \forall n \in \mathbb{N}, \forall\{c_i\}_{i=1}^{\infty} \in \Theta$$

will be called λ-BK-space. A BK-space is called a CB-space if the set of the canonical vectors forms a basis.

Definition 2.7. Let $\{X_s, \|.\|_s\}_{s\in\mathbb{N}}$ be a family of Banach spaces, satisfying (2.3)-(2.5) and let $\{\Theta_s, |||.|||_s\}_{s\in\mathbb{N}}$ be a family of BK-spaces, satisfying (2.3)-(2.5). A sequence $\{g_i\}_{i=1}^{\infty} \subseteq X_F^*$ is called a pre-Fréchet frame (a pre-F-frame) for X_F with respect to Θ_F if for every $s \in \mathbb{N}$ there exist constants $0 < A_s \leq B_s < \infty$ such that

$$\{g_i(f)\}_{i=1}^{\infty} \in \Theta_F, \tag{2.6}$$

$$A_s\|f\|_s \leq |||\{g_i(f)\}_{i=1}^{\infty}|||_s \leq B_s\|f\|_s, \tag{2.7}$$

for all $f \in X_F$. The constants A_s and B_s are called lower and upper bounds for $\{g_i\}_{i=1}^{\infty}$. The pre-F-frame is called tight if $A_s = B_s$ for all $s \in \mathbb{N}$. Moreover, if there exists a F-bounded operator $S : \Theta_F \to X_F$ so that $S(\{g_i(f)\}_{i=1}^{\infty}) = f$ for all $f \in X_F$, then a pre-F-frame $\{g_i\}_{i=1}^{\infty}$ is called a Fréchet frame (or F-frame) for X_F with respect to Θ_F and S is called an F-frame operator of $\{g_i\}_{i=1}^{\infty}$. When (2.6) and at least the upper inequality in (2.7) hold, then $\{g_i\}_{i=1}^{\infty}$ is called a F-Bessel sequence for X_F with respect to Θ_F.

Since X_F is dense in X_s for all $s \in \mathbb{N}$, g_i has a unique continuous extension on X_s, we show it by g_i^s. Thus $g_i^s \in X_s^*$ and $g_i^s = g_i$ on X_F.

The following Theorem gives some conditions, under which an element can be expanded by some elementary vectors.

Theorem 2.8. [15, 16] Let $\{X_s, \|.\|_s\}_{s\in\mathbb{N}}$ be a family of Banach spaces, satisfying (2.3)-(2.5) and let $\{\Theta_s, |||.|||_s\}_{s\in\mathbb{N}}$ be a family of CB-spaces, satisfying (2.3)-(2.5) and we assume that Θ_s^* is a CB-space for every

$s \in \mathbb{N}$. Let $\{g_i\}_{i=1}^{\infty}$ be a pre-F-frame for X_F with respect to Θ_F. There exists a family $\{f_i\}_{i=1}^{\infty} \subseteq X_F$ such that

(i) $f = \sum_{i=1}^{\infty} g_i(f) f_i$ and $g = \sum_{i=1}^{\infty} g(f_i) g_i$, $\forall f \in X_F$ and $\forall g \in X_F^*$;
(ii) $f = \sum_{i=1}^{\infty} g_i^s(f) f_i$ and $g = \sum_{i=1}^{\infty} g(f_i) g_i^s$, $\forall f \in X_s$ and $\forall g \in X_s^*, \forall s \in \mathbb{N}$;
(iii) for every $s \in \mathbb{N}$, $\{f_i\}_{i=1}^{\infty}$ is a Θ_s^*-frame for X_s^*.

if and only if there exists a continuous projection U from Θ_F onto its subspace $\big\{\{g_i(f)\}_{i=1}^{\infty} : f \in X_F\big\}$.

The following proposition shows that the pre-Fréchet Besselness is equivalent to the F-boundedness of an operator.

Proposition 2.9. Let $\{X_s, \|.\|_s\}_{s\in\mathbb{N}}$ be a family of Banach spaces, satisfying (2.3)-(2.5) and let $\{\Theta_s, |||.|||_s\}_{s\in\mathbb{N}}$ be a family of CB-spaces, satisfying (2.3)-(2.5) and we assume that Θ_s^* is a CB-space for every $s \in \mathbb{N}$. The family $\{g_i\}_{i=1}^{\infty} \subseteq X_F^*$ is a pre-Fréchet Bessel sequences for X_F with respect to Θ_F if and only if the operator $T : \Theta_F^* \to X_F^*$ defined by $T\{d_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} d_i g_i$ is well defined and $\|T\|_s \leq B_s$, for all $s \in \mathbb{N}$.

Proof. First, suppose that $\{g_i\}_{i=1}^{\infty} \subseteq X_F^*$ is a pre-Fréchet Bessel sequences for X_F with respect to Θ_F. Define the operator

$$R : X_F \to \Theta_F$$

by

$$Rf = \{g_i(f)\}_{i=1}^{\infty}.$$

Since $\{g_i(f)\}_{i=1}^{\infty}$ is a pre-Fréchet Bessel sequence , so $\|R\|_s \leq B_s$ for every $s \in \mathbb{N}$. The adjoint of R is in the form $R^* : \Theta_F^* \to X_F^*$ and

$$R^*(e_j)f = e_j(Rf) = e_j(\{g_i f\}_{i=1}^{\infty}) = g_j f,$$

so $R^* e_j = g_j$. Put $T = R^*$, then $\|T\|_s \leq B_s$ and

$$T\{d_i\}_{i=1}^{\infty} = T(\sum_{i=1}^{\infty} d_i e_i) = \sum_{i=1}^{\infty} d_i T e_i = \sum_{i=1}^{\infty} d_i R^* e_i = \sum_{i=1}^{\infty} d_i g_i.$$

Conversely, suppose the operator $T : \Theta_F^* \to X_F^*$ defined by $T\{d_i\}_{i=1}^{\infty} = \sum_{i=1}^{\infty} d_i g_i$ is well defined and s-bounded for all $s \in \mathbb{N}$. It is clear that $Te_i = g_i$ and $T^* : X_F^{**} \to \Theta_F^{**}$ is $(T^* f)e_i = f(Te_i) = f(g_i)$. Therefore $\{g_i(f)\}_{i=1}^{\infty} = \{(T^* f)e_i\}_{i=1}^{\infty}$, i.e. $\{g_i(f)\}_{i=1}^{\infty} \in \Theta_F$. The s- boundedness of T imply that $|||\{g_i(f)\}_{i=1}^{\infty}|||_s \leq B_s$. □

Similar to Theorem 2.5, the following Theorem shows that for any Fréchet space X_F we can construct a Fréchet frame.

Theorem 2.10. Let $\{X_s, \|.\|_s\}_{s\in\mathbb{N}}$ be a family of separable Banach spaces satisfying (2.3)-(2.5). Let $X_F := \bigcap_{s\in\mathbb{N}} X_s$. Then X_F can be equipped with a Fréchet frame with respect to an appropriately sequence space.

Proof. Since X_s is a separable Banach space for any $s \in \mathbb{N}$, there is a sequence $\{x_i^s\}_{i=1}^{\infty} \subseteq X_s$, such that $\overline{\{x_i^s\}_{i=1}^{\infty}} = X_s$. For any $f_s \in X_s$ there is a subsequence $x_{k_i}^s \to f_s$ as $i \to \infty$. By Hahn-Banach Theorem, there is $g_i^s \in X_s^*$ such that $g_i^s(x_i^s) = \|x_i^s\|$ and $\|g_i^s\| = 1$. Now,

$$\|x_{k_i}^s\| = |g_{k_i}^s(x_{k_i})| \leq \|g_{k_i}^s(f^s)\| + \|f^s - x_{k_i}^s\|,$$

so $\|f^s\| \leq \sup \|g_i(f^s)\|$. Since we also have $\|f^s\| \geq \sup_i \|g_i(f^s)\|$, therefore

$$\|f^s\| = \sup_i \|g_i^s(f^s)\|, \quad \forall f^s \in X_s.$$

Let $\Theta_s \subseteq \ell^\infty$ and $\Theta_s = \big\{\{g_i(f^s)\}_{i=1}^{\infty} : f^s \in X_s\big\}$, then $\{\Theta_s\}_{s\in\mathbb{N}}\Theta_s$ satisfies (2.3)-(2.5). Let $\Theta_F := \bigcap_{s\in\mathbb{N}}$ and $S(\{g_i(f)\}_{i=1}^{\infty}) = f$. Then $(\{g_i^s\}_{i=1}^{\infty}, S)$ is a Fréchet frame for X_F with respect Θ_F. □

3. Perturbation of Fréchet Frames

Like the results about p-frames [20], Banach frames and atomic decompositions [6, 11], generalized frames [14], continuous frames [19], we study perturbations of Fréchet frames. We need the following assertion.

Lemma 3.1. [12] Let $U : X \to X$ be a linear operator and assume that there exist constants $\lambda_1, \lambda_2 \in [0, 1[$ such that

$$\|x - Ux\| \leq \lambda_1\|x\| + \lambda_2\|Ux\|, \quad \forall x \in X.$$

Then U is invertible and

$$\frac{1-\lambda_1}{1+\lambda_2}\|x\| \leq \|Ux\| \leq \frac{1+\lambda_1}{1-\lambda_2}\|x\|$$

$$\frac{1-\lambda_2}{1+\lambda_1}\|x\| \leq \|U^{-1}x\| \leq \frac{1+\lambda_2}{1-\lambda_1}\|x\|$$

for all $x \in X$.

The simplest assertion about perturbation of F-Bessel sequences is:

Proposition 3.2. Let X_F be a Fréchet space satisfying (1)-(3) and let Θ_F be a Fréchet sequence space satisfying (1)-(3) so that Θ_s is a reflexive CB-space for every s. Let $\{g_i\}$ be an Fréchet Bessel sequence for X_F with respect to Θ_F with bounds B_s and let $\{f_i\} \subset X_F^*$.

Assume that $\{(g_i - f_i)(f)\} \in \Theta_F$, $\forall f \in X_F$, and $\exists\, \widetilde{\mu}_s \geq 0$ such that

$$|||\{(g_i - f_i)(f)\}|||_s \leq \widetilde{\mu}_s\|f\|_s, \ \forall f \in X_F \tag{3.1}$$

(i.e. $\{g_i - f_i\}$ is an Fréchet Bessel sequence for X_F w.r.t. Θ_F). Then $\{f_i\}$ is an F-Bessel sequence for X_F w.r.t. Θ_F with bounds $B_s + \widetilde{\mu}_s$.

The converse also holds.

Proof. It is clear that $\{(f_i)(f)\} = \{(-g_i + f_i)(f)\} + \{(g_i - f_i)(f)\} \in \Theta_F$ for all $f \in X_F$, also

$$|||\{f_i(f)\}|||_s \leq |||\{(g_i - f_i)(f)\}|||_s + |||\{(g_i - f_i)(f)\}|||_s \leq (B_s + \widetilde{\mu}_s)\|f\|$$

for all $s \in \mathbb{N}, f \in X_F$. □

The following Theorem generalizes a Theorem of [11] to Fréchet frames and gives a necessary and sufficient condition for the stability of Fréchet frames.

Theorem 3.3. Let $\big(\{g_i\}, S\big)$ be a Fréchet frame for X_F with respect to Θ_F with bounds A_s and B_s. Let $\{h_i\} \subseteq X_F^*$ such that $\{h_i(f)\} \in \Theta_F$ for all $f \in X_F$ and let $D : \Theta_F \to \Theta_F$ be a continuous linear operator such that $D\{h_n(f)\} = \{g_n(f)\}$, $f \in X_F$. Then there exists an operator $V : \Theta_F \to X_F$ such that $(\{h_n\}, V)$ is a Fréchet frame if and only if for each $s \in \mathbb{N}$ there exists $\lambda_s > 0$ such that

$$|||\{(g_n - h_n)(f)\}|||_s \leq \lambda_s min\{|||\{g_n(f)\}|||_s, |||\{(h_n(f))\}|||_s\} \tag{3.2}$$

for all $f \in X_F$.

Proof. Suppose there exists λ_s for every $s \in \mathbb{N}$ such that (3.2) holds. By assumption, $\{h_i(f)\} \in \Theta_F$ for all $f \in X_F$. For any $f \in X_F$ and $s \in \mathbb{N}$,

$$\begin{aligned}
A_s\|f\|_s &\leq |||\{g_n(f)\}|||_s \\
&\leq |||(g_n - h_n)(f)\}|||_s + |||\{h_n(f)\}|||_s \\
&\leq \lambda_s|||\{h_n(f)\}|||_s + |||\{h_n(f)\}|||_s \\
&= (1 + \lambda_s)|||\{h_n(f)\}|||_s \\
&\leq (1 + \lambda_s)\big(|||\{(g_n - h_n)(f)\}|||_s + |||\{g_n(f)\}|||_s\big) \\
&\leq (1 + \lambda_s)^2|||\{g_n(f)\}|||_s \\
&\leq (1 + \lambda_s)^2 B_s\|f\|_s.
\end{aligned}$$

So we have

$$\frac{A_s}{1 + \lambda_s}\|f\|_s \leq |||\{h_n(f)\}|||_s \leq (1 + \lambda_s)B_s\|f\|_s.$$

Let $V = SD$. Then $V(\{h_n(f)\}) = SD(\{h_n(f)\}) = f$, i.e. $\big(\{h_n\}, V\big)$ is a Fréchet frame for X_F with respect to Θ_F.
Conversely, suppose $\big(\{g_i\}, S\big)$ and $\big(\{h_n\}, V\big)$ are Fréchet frames for X_F with respect to Θ_F with bounds A_s, B_s and A'_s, B'_s , respectively. Then, by using the inequalities, we get

$$|||\{(g_n - h_n)(f)\}|||_s \leq \big(1 + \frac{B'_s}{A_s}\big)|||\{g_n(f)\}|||_s, \quad f \in X_F$$

and

$$|||\{(g_n - h_n)(f)\}|||_s \leq \big(1 + \frac{B_s}{A'_s}\big)|||\{h_n(f)\}|||_s, \quad f \in X_F.$$

Choose $\lambda_s := Max\{1 + \frac{B'_s}{A_s}, 1 + \frac{B_s}{A'_s}\}$, therefore (3.2) holds for any $f \in X_F$. □

Theorem 3.4. Let $\big(\{g_i\}, S\big)$ be a Fréchet frames for X_s with respect to Θ_s. Let $\{h_i\} \in X_F^*$ such that $\{h_i(f)\} \in \Theta_F$ for all $f \in X_F$ and let $D : \Theta_F \to \Theta_F$ be a continuous (or F-bounded) linear operator such that $D\{g_n(f)\} = \{h_n(f)\}$, $f \in X_F$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers for which $0 < \inf \alpha_n \leq \sup \alpha_n < \infty$ and $0 < \inf \beta_n \leq \sup \beta_n < \infty$. If for any $s \in \mathbb{N}$ there exist non negative scalers $\lambda_s, \mu_s \in [0, 1[$ and γ_s such that

(i) $\|S\|_s \gamma_s < (1 - \lambda_s)(\inf \alpha_n)$
(ii) $|||\{(\alpha_n g_n - \beta_n h_n)f\}|||_s \leq \lambda_s |||\{(\alpha_n g_n)f\}|||_s + \mu_s |||\{(\beta_n h_n)f\}|||_s + \gamma_s \|f\|_s, \forall f \in X_F,$

then there is a F-bounded operator $V : \Theta_F \to X_F$ such that $(\{h_i\}, V)$ is a Fréchet frame for X_F with respect to Θ_F.

Proof. Let $W : X_F \to \Theta_F$ with $Wf := \{g_i(f)\}$ for all $f \in X_F$. Then $SW : \Theta_F \to \Theta_F$ is an identity operator and for all $s \in \mathbb{N}$

$$\|f\|_s = \|SWf\|_s \leq \|S\|_s |||\{g_i(f)\}|||_s.$$

Now,

$$\begin{aligned} |||\{(\beta_n h_n)f\}|||_s &\leq |||\{(\alpha_n g_n)f\}|||_s + |||\{(\alpha_n g_n - \beta_n h_n)f\}|||_s \\ &\leq |||\{(\alpha_n g_n)f\}|||_s + \lambda_s |||\{(\alpha_n g_n)f\}|||_s \\ &+ \mu_s |||\{(\beta_n h_n)f\}|||_s + \gamma_s \|f\|_s \end{aligned}$$

for all $s \in \mathbb{N}$ and $f \in X_F$. Therefore

$$(1 - \mu_s)|||\{(\beta_n h_n)f\}|||_s \leq \big((1 + \lambda_s)\|W\|_s(sup\alpha_n) + \gamma_s\big)\|f\|_s,$$

also

$$\begin{aligned} (1 - \mu_s)\big(\inf \beta_n\big)|||\{(h_n)f\}|||_s &\leq (1 - \mu_s)|||\{(\beta_n h_n)f\}|||_s \\ &\leq \big((1 + \lambda_s)\|W\|_s(\sup \alpha_n) + \gamma_s\big)\|f\|_s, \end{aligned}$$

By using (2), we have:

$$\begin{aligned} (1 + \mu_s)|||\{(\beta_n h_n)f\}|||_s &\geq (1 - \lambda_s)|||\{(\alpha_n g_n)f\}|||_s - \gamma_s \|f\|_s \\ &\geq \big((1 - \lambda_s)\|S\|_s^{-1}(\inf \alpha_n) - \gamma_s\big)\|f\|_s \end{aligned}$$

for all $f \in X_F$ and $s \in \mathbb{N}$. Therefore

$$\begin{aligned} (1 + \mu_s)\big(\sup \beta_n\big)|||\{(h_n)f\}|||_s &\geq (1 + \mu_s)|||\{(\beta_n h_n)f\}|||_s \\ &\geq \big((1 - \lambda_s)\|S\|^{-1}\big(\inf \alpha_n\big) - \gamma_s\big)\|f\|_s \end{aligned}$$

for all $f \in X_F$ and $s \in \mathbb{N}$. The above inequality shows the frame bounds. Let $V = SD$. Then V is a bounded operator that $V\{h_n(f)\} = f$, for all $f \in X_F$. Therefore $(\{h_i\}, V)$ is a Fréchet frame for X_F with respect to Θ_F. □

Theorem 3.5. Let $(\{g_i\}, V)$ be a Fréchet frame for X_F with respect to Θ_F. Suppose $\lambda_{1_s}, \lambda_{2_s}, \mu_s \geq 0$ such that $\max\{\lambda_{2_s}, \lambda_{1_s} + \mu_s B_s\} < 1$ for all $s \in \mathbb{N}$ and $S : \Theta_F \to X_F$ a continuous operator such that for any $\{c_i\} \in \Theta_F$ and $s \in \mathbb{N}$

$$\|S\{c_i\} - V\{c_i\}\|_s \leq \lambda_1 \|V\{c_i\}\|_s + \lambda_{2_s} \|S\{c_i\}\|_s + \mu_s |||\{c_i\}|||_s \tag{3.3}$$

then there exists a $\{h_i\} \subseteq X_F^*$ such that $(\{h_i\}, S)$ is a Fréchet frame of X_F with respect to Θ_F.

Proof. For $f \in X_F$, let $c_i = g_i(f)$ in (3.3), then we have

$$\|S\{g_i(f)\} - V\{g_i(f)\}\|_s \leq \lambda_{1_s} \|V\{g_i(f)\}\|_s + \lambda_{2_s} \|S\{g_i(f)\}\|_s + \mu_s |||\{g_i(f)\}|||_s$$

since $V(\{g_i(f)\}) = f$, so

$$\|S\{g_i(f)\} - f\|_s \leq \lambda_{1_s} \|f\}\|_s + \lambda_{2_s} \|S\{g_i(f)\}\|_s + \mu_s B_S \|f|_s.$$

Let $L(f) := S\{g_i(f)\}$, so

$$\|f - Lf\|_s \leq \lambda_{1_s} \|f\|_s + \lambda_{2_s} \|Lf\|_s + \mu_s B_S \|f|_s$$

or

$$\|f - Lf\|_s \leq (\lambda_{1_s} + \mu_s B_s)\|f\|_s + \lambda_{2_s} \|Lf\|_s.$$

Lemma 3.1 results that the operator L is invertible and

$$\frac{1 - \lambda_{2_s}}{1 + \lambda_{1_s} + \mu B_s} \|f\|_s \leq \|L^{-1} f\|_s \leq \frac{1 + \lambda_{2_s}}{1 - (\lambda_{1_s} + \mu_s) B_s} \|f\|_s$$

and $f = LL^{-1} f = S(\{g_i(L^{-1} f)\})$. It is clear that $\{g_i(L^{-1} f)\} \subseteq X_F^*$. By choosing $h_i = g_i \circ L^{-1}$, we have

$$|||\{h_i(f)\}|||_s = |||\{g_i(L^{-1} f)\}|||_s \geq A_s |||L^{-1}|||_s \geq \frac{A_s(1 - \lambda_{2_s})}{1 + \lambda_{1_s} + \mu_s B_s} \|f\|_s$$

and

$$|||\{h_i(f)\}|||_s \leq B_s \|L^{-1} f\|_s \leq \frac{B_s(1 + \lambda_{2_s})}{1 - (\lambda_{1_s} + \mu_s B_s)} \|f\|_s.$$

□

Theorem 3.6. Let $(\{g_i\}, S)$ be a F-frame for X_F with respect to Θ_F. Let $\{h_i\} \subset X_F^*$. If for every $s \in \mathbb{N}$ there exist $\lambda_s, \mu_s \geq 0$ such that

(i) $\lambda_s \|U\| + \mu_s \leq \|S\|_s^{-1}$,
(ii) $|||\{(g_i - h_i)f\}|||_s \leq \lambda_s |||\{(g_i)f\}|||_s + \mu_s \|f\|_s$ for all $s \in \mathbb{N}$ and $f \in X_F$.

Then there is a continuous operator T such that $(\{h_i\}, T)$ is a F-frame for X_F with respect to Θ_F.

Proof. It is a direct result of (2) that the operator $V : X_F \to \Theta_F$ defined by $Vf := \{h_i(f)\}$ is bounded and

$$|||Uf - Vf|||_s \leq \lambda_s |||Uf|||_s + \mu_s \|f\|_s$$

for all $s \in \mathbb{N}$ and $f \in X_F$. Therefore,

$$|||Vf|||_s \leq (\|U\|_s + \lambda_s \|U\|_s + \mu_s)\|f\|_s \tag{3.4}$$

for all $s \in \mathbb{N}$ and $f \in X_F$. Also, $SU = I$ imply that

$$\|I - SV\|_s \leq \|S\|_s(\lambda_s \|U\|_s + \mu_s) < 1.$$

Therefore SV is invertible and $\|(SV)^{-1}\|_s \leq \big(1 - (\lambda\|U\|_s + \mu)\|S\|_s)\big)^{-1}$. Finally $T = (SV)^{-1}S$ and $TV = I$. For all $f \in X_F$ and $s \in \mathbb{N}$

$$\|f\|_s \leq \|T\|_s . |||Vf|||_s \leq \frac{\|S\|}{1 - (\lambda_s \|U\|_s + \mu_s)\|S\|_s} |||Vf|||_s. \tag{3.5}$$

(3.4) and (3.5) which imply that for every $s \in \mathbb{N}$ and $f \in X_F$:

$$\frac{1 - (\lambda_s \|U\|_s + \mu_s)\|S\|_s}{\|S\|_s} \|f\|_s \leq |||\{h_i(f)\}|||_s \leq (\|U\|_s + \lambda_s \|U\|_s + \mu_s)\|f\|_s.$$

So $(\{h_i\}, T)$ is a F-frame for X_F with respect to Θ_F. $\square$

4. Perspective

However the concept of Fréchet frame is new and there are few papers in this area, but in my opinion it can be generalized in more general setting and some concepts like g-frame, controlled frames, controlled g-frames [18], multiplier of frames [2] and etc may be extended to Fréchet frames.

Acknowledgment: Some of the results in this paper were obtained during the author's visiting at the Acoustics Research Institute, Austrian Academy of Sciences, Austria (Summers 2009, 2010 and 2011). He thanks this institute for hospitality. Also, he would like to thank Peter Balazs and D. T. Stoeva for useful discussions, comments and suggestions. This work was partly supported by the WWTF project MULAC ('Frame Multipliers: Theory and Application in Acoustics; MA07-025)

References

1. A. Aldroubi, Q. Sun and W. Tang, *p-frames and shift invariant subspaces of L^p*, J. Fourier Anal. Appl. 7 no. 1, (2001) 1-22.
2. P. Balazs, D. Bayer, A. Rahimi, *Multipliers for continuous frames in Hilbert spaces,* Journal of Physics A: Mathematical and Theoretical, 45 (2012) .
3. P. G. Casazza and O. Christensen, *Perturbation of operators and application to frame theory,* J. Fourier Anal. Appl. 3(5) (1997) 543–557.
4. P. G. Casazza, O. Christensen and D. T. Stoeva, *Frame expansions in separable Banach spaces*, J. Math. Anal. Appl. 307 (2005) 710723.

5. P. G. Casazza, D. Han and D. Larson, *Frames for Banach spaces*, Contemp. Math. 247 (1999) 149182.
6. O. Christensen, *An Introduction to Frame and Riesz Bases,* Birkhauser 2002.
7. O. Christensen, *Frame perturbations,* Proc. Amer. Math. Soc. 123 (1995) 12171220.
8. O. Christensen, *A Paley-Wiener theorem for frames,* Proc. Amer. Math. Soc. 123 (1995) 21992201.
9. R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series,* Trans. Amer. Math. Soc. 72 (1952) 341-366.
10. K. Gröchenig, *Describing functions: atomic decompositions versus frames,* Monatsh. Math. 112 (1) (1991) 142.
11. P. K. Jain, Sh. K. Kaushik and L. K. Vashisht, *On Stability of Banach Frames, Bull. Korean Math. Soc.* 44 (2007) 73-81.
12. T. Kato, *Perturbation Theory for Linear Operators,* Springer, New York, 1976.
13. R. Meise, D. Vogt, *Introduction to Functional Analysis,* Clarendon Press, Oxford, 1997.
14. A. Najati and A. Rahimi, *Generalized frames in Hilbert spaces,* Bull. Iran. Math. Soc. Vol. 35 No. 1 (2009) 97-109.
15. S. Pilipović, D.T. Stoeva and N. Teofanov, *Frames for Fréchet spaces,* Bull. Cl. Sci. Math. Nat., 32 (2007) 6984.
16. S. Pilipović and D.T. Stoeva, *Series expansions in Frechet spaces and their duals; construction of Frechet frames,* Journal of Approximation Theory 163 (2011) 17291747.
17. A. Rahimi, *Frames and Their Generalizations in Hilbert and Banach Spaces ,* Lambert Academic Publishing, 2011.
18. A. Rahimi and A. H. Fereydooni, *Controlled G-Frames And Their G-Multipliers In Hilbert Spaces,* Anal. Stii. Univ. OVIDIUS CON. Seria Math, (accepted paper).
19. A. Rahimi, A. Najati and Y. N. Dehghan, *Continuous frames in Hilbert spaces,* Meth. Func. Anal. Top. Vol. 12, No. 2 (2006) 170-182.
20. D. T. Stoeva, *Perturbation of frames in Banach spaces,* Asian- European Journal of Mathematics 5, N0.1 (2012), 1250011 (15 pages), DOI: 10.1142/S1793557112500118.

Department of mathematics, University of Maragheh, Maragheh, Iran.
E-mail address: rahimi@maragheh.ac.ir