INVARINANCE OF FRÉCHET FRAMES UNDER PERTURBATION

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Abstract. Motivating the perturbations of frames in Hilbert and Banach spaces, in this paper we introduce the invariance of Fréchet frames under perturbation. Also we show that for any Fréchet spaces, there is a Fréchet frame and any element in these spaces has a series expansion.

1. Introduction

Historically, theory of frames appeared in the paper of Duffin and Schaeffer in 1952. Around 1986, Daubechies, Grossmann, Meyer and others reconsidered ideas of Duffin and Schaeffer [9], and started to develop the wavelet and frame theory. Frames for Banach spaces were introduced by K. Gröchenig [10] and subsequently many mathematicians have contributed to this theory [1, 4, 5, 17]. There are some complete spaces which are not Banach spaces (like Fréchet spaces). This was the main motivation for frames on Fréchet spaces. The concept of Fréchet frames investigated by Pilipović and Stoeva in [15, 16]. Like Hilbert and Banach spaces, we will show that for any Fréchet space we can find a Fréchet frame and every element in a Fréchet space has a series expansion [4].

In this manuscript we are interested in the problem of finding conditions under which the perturbation of a Fréchet frame is also a Fréchet frame. The following result [3], is one of the most general and also typical results about frame perturbations for the whole space $H$ which generalizes the main results in [7, 8].

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Definition 2.3. Let \( \mathcal{S} \) be a sequence in \( \mathcal{H} \) with bounds \( A \) and \( B \), and \( \mathcal{G} := \{ g_i \}_{i \in I} \) is a sequence in \( \mathcal{H} \). Suppose that there exist non-negative \( \lambda_1, \lambda_2 \) and \( \mu \) with \( \lambda_2 < 1 \) such that

\[
\| \sum_i c_i (f_i - g_i) \| \leq \lambda_1 \| \sum_i c_i f_i \| + \lambda_2 \| \sum_i c_i g_i \| + \mu \| c \|
\]

for each finitely supported \( c \in \ell^2(\mathbb{N}) \) and \( \lambda_1 + \frac{\mu}{\sqrt{A}} < 1 \). Then \( \mathcal{G} \) is a frame for \( \mathcal{H} \) with bounds

\[
A(1 - \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{A}}}{1 + \lambda_2})^2 \quad \text{and} \quad B(1 + \frac{\lambda_1 + \lambda_2 + \frac{\mu}{\sqrt{B}}}{1 - \lambda_2})^2.
\]

2. Fréchet Frames

Throughout this paper, \( (X, \| \cdot \|) \) is a Banach space and \( (X^*, \| \cdot \|^*) \) is its dual, \( (\Theta, \| \cdot \|) \) is a Banach sequence space and \( (\Theta^*, \| \cdot \|^*) \) is the dual of \( \Theta \).

Definition 2.1. A sequence space \( X_d \) is called a Banach space of scalar valued sequences or briefly a BK-space, if it is a Banach space and the coordinate functionals are continuous on \( X_d \), i.e., the relations \( x_n = \{ \alpha_j^{(n)} \}, x = \{ \alpha_j \} \in X_d, \lim x_n = x \) imply \( \lim \alpha_j^{(n)} = \alpha_j \).

Definition 2.2. Let \( X \) be a Banach space and \( X_d \) be a BK-space. A countable family \( \{ g_i \}_{i \in I} \) in the dual \( X^* \) is called an \( X_d \)-frame for \( X \) if

(i) \( \{ g_i(f) \}_{i=1}^{\infty} \in X_d, \forall f \in X \);

(ii) the norms \( \| f \|_X \) and \( \| \{ g_i(f) \}_{i=1}^{\infty} \|_{X_d} \) are equivalent, i.e., there exist constants \( A, B > 0 \) such that

\[
A \| f \|_X \leq \| \{ g_i(f) \}_{i=1}^{\infty} \|_{X_d} \leq B \| f \|_X, \quad \forall f \in X
\]

\( A \) and \( B \) are called \( X_d \)-frame bounds. If at least (1) and the upper condition in (2) are satisfied, \( \{ g_i \}_{i=1}^{\infty} \) is called an \( X_d \)-Bessel sequence for \( X \).

If \( X \) is a Hilbert space and \( X_d = \ell^2 \), (2) means that \( \{ g_i \}_{i=1}^{\infty} \) is a frame, and in this case it is well known that there exists a sequence \( \{ f_i \}_{i=1}^{\infty} \) in \( X \) such that \( f = \sum_{i=1}^{\infty} \langle f, f_i \rangle g_i = \sum \langle f, g_i \rangle f_i \).

Similar reconstruction formulas are not always available in the Banach space setting. This is the reason behind the following definition:

Definition 2.3. Let \( X \) be a Banach space and \( X_d \) a sequence space. Given a bounded linear operator \( S : X_d \rightarrow X \), and an \( X_d \)-frame \( \{ g_i \}_{i=1}^{\infty} \subseteq X^* \), we say that \( \{ g_i \}_{i=1}^{\infty}, S \) is a Banach frame for \( X \) with respect to \( X_d \) if

\[
S(\{ g_i \}) = f, \quad \forall f \in X.
\]
Note that (2.1) can be considered as some kind of generalized reconstruction formula, in the sense that it tells how to come back to \( f \in X \) based on the coefficients \( \{ g_i(f) \}_{i=1}^{\infty} \).

There is a relationship between these definitions, a Banach frame is an atomic decomposition if and only if the unit vectors form a basis for the space \( X_d \). The following Proposition states this result.

**Proposition 2.4.** [5] Let \( X \) be a Banach space and \( X_d \) be a BK-space. Let \( \{ y_i \}_{i=1}^{\infty} \subseteq X^* \) and \( S : X_d \rightarrow X \) be given. Let \( \{ e_i \}_{i=1}^{\infty} \) be the unit vectors in \( X_d \). Then the following are equivalent:

(i) \( (\{ y_i \}_{i=1}^{\infty}, S) \) is a Banach frame for \( X \) with respect to \( X_d \) and \( \{ e_i \}_{i=1}^{\infty} \) is a Schauder basis for \( X_d \).

(ii) \( (\{ y_i \}_{i=1}^{\infty}, \{ S(e_i) \}_{i=1}^{\infty}) \) is an atomic decomposition for \( X \) with respect to \( X_d \).

It is known [5] that every separable Banach space has a Banach frame.

**Theorem 2.5.** Every separable Banach space has a Banach frame with bounds \( A = B = 1 \).

The main motivation of Fréchet frames comes from some sequences \( \{ g_i \}_{i=1}^{\infty} \) which are not Bessel sequences but they give rise to series expansions. For Banach space \( X \), let \( \{ g_i \}_{i=1}^{\infty} \subseteq X^* \) be given and let there exist \( \{ f_i \}_{i=1}^{\infty} \subseteq X \) such that the following series expansion in \( X \) holds

\[
(2.2) \quad f = \sum_{i=1}^{\infty} g_i(f) f_i, \quad \forall f \in X.
\]

Validity of (2.2) does not imply that \( \{ g_i \}_{i=1}^{\infty} \) is a Banach frame for \( X \) with respect to the given sequence space. As one can see in the following examples.

**Example 2.6.** Let \( \{ e_i \}_{i=1}^{\infty} \) be an orthonormal basis for the Hilbert space \( \mathcal{H} \). Consider the sequence \( \{ g_i \}_{i=1}^{\infty} = \{ e_1, e_1, 2e_2, 3e_3, 4e_4, \ldots \} \). This sequence is not a Banach frame for \( \mathcal{H} \) with respect to \( \ell^2 \). However, the series expansion in \( \mathcal{H} \) in the form (2.2) is \( \{ f_i \}_{i=1}^{\infty} = \{ e_1, 0, \frac{1}{2} e_2, \frac{1}{3} e_3, \frac{1}{4} e_4, \ldots \} \).

Validity of (2.2) implies that \( \{ g_i \}_{i=1}^{\infty} \) is a Banach frame for \( X \) with respect to the sequence space \( \{ \{ c_i \}_{i=1}^{\infty} : \sum_{i=1}^{\infty} c_i f_i \text{ converges} \} \).

Recall, a complete locally convex space which has a countable fundamental system of seminorms is called a Fréchet space.

Let \( \{ Y_s, \| \cdot \|_s \}_{s \in \mathbb{N}} \) be a sequence of separable Banach spaces such that

\[
(2.3) \quad \{ 0 \} \neq \cap_{s \in \mathbb{N}} Y_s \subseteq \ldots \subseteq Y_2 \subseteq Y_1 \subseteq Y_0
\]

\[
(2.4) \quad \| \cdot \|_0 \leq \| \cdot \|_1 \leq \| \cdot \|_2 \leq \ldots
\]

\[
(2.5) \quad Y_F := \cap_{s \in \mathbb{N}} Y_s \text{ is dense in } Y_s \text{ \forall } s \in \mathbb{N}.
\]
Then $Y_F$ is a Fréchet space with the sequence of norms $\| \cdot \|_s$, $s \in \mathbb{N}$. We use the above sequences in two cases: $Y_s = X_s$ and $Y_s = \Theta_s$. Let $\{X_s, \| \cdot \|_s\}_{s \in \mathbb{N}}$ and $\{\Theta_s, \| \cdot \|_s\}_{s \in \mathbb{N}}$ be sequences of Banach and Banach sequence spaces, which satisfy (2.3)-(2.5). For fixed $s \in \mathbb{N}$, an operator $V : \Theta_F \rightarrow X_F$ is $s$-bounded if there exist constants $K_s > 0$ such that $\|V\{c_i\}_{i=1}^\infty\|_s \leq K_s\|\{c_i\}_{i=1}^\infty\|_s$ for all $\{c_i\}_{i=1}^\infty \in \Theta_F$. If $V$ is $s$-bounded for every $s \in \mathbb{N}$, then $V$ is called $F$-bounded. Note that an $F$-bounded operator is continuous but the converse does not hold in general. The book of R. Meise, D. Vogt is a very useful textbook about Fréchet spaces [13].

The Banach sequence space $\Theta$ is called solid if the condition $\{c_i\}_{i=1}^\infty \in \Theta$ and $|d_i| \leq |c_i|$ for all $i \in \mathbb{N}$, imply that $\{d_i\}_{i=1}^\infty \in \Theta$ and $\|\{d_i\}_{i=1}^\infty\|_\Theta \leq \lambda \|\{c_i\}_{i=1}^\infty\|_\Theta$. A BK-space which contains all the canonical vectors $e_i$ and for which there exists a constant $\lambda \geq 1$ such that

$$\|\{c_i\}_{i=1}^\infty\|_\Theta \leq \lambda \|\{c_i\}_{i=1}^\infty\|_{\Theta}, \quad \forall n \in \mathbb{N}, \forall \{c_i\}_{i=1}^\infty \in \Theta$$

will be called $\lambda$-BK-space. A BK-space is called a CB-space if the set of the canonical vectors forms a basis.

**Definition 2.7.** Let $\{X_s, \| \cdot \|_s\}_{s \in \mathbb{N}}$ be a family of Banach spaces, satisfying (2.3)-(2.5) and let $\{\Theta_s, \| \cdot \|_s\}_{s \in \mathbb{N}}$ be a family of BK-spaces, satisfying (2.3)-(2.5). A sequence $\{g_i\}_{i=1}^\infty \subseteq X_F^*$ is called a pre-Fréchet frame (a pre-F-frame) for $X_F$ with respect to $\Theta_F$ if for every $s \in \mathbb{N}$ there exist constants $0 < A_s \leq B_s < \infty$ such that

$$\{g_i(f)\}_{i=1}^\infty \in \Theta_F,$$

(2.6)

$$A_s\|f\|_s \leq \|\{g_i(f)\}_{i=1}^\infty\|_s \leq B_s\|f\|_s,$$

(2.7)

for all $f \in X_F$. The constants $A_s$ and $B_s$ are called lower and upper bounds for $\{g_i\}_{i=1}^\infty$. The pre-F-frame is called tight if $A_s = B_s$ for all $s \in \mathbb{N}$. Moreover, if there exists a $F$-bounded operator $S : \Theta_F \rightarrow X_F$ so that $S(\{g_i(f)\}_{i=1}^\infty) = f$ for all $f \in X_F$, then a pre-F-frame $\{g_i\}_{i=1}^\infty$ is called a Fréchet frame (or F-frame) for $X_F$ with respect to $\Theta_F$ and $S$ is called an F-frame operator of $\{g_i\}_{i=1}^\infty$. When (2.6) and at least the upper inequality in (2.7) hold, then $\{g_i\}_{i=1}^\infty$ is called a F-Bessel sequence for $X_F$ with respect to $\Theta_F$.

Since $X_F$ is dense in $X_s$ for all $s \in \mathbb{N}$, $g_i$ has a unique continuous extension on $X_s$, we show it by $g_i^s$. Thus $g_i^s \in X_s^*$ and $g_i^s = g_i$ on $X_F$.

The following Theorem gives some conditions, under which an element can be expanded by some elementary vectors.

**Theorem 2.8.** [15, 16] Let $\{X_s, \| \cdot \|_s\}_{s \in \mathbb{N}}$ be a family of Banach spaces, satisfying (2.3)-(2.5) and let $\{\Theta_s, \| \cdot \|_s\}_{s \in \mathbb{N}}$ be a family of CB-spaces, satisfying (2.3)-(2.5) and we assume that $\Theta_s^*$ is a CB-space for every
with a Fréchet frame with respect to an appropriately sequence space.

\[ \sum_{i=1}^{\infty} X_i \]

so that

\[ \forall f \in X_F \text{ and } \forall g \in X_F^*; \]

\[ \sum_{i=1}^{\infty} g_i^* (f_i) \text{ and } g = \sum_{i=1}^{\infty} g_i^* g_i; \]

\[ \forall f \in X_s \text{ and } \forall g \in X_s^*; \]

\[ \forall s \in \mathbb{N}; \]

\[ \forall s \in \mathbb{N}. \{ f_i \}_{i=1}^{\infty} \text{ is a } \Theta_s^*\text{-frame for } X_s^*; \]

if and only if there exists a continuous projection \( U \) from \( \Theta_F \) onto its subspace \( \{ \{ g_i(f) \}_{i=1}^{\infty} : f \in X_F \} \).

The following proposition shows that the pre-Fréchet Besselness is equivalent to the F-boundedness of an operator.

**Proposition 2.9.** Let \( \{ X_s, \| \cdot \|_s \} \) be a family of Banach spaces, satisfying (2.3)-(2.5) and let \( \{ \Theta_s, \| \cdot \|_s \} \) be a family of CB-spaces, satisfying (2.3)-(2.5) and we assume that \( \Theta_s \) is a CB-space for every \( s \in \mathbb{N} \).

The family \( \{ g_i \}_{i=1}^{\infty} \subseteq X_F^* \) is a pre-Fréchet Bessel sequences for \( X_F \) with respect to \( \Theta_F \) if and only if the operator \( T : \Theta_F^* \to X_F^* \) defined by

\[ T\{ d_i \}_{i=1}^{\infty} = \sum_{i=1}^{\infty} d_i g_i \]

is well defined and \( \| T \|_s \leq B_s \), for all \( s \in \mathbb{N} \).

**Proof.** First, suppose that \( \{ g_i \}_{i=1}^{\infty} \subseteq X_F^* \) is a pre-Fréchet Bessel sequences for \( X_F \) with respect to \( \Theta_F \). Define the operator

\[ R : X_F \to \Theta_F \]

by

\[ Rf = \{ g_i(f) \}_{i=1}^{\infty}. \]

Since \( \{ g_i(f) \}_{i=1}^{\infty} \) is a pre-Fréchet Bessel sequence, so \( \| R \|_s \leq B_s \) for every \( s \in \mathbb{N} \). The adjoint of \( R \) is in the form \( R^* : \Theta_F^* \to X_F^* \) and

\[ R^*(e_j) f = e_j (Rf) = e_j (\{ g_i(f) \}_{i=1}^{\infty}) = g_j f, \]

so \( R^* e_j = g_j \). Put \( T = R^* \), then \( \| T \|_s \leq B_s \) and

\[ T\{ d_i \}_{i=1}^{\infty} = T(\sum_{i=1}^{\infty} d_i e_i) = \sum_{i=1}^{\infty} d_i T e_i = \sum_{i=1}^{\infty} d_i R^* e_i = \sum_{i=1}^{\infty} d_i g_i. \]

Conversely, suppose the operator \( T : \Theta_F^* \to X_F^* \) defined by \( T\{ d_i \}_{i=1}^{\infty} = \sum_{i=1}^{\infty} d_i g_i \) is well defined and \( s \)-bounded for all \( s \in \mathbb{N} \). It is clear that \( T e_i = g_i \) and \( T^* : X_F^{**} \to \Theta_F^{**} \) is \( (T^* f) e_i = f(T e_i) = f(g_i) \). Therefore \( \{ g_i(f) \}_{i=1}^{\infty} = \{ (T^* f) e_i \}_{i=1}^{\infty} \), i.e. \( \{ g_i(f) \}_{i=1}^{\infty} \in \Theta_F \). The \( s \)-boundedness of \( T \) imply that \( \| \{ g_i(f) \}_{i=1}^{\infty} \|_s \leq B_s. \)

Similar to Theorem 2.5, the following Theorem shows that for any Fréchet space \( X_F \) we can construct a Fréchet frame.

**Theorem 2.10.** Let \( \{ X_s, \| \cdot \|_s \} \) be a family of separable Banach spaces satisfying (2.3)-(2.5). Let \( X_F := \bigcap_{s \in \mathbb{N}} X_s \). Then \( X_F \) can be equipped with a Fréchet frame with respect to an appropriately sequence space.
Proof. Since \( X_s \) is a separable Banach space for any \( s \in \mathbb{N} \), there is a sequence \( \{ x_i^s \}_{i=1}^{\infty} \subseteq X_s \), such that \( \overline{\{ x_i^s \}_{i=1}^{\infty}} = X_s \). For any \( f_s \in X_s \) there is a subsequence \( x_{k_i}^s \to f_s \) as \( i \to \infty \). By Hahn-Banach Theorem, there is \( g_i^s \in X_s^* \) such that \( g_i^s(x_{k_i}^s) = \| x_{k_i}^s \| \) and \( \| g_i^s \| = 1 \). Now,

\[
\| x_{k_i}^s \| = |g_{k_i}^s(x_{k_i}^s)| \leq \| g_{k_i}^s(f^s) \| + \| f^s - x_{k_i}^s \| ,
\]

so \( \| f^s \| \leq \sup_i \| g_i(f^s) \| \). Since we also have \( \| f^s \| \geq \sup_i \| g_i(f^s) \| \), therefore

\[
\| f^s \| = \sup_i \| g_i(f^s) \| , \quad \forall f^s \in X_s .
\]

Let \( \Theta_s \subseteq f^\infty \) and \( \Theta_s = \{ \{ g_i(f^s) \}_{i=1}^{\infty} : f^s \in X_s \} \), then \( \{ \Theta_s \}_{s \in \mathbb{N}} \Theta_s \) satisfies (2.3)-(2.5). Let \( \Theta_F := \bigcap_{k \in \mathbb{N}} S(\{ g_i(f) \}_{i=1}^{\infty}) = f \). Then \( \{ g_i^s \}_{i=1}^{\infty}, S \) is a Fréchet frame for \( X_F \) with respect \( \Theta_F \). \( \Box \)

3. Perturbation of Fréchet Frames

Like the results about \( p \)-frames \([20]\), Banach frames and atomic decompositions \([6, 11]\), generalized frames \([14]\), continuous frames \([19]\), we study perturbations of Fréchet frames. We need the following assertion.

Lemma 3.1. \([12]\) Let \( U : X \to X \) be a linear operator and assume that there exist constants \( \lambda_1, \lambda_2 \in [0, 1] \) such that

\[
\| x - U x \| \leq \lambda_1 \| x \| + \lambda_2 \| U x \| , \quad \forall x \in X .
\]

Then \( U \) is invertible and

\[
\frac{1 - \lambda_1}{1 + \lambda_2} \| x \| \leq \| U x \| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \| x \|
\]

\[
\frac{1 - \lambda_2}{1 + \lambda_1} \| x \| \leq \| U^{-1} x \| \leq \frac{1 + \lambda_2}{1 - \lambda_1} \| x \|
\]

for all \( x \in X \).

The simplest assertion about perturbation of \( F \)-Bessel sequences is:

Proposition 3.2. Let \( X_F \) be a Fréchet space satisfying (1)-(3) and let \( \Theta_F \) be a Fréchet sequence space satisfying (1)-(3) so that \( \Theta_s \) is a reflexive \( CB \)-space for every \( s \). Let \( \{ g_i \} \) be an Fréchet Bessel sequence for \( X_F \) with respect to \( \Theta_F \) with bounds \( B_s \) and let \( \{ f_i \} \subseteq X_F^* \).

Assume that \( \{ (g_i - f_i)(f) \} \in \Theta_F \), \( \forall f \in X_F \), and \( \exists \tilde{\mu}_s \geq 0 \) such that

\[
(3.1) \quad \|\{(g_i - f_i)(f)\}\|_s \leq \tilde{\mu}_s \| f \|_s , \quad \forall f \in X_F
\]

(i.e. \( \{ g_i - f_i \} \) is an Fréchet Bessel sequence for \( X_F \) w.r.t. \( \Theta_F \)). Then \( \{ f_i \} \) is an \( F \)-Bessel sequence for \( X_F \) w.r.t. \( \Theta_F \) with bounds \( B_s + \tilde{\mu}_s \).

The converse also holds.
Theorem 3.3. Let \( \{g_i\}, S \) be a Fréchet frame for \( X_F \) with respect to \( \Theta_F \) for all \( f \in X_F \). Let \( \{h_i\} \subseteq X_F \) such that \( \{h_i(f)\} \in \Theta_F \) for all \( f \in X_F \). Suppose there exists an operator \( V : \Theta_F \rightarrow X_F \) such that \( \{h_n(f)\} \) is a Fréchet frame if and only if for each \( s \in \mathbb{N} \) there exists \( \lambda_s > 0 \) such that

\[
\|\{(g_n - h_n)(f)\}\|_s \leq \lambda_s \min\{\|\{g_n(f)\}\|_s, \|\{(h_n(f))\}\|_s\}
\]

for all \( f \in X_F \).

Proof. Suppose there exists \( \lambda_s \) for every \( s \in \mathbb{N} \) such that (3.2) holds. By assumption, \( \{h_i(f)\} \in \Theta_F \) for all \( f \in X_F \). For any \( f \in X_F \) and \( s \in \mathbb{N} \),

\[
A_s\|f\|_s \leq \|\{g_n(f)\}\|_s
\]

\[
\leq \|\{g_n - h_n)(f)\}\|_s + \|\{h_n(f)\}\|_s
\]

\[
\leq \lambda_s\|\{h_n(f)\}\|_s + \|\{h_n(f)\}\|_s
\]

\[
= (1 + \lambda_s)\|\{h_n(f)\}\|_s
\]

\[
\leq (1 + \lambda_s)^2\|\{g_n(f)\}\|_s
\]

\[
\leq (1 + \lambda_s)^2B_s\|f\|_s.
\]

So we have

\[
A_s\frac{\|f\|_s}{1 + \lambda_s} \leq \|\{h_n(f)\}\|_s \leq (1 + \lambda_s)B_s\|f\|_s.
\]

Let \( V = SD \). Then \( V(\{h_n(f)\}) = SD(\{h_n(f)\}) = f \), i.e. \( \{h_n\}, V \) is a Fréchet frame for \( X_F \) with respect to \( \Theta_F \).

Conversely, suppose \( \{g_i\}, S \) and \( \{h_n\}, V \) are Fréchet frames for \( X_F \) with respect to \( \Theta_F \) with bounds \( A_s, B_s \) and \( A'_s, B'_s \), respectively. Then, by using the inequalities, we get

\[
\|\{g_n - h_n)(f)\}\|_s \leq (1 + \frac{B'_s}{A_s})\|\{g_n(f)\}\|_s, \quad f \in X_F
\]
and
\[ \|\{(g_n - h_n)(f)\}\|_s \leq (1 + \frac{B_s}{A'_s})\|\{h_n(f)\}\|_s, \quad f \in X_F. \]

Choose \( \lambda_s := \max\{1 + \frac{B_s'}{A'_s}, 1 + \frac{B_s}{A'_s}\} \), therefore (3.2) holds for any \( f \in X_F \).

**Theorem 3.4.** Let \( \{(g_i), S\} \) be a Fréchet frames for \( X_s \) with respect to \( \Theta_s \). Let \( \{h_i\} \in X_F^s \) such that \( \{h_i(f)\} \in \Theta_F \) for all \( f \in X_F \) and let \( D : \Theta_F \to \Theta_F \) be a continuous (or F-bounded) linear operator such that \( D\{g_n(f)\} = \{h_n(f)\}, f \in X_F \). Let \( \{\alpha_n\} \) and \( \{\beta_n\} \) be sequences of positive numbers for which \( 0 < \inf \alpha_n \leq \sup \alpha_n < \infty \) and \( 0 < \inf \beta_n \leq \sup \beta_n < \infty \). If for any \( s \in \mathbb{N} \) there exist non negative scalars \( \lambda_s, \mu_s \in [0,1] \) and \( \gamma_s \) such that

(i) \( \|S\|_s \gamma_s < (1 - \lambda_s)(\inf \alpha_n) \)

(ii) \( \|\{(\alpha_n g_n - \beta_n h_n)f\}\|_s \leq \lambda_s \|\{(\alpha_n g_n)f\}\|_s + \mu_s \|\{(\beta_n h_n)f\}\|_s + \gamma_s\|f\|s, \forall f \in X_F, \)

then there is a F-bounded operator \( V : \Theta_F \to X_F \) such that \( \{(h_i), V\} \) is a Fréchet frame for \( X_F \) with respect to \( \Theta_F \).

**Proof.** Let \( W : X_F \to \Theta_F \) with \( Wf := \{g_i(f)\} \) for all \( f \in X_F \). Then \( SW : \Theta_F \to \Theta_F \) is an identity operator and for all \( s \in \mathbb{N} \)

\[ \|f\|_s = \|SWf\|_s \leq \|S\|_s\|\{g_i(f)\}\|_s. \]

Now,
\[
\|\{(\beta_n h_n)f\}\|_s \leq \|\{(\alpha_n g_n)f\}\|_s + \|\{(\alpha_n g_n - \beta_n h_n)f\}\|_s \\
\leq \|\{(\alpha_n g_n)f\}\|_s + \lambda_s \|\{(\alpha_n g_n)f\}\|_s + \mu_s \|\{(\beta_n h_n)f\}\|_s + \gamma_s\|f\|s
\]

for all \( s \in \mathbb{N} \) and \( f \in X_F \). Therefore

\[ (1 - \mu_s)\|\{(\beta_n h_n)f\}\|_s \leq (1 + \lambda_s)\|W\|_s(\sup \alpha_n) + \gamma_s\|f\|s, \]

also
\[ (1 - \mu_s)(\inf \beta_n)\|\{(h_n)f\}\|_s \leq (1 - \mu_s)\|\{(\beta_n h_n)f\}\|_s \leq (1 + \lambda_s)\|W\|_s(\sup \alpha_n) + \gamma_s\|f\|s, \]

By using (2), we have:
\[ (1 + \mu_s)\|\{(\beta_n h_n)f\}\|_s \geq (1 - \lambda_s)\|\{(\alpha_n g_n)f\}\|_s - \gamma_s\|f\|s \geq (1 - \lambda_s)\|S\|^{-1}_s(\inf \alpha_n) - \gamma_s\|f\|s, \]

for all \( f \in X_F \) and \( s \in \mathbb{N} \). Therefore
\[ (1 + \mu_s)(\sup \beta_n)\|\{(h_n)f\}\|_s \geq (1 + \mu_s)\|\{(\beta_n h_n)f\}\|_s \geq (1 - \lambda_s)\|S\|^{-1}_s(\inf \alpha_n) - \gamma_s\|f\|s \]
for all \( f \in X_F \) and \( s \in \mathbb{N} \). The above inequality shows the frame bounds. Let \( V = SD \). Then \( V \) is a bounded operator that \( V\{h_t(f)\} = f \), for all \( f \in X_F \). Therefore \( \{h_t\}, V \) is a Fréchet frame for \( X_F \) with respect to \( \Theta_F \).

**Theorem 3.5.** Let \( \{g_t\}, V \) be a Fréchet frame for \( X_F \) with respect to \( \Theta_F \). Suppose \( \lambda_1, \lambda_2, \mu_s \geq 0 \) such that \( \max\{\lambda_2, \lambda_1 + \mu_s B_s\} < 1 \) for all \( s \in \mathbb{N} \) and \( S : \Theta_F \rightarrow X_F \) a continuous operator such that for any \( \{c_i\} \in \Theta_F \) and \( s \in \mathbb{N} \)

\[
(3.3) \quad \|S\{c_i\} - V\{c_i\}\|_s \leq \lambda_1\|V\{c_i\}\|_s + \lambda_2\|S\{c_i\}\|_s + \mu_s\|\{c_i\}\|_s
\]

then there exists a \( \{h_t\} \subseteq X_F^* \) such that \( \{h_t\}, S \) is a Fréchet frame of \( X_F \) with respect to \( \Theta_F \).

**Proof.** For \( f \in X_F \), let \( c_i = g_t(f) \) in (3.3), then we have

\[
\|S\{g_t(f)\} - V\{g_t(f)\}\|_s \leq \lambda_1\|V\{g_t(f)\}\|_s + \lambda_2\|S\{g_t(f)\}\|_s + \mu_s\|\{g_t(f)\}\|_s
\]

since \( V\{g_t(f)\} = f \), so

\[
\|S\{g_t(f)\} - f\|_s \leq \lambda_1\|f\|_s + \lambda_2\|S\{g_t(f)\}\|_s + \mu_s B_s\|f\|_s.
\]

Let \( L(f) := S\{g_t(f)\} \), so

\[
\|f - Lf\|_s \leq \lambda_1\|f\|_s + \lambda_2\|Lf\|_s + \mu_s B_s\|f\|_s
\]

or

\[
\|f - Lf\|_s \leq (\lambda_1 + \mu_s B_s)\|f\|_s + \lambda_2\|Lf\|_s.
\]

Lemma 3.1 results that the operator \( L \) is invertible and

\[
\frac{1 - \lambda_2}{1 + \lambda_1 + \mu_s B_s}\|f\|_s \leq \|L^{-1}f\|_s \leq \frac{1 + \lambda_2}{1 - (\lambda_1 + \mu_s B_s)}\|f\|_s
\]

and \( f = LL^{-1}f = S\{g_t(L^{-1}f)\} \). It is clear that \( \{g_t(L^{-1}f)\} \subseteq X_F^* \).

By choosing \( h_t = g_t \circ L^{-1} \), we have

\[
\|\{h_t(f)\}\|_s = \|\{g_t(L^{-1}f)\}\|_s \geq A_s\|L^{-1}f\|_s \geq \frac{A_s(1 - \lambda_2)}{1 + \lambda_1 + \mu_s B_s}\|f\|_s
\]

and

\[
\|\{h_t(f)\}\|_s \leq B_s\|L^{-1}f\|_s \leq \frac{B_s(1 + \lambda_2)}{1 - (\lambda_1 + \mu_s B_s)}\|f\|_s.
\]

**Theorem 3.6.** Let \( \{g_t\}, S \) be a F-frame for \( X_F \) with respect to \( \Theta_F \). Let \( \{h_t\} \subseteq X_F^* \). If for every \( s \in \mathbb{N} \) there exist \( \lambda_s, \mu_s \geq 0 \) such that

(i) \( \lambda_s\|U\| + \mu_s \leq \|S\|^{-1} \),

(ii) \( \|\{g_t - h_t\}f\|_s \leq \lambda_s\|\{g_t\}f\|_s + \mu_s\|f\|_s \) for all \( s \in \mathbb{N} \) and \( f \in X_F \).

Then there is a continuous operator \( T \) such that \( \{h_t, T\} \) is a F-frame for \( X_F \) with respect to \( \Theta_F \).
Proof. It is a direct result of (2) that the operator \( V : X_F \rightarrow \Theta_F \) defined by \( Vf := \{ h_i(f) \} \) is bounded and
\[
\| Uf - Vf \|_s \leq \lambda_s \| Uf \|_s + \mu_s \| f \|_s
\]
for all \( s \in \mathbb{N} \) and \( f \in X_F \). Therefore,
\[
(3.4) \quad \| Vf \|_s \leq (\| U \|_s + \lambda_s \| U \|_s + \mu_s) \| f \|_s
\]
for all \( s \in \mathbb{N} \) and \( f \in X_F \). Also, \( SU = I \) imply that
\[
\| I - SV \|_s \leq \| S \|_s (\lambda_s \| U \|_s + \mu_s) < 1.
\]
Therefore \( SV \) is invertible and \( \| (SV)^{-1} \|_s \leq (1 - (\lambda \| U \|_s + \mu)\| S \|_s))^{-1} \).
Finally \( T = (SV)^{-1}S \) and \( TV = I \). For all \( f \in X_F \) and \( s \in \mathbb{N} \)
\[
(3.5) \quad \| f \|_s \leq \| T \|_s \| Vf \|_s \leq \frac{\| S \|}{1 - (\lambda_s \| U \|_s + \mu_s)\| S \|_s} \| Vf \|_s.
\]
(3.4) and (3.5) which imply that for every \( s \in \mathbb{N} \) and \( f \in X_F \):
\[
\frac{1}{\| S \|_s} \| f \|_s \leq \| \{ h_i(f) \} \|_s \leq (\| U \|_s + \lambda_s \| U \|_s + \mu_s)\| f \|_s.
\]
So \( \{ h_i \}, T \) is a F-frame for \( X_F \) with respect to \( \Theta_F \).

4. Perspective

However the concept of Fréchet frame is new and there are few papers in this area, but in my opinion it can be generalized in more general setting and some concepts like g-frame, controlled frames, controlled g-frames [18], multiplier of frames [2] and etc may be extended to Fréchet frames.

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