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SANDWICH-TYPE THEOREMS FOR A CLASS OF INTEGRAL OPERATORS WITH SPECIAL PROPERTIES

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ABSTRACT. In the present paper, we prove subordination and superordination and sandwich-type properties of a certain integral operators for univalent functions on open unit disc, moreover the special behavior of this class are investigated.

1. INTRODUCTION

Let $H(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}.$$

For $a \in \mathbb{C}$, let

$$H[a, n] = \{f \in H(\mathbb{U}) : f(z) = a + a_n z^n + \cdots\}.$$

Let f and F be members of the analytic class $H(\mathbb{U})$. The function f is said to be *subordinate* to F, or F is *superordinate* to f, written $f(z) \prec F(z)$, if there exist a function ω analytic in $\mathbb{U}$, with

$$\omega(0) = 0 \quad \text{and} \quad |w(z)| < 1, (z \in \mathbb{U}),$$

such that

$$f(z) = F(\omega(z)), (z \in \mathbb{U})$$

In this case we write

$$f \prec F \quad (z \in \mathbb{U}) \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{U}).$$

If the function F is univalent in $\mathbb{U}$ then we have

$$f \prec F \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subseteq F(\mathbb{U}).$$

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Definition 1.1. [3] Let

$$\phi : \mathbb{C}^2 \to \mathbb{C}$$

and let h be univalent in $\mathbb{U}$. If p is analytic in $\mathbb{U}$ and satisfies the following differential subordination:

$$\phi(p(z), zp'(z)) \prec h(z) \quad (z \in \mathbb{U}), \tag{1.1}$$

then p is called a solution of the differential subordination. A univalent function q is called a dominant of the solution of the differential subordination if $p \prec q$, for all p satisfying the differential subordination (1.1). A dominant q̃ that satisfies q̃ $\prec q$ for all domainants q of (1.1) is said to be the best dominant.

Definition 1.2. [5] Let

$$\phi : \mathbb{C}^2 \to \mathbb{C}$$

and let h be analytic in $\mathbb{U}$. If p and $\phi(p(z), zp'(z))$ are univalent in $\mathbb{U}$ and satisfy the following differential superordination:

$$h(z) \prec \phi(p(z), zp'(z)) \quad (z \in \mathbb{U}), \tag{1.2}$$

then p is called a solution of the differential superordination. An analytic function q is called a subordinant of the solution of the differential superordination if $q \prec p$, for all p satisfying the differential superordination (1.2). A univalent subordinant q̃ that satisfies $q \prec$ q̃ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 1.3. [5] We denote by Q the set of functions f that are analytic and injective on $\bar{\mathbb{U}} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial\mathbb{U}; \lim_{z \to \zeta} f(z) = +\infty\}, \tag{1.3}$$

and such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{U} \setminus E(f)$.

Definition 1.4. [3] Let $K(\alpha)$, $\alpha < 1$ be class of convex functions of order α in $\mathbb{U}$ it is defined as follows

$$K(\alpha) = \{f \in A : \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha, z \in \mathbb{U}\},$$

where $K(0) \equiv K$, is the class of convex and univalent functions in the unit disk.
Moreover, the class of starlike functions of order α in $\mathbb{U}$, $\alpha < 1$ defined by $S^*(\alpha)$,

$$S^*(\alpha) = \{f \in A : \mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, z \in \mathbb{U}\},$$

where, $S^*(0) \equiv S^*$ represents the class of starlike and univalent functions in the unit disk.

Definition 1.5. [3] Functions in the class

$$R = \{f \in A : \mathrm{Re}\{f'(z)\} > 0, z \in \mathbb{U}\}$$

are called functions with bounded turning(rotation).
It is easily shown that $R \subset S^*$.

In present paper we consider the integral operator

$$\Upsilon_{g,\gamma}[f](z) = \frac{\gamma+1}{[g(z)]^\gamma} \int_0^z f(t)[g(t)]^{\gamma-1} g'(t)dt, \tag{1.4}$$

where $\gamma \in \mathbb{C}$, $g(z) \in H(\mathbb{U})$, $\frac{g(z)g'(z)}{z} \neq 0$ in $\mathbb{U}$, moreover f is analytic and $f(0) = f'(0) - 1 = 0$.
In 1995, P. T. Mocanu, et al in [7] introduced this operator and they obtained certain sufficient conditions on g and γ so that $\Upsilon_{g,\gamma}[R] \subset R$, $\Upsilon_{g,\gamma}[R] \subset S^*$ and $\Upsilon_{g,\gamma}[S^*] \subset S^*$. Later Miller and Mocanu in [3] mentioned some other properties for this operator. Here we present Sandwich-type Theorem for this operator, moreover some special cases for g and γ would be deduced some interesting results.

2. Preliminaries

We will need the following lemmas to prove our main results.
Let $c \in \mathbb{C}$ with $\mathrm{Re}\, c > 0$, let $n \in \mathbb{N}^*$ and let

$$C_n = C_n(c) = \frac{n}{\mathrm{Re}\, c}\left[|c|\sqrt{1 + 2\,\mathrm{Re}\left(\frac{c}{n}\right)} + \mathrm{Im}\, c\right].$$

If R is the univalent function $R(z) = \dfrac{2C_n z}{1-z^2}$, then the *open door function* $R_{c,n}$ is defined by

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right),\ z \in \mathbb{U},$$

where $b = R^{-1}(c)$.
Remark that $R_{c,n}$ is univalent in $\mathbb{U}$, $R_{c,n}(0) = c$ and $R_{c,n}(\mathbb{U}) = R(\mathbb{U})$ is the complex plane slit along the half-lines $\mathrm{Re}\, w = 0$, $\mathrm{Im}\, w \geq C_n$ and $\mathrm{Re}\, w = 0$, $\mathrm{Im}\, w \leq -C_n$.
Moreover, if $c > 0$, then $C_{n+1} > C_n$ and $\lim\limits_{n\to\infty} C_n = \infty$, hence $R_{c,n} \prec R_{c,n+1}$ and $\lim\limits_{n\to\infty} R_{c,n}(\mathbb{U}) = \mathbb{C}$. We will use the notation $R_c \equiv R_{c,1}$.
Let denote the class of functions

$$A_n = \{f \in H(\mathbb{U}) : f(z) = z + a_{n+1}z^{n+1} + \cdots\},$$

and let $A \equiv A_1$.

Lemma 2.1 (Integral Existence Theorem). [3] Let $\phi, \Phi \in H[1,n]$ with $\phi(z) \neq 0$, $\Phi(z) \neq 0$ for $z \in \mathbb{U}$. Let $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, $\alpha+\delta = \beta+\gamma$ and $\mathrm{Re}(\alpha+\delta) > 0$. If the function $f(z) = z + a_{n+1}z^{n+1} + \cdots \in A_n$ and satisfies

$$\alpha \frac{zf'(z)}{f(z)} + \frac{z\phi'(z)}{\phi(z)} + \delta \prec R_{\alpha+\delta,n}(z)$$

then

$$F(z) = \left[\frac{\beta+\gamma}{z^\gamma \Phi(z)} \int_0^z f^\alpha(t)\phi(t)t^{\gamma-1}\,\mathrm{d}\,t\right]^{1/\beta} = z + b_{n+1}z^{n+1} + \cdots \in A_n,$$

$\dfrac{F(z)}{z} \neq 0,\ z \in \mathbb{U}$, and

$$\mathrm{Re}\left[\beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in \mathbb{U}.$$

(All powers are principal ones).

Lemma 2.2. Let $\gamma \in \mathbb{C}$ and $\mathrm{Re}(1+\gamma) > 0$, for $z \in \mathbb{U}$, if the function $f(z) \in A$ and satisfies

$$\frac{zf'(z)}{f(z)} + (\gamma - 1)\left(\frac{zg'(z)}{g(z)}\right) + \frac{zg''(z)}{g'(z)} + 1 \prec R_{1+\gamma}(z),$$

then

$$F(z) = \frac{\gamma+1}{[g(z)]^\gamma} \int_0^z f(t)[g(t)]^{\gamma-1}g'(t)dt \in A$$

$\frac{F(z)}{z} \neq 0,\ z \in \mathbb{U}$ and

$$\mathrm{Re}\left[\frac{zF'(z)}{F(z)} + \frac{\gamma z g'(z)}{g(z)}\right] > 0, z \in \mathbb{U}.$$

Lemma 2.3. [4] Suppose that the function

$$H : \mathbb{C}^2 \longrightarrow \mathbb{C}$$

satisfies the following condition:

$$\mathrm{Re}\, H(is,t) \leq 0$$

for all real s and for all

$$t \geq -\frac{1}{2}n(1+s^2), \quad (n \in \mathbb{N})$$

if the function

$$p(z) = 1 + p_n(z)z^n + \ldots$$

is analytic in $\mathbb{U}$ and

$$\mathrm{Re}\{H(p(z), zp'(z))\} > 0, (z \in \mathbb{U}),$$

then $\mathrm{Re}\,\{p(z)\} > 0$.

A function $L(z;t) : \mathbb{U} \times [0,+\infty) \to \mathbb{C}$ is called *a subordination (or a Loewner) chain* if $L(\cdot;t)$ is analytic and univalent in $\mathbb{U}$ for all $t \geq 0$, $L(z;\cdot)$ is continuously differentiable on $[0,\infty)$ for all $z \in \mathbb{U}$, and $L(z;s) \prec L(z;t)$ when $0 \leq s \leq t$.

Lemma 2.4. [8] The function

$$L(z;t) = a_1(t)z + \ldots,$$

with $a_1(t) \neq 0$ and $\lim\limits_{t\to+\infty} |a_1(t)| = \infty$, is a subordination chain if and only if

$$\mathrm{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] > 0,\ z \in \mathbb{U},\ t \geq 0.$$

Lemma 2.5. [3] Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and let $h \in H(\mathbb{U})$, with $h(0) = c$. If $\mathrm{Re}[\beta h(z)+\gamma] > 0,\ z \in \mathbb{U}$, then the solution of the differential equation

$$q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z), \tag{2.1}$$

with $q(0) = c$, is analytic in U and satisfies $\mathrm{Re}[\beta q(z) + \gamma] > 0,\ z \in \mathbb{U}$.

Lemma 2.6. [5] Let $q \in H[a,1]$, let $\chi : \mathbb{C}^2 \to \mathbb{C}$ and set $\chi(q(z), zq'(z)) \equiv h(z)$. If $L(z,t) = \chi(q(z), tzq'(z))$ is a subordination chain and $p \in H[a,1] \cap Q$, then

$$h(z) \prec \chi(p(z), zp'(z)) \quad \text{implies} \quad q(z) \prec p(z).$$

Furthermore, if $\chi(q(z), zq'(z)) = h(z)$ has a univalent solution $q \in Q$, then q is the best subordinant.

Lemma 2.7. [3] Let $p \in Q$ with $p(0) = a$ and let

$$q(z) = a + a_n z^n + \ldots$$

be analytic in $\mathbb{U}$ with $q(z) \neq a$ and $n \in \mathbb{N}$. If q is not subordinate to p, then there exist points

$$z_0 = r_0 \exp(i\theta) \in \mathbb{U} \quad \text{and} \quad \zeta_0 \in \partial\mathbb{U} \setminus E(f),$$

for which

$$q(\mathbb{U}_{r_0}) \subseteq p(\mathbb{U}) \quad , \quad q(z_0) = p(\zeta_0)$$

and

$$z_0 q'(z_0) = m\zeta_0 p'(\zeta_0), \quad (m \geq n).$$

Lemma 2.8. Let $\alpha = \frac{3-4\ln 2}{2(2\ln 2-1)} = 0294\ldots$, $\delta = 1.2468\ldots$ Moreover let $\omega = 1 + \delta 2^{2(1-\alpha)} = 4.3162\ldots$, let one of the following cases satisfies

a: Let in (1.4) we denote $\frac{zg'(z)}{z} = \exp(\lambda z)$, $|\lambda| \leq \frac{\delta}{\omega+1} = 0.2345\ldots$, we will have $F(z) = \displaystyle\int_0^z \frac{f(t)}{t}\exp(\lambda t)dt$.

b: Let in (1.4) we denote $\frac{zg'(z)}{z} = 1 + \lambda z$, $|\lambda| \le \frac{\delta}{\delta+\omega+1} = 0.19\ldots$, we deduce $F(z) = \displaystyle\int_0^z \frac{f(t)}{t}(1+\lambda t)dt$,

then by [7], if $f \in R$ then $F \in S^*$.

3. Main results

At first we need to determine the subset $\mathcal{K} \subset H(\mathbb{U})$ such that the integral operator $\Upsilon_{g,\gamma}$ given by (1.4) will be well-defined. By Lemma 2.2 we can determine $\mathcal{K}$ where the integral operator $\Upsilon_{g,\gamma}$ is well-defined.

$$\mathcal{K} = \mathcal{K}_\gamma^g = \left\{ f \in A : \frac{zf'(z)}{f(z)} + (\gamma-1)\left(\frac{zg'(z)}{g(z)}\right) + \frac{zg''(z)}{g'(z)} + 1 \prec R_{1+\gamma}(z) \right\}.$$

So if $F = \Upsilon_{g,\gamma}[f]$, then $f \in \mathcal{K}_\gamma^g$ implies $F \in A$, $\dfrac{F(z)}{z} \neq 0$, $z \in \mathbb{U}$, and $\operatorname{Re}\left[\dfrac{zF'(z)}{F(z)} + \gamma\dfrac{zg'(z)}{g(z)}\right] > 0$, $z \in \mathbb{U}$.

Theorem 3.1. Let $\gamma \in \mathbb{C}$ with , $0 < \gamma \le 1$, and let $f_1 \in \mathcal{K}_\gamma^g$, such that $\frac{f_1(z)}{z} \neq 0$, for $z \in \mathbb{U}$. Suppose that

$$\operatorname{Re}\left[1 + \frac{zu_1''(z)}{u_1'(z)}\right] > -\frac{\gamma}{2},\ z \in \mathbb{U}, \tag{3.1}$$

where $u_1(z) = g(z)^{\gamma-1}f_1(z)g'(z)$.

Let $f_2 \in \mathcal{K}_\gamma^g$ such that $g(z)^{\gamma-1}f_2(z)g'(z)$ is univalent in $\mathbb{U}$ and

$$g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_2](z)}{z}\right) \in\ Q.$$

Then

$$g(z)^{\gamma-1}f_1(z)g'(z) \prec g(z)^{\gamma-1}f_2(z)g'(z)$$

implies

$$g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right) \prec g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_2](z)}{z}\right),$$

and the function $g(z)^\gamma\left(\dfrac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right)$ is the best subordinant.

Proof. Let $F_1 = \Upsilon_{g,\gamma}[f_1]$, $F_2 = \Upsilon_{g,\gamma}[f_2]$, $u_1(z) = g(z)^{\gamma-1}f_1(z)g'(z)$, $u_2(z) = g(z)^{\gamma-1}f_2(z)g'(z)$, $U_1(z) = g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right)$ and $U_2(z) = g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_2](z)}{z}\right)$, we should prove that $u_1(z) \prec u_2(z)$ implies $U_1(z) \prec U_2(z)$.

Because $f_1, f_2 \in \mathcal{K}_\gamma^g$, then $u_1, u_2 \in A$ and by Lemma 2.2 we have $F_1(z)/z \neq 0$ and $F_2(z)/z \neq 0$, $z \in \mathbb{U}$, hence $U_1, U_2 \in H(\mathbb{U})$ and moreover $U_1, U_2 \in A$.

If we differentiate the relation $F_1(z) = \Upsilon_{g,\gamma}[f_1](z)$ we have

$$g(z)^\gamma \frac{F_1(z)}{z}\left(\frac{zF_1'(z)}{F_1(z)} + \gamma\frac{zg'(z)}{g(z)}\right) = (\gamma+1)f_1(z)g(z)^{\gamma-1}g'(z) \tag{3.2}$$

Since $U_1(z) = g(z)^\gamma \dfrac{F_1(z)}{z}$, by differentiating this relation we obtain

$$\frac{zF_1'(z)}{F_1(z)} + \gamma\frac{zg'(z)}{g(z)} = \frac{zU_1'(z)}{U_1(z)} + 1,$$

by replacing in (3.2) we obtain

$$u_1(z) = \frac{1}{1+\gamma}U_1(z) + \frac{1}{1+\gamma}zU_1'(z) = \chi(U_1(z), zU_1'(z)). \tag{3.3}$$

If we suppose

$$L(z;t) = \frac{1}{1+\gamma}U_1(z) + \frac{t}{1+\gamma}zU_1'(z), \tag{3.4}$$

obviously $L(z;1) = u_1(z)$. Let $L(z;t) = a_1(t)z + \dots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \left(\frac{1+t}{1+\gamma}\right)U_1'(0) = \frac{1+t}{1+\gamma},$$

thus $\lim\limits_{t\to+\infty} |a_1(t)| = +\infty$, since $\gamma > 0$ we obtain $a_1(t) \neq 0$, $\forall t \geq 0$.

Now, by Lemma 2.4 we will prove that $L(z;t)$ is a subordination chain. From (3.4), we can obtain

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = t\operatorname{Re}\left[1 + \frac{zU_1''(z)}{U_1'(z)}\right] + 1.$$

We need to show that

$$\operatorname{Re}\left[1 + \frac{zU''(z)}{U'(z)}\right] > 0,\ z \in \mathbb{U}. \tag{3.5}$$

Denote $q(z) = 1 + \dfrac{zU_1''(z)}{U_1'(z)}$, if we differentiate (3.3) we have

$$u_1'(z) = \frac{1}{1+\gamma}U_1'(z) + \frac{1}{1+\gamma}\left[U_1'(z) + zU_1''(z)\right],$$

and by computing the logarithmical derivative of the above equality we have

$$q(z) + \frac{zq'(z)}{1+q(z)} = 1 + \frac{zu_1''(z)}{u_1'(z)} \equiv H(z). \tag{3.6}$$

From (3.1) we have

$$\mathrm{Re}\,[H(z)+\gamma] > \frac{\gamma}{2} > 0,\ z \in \mathbb{U},$$

now by using Lemma 2.5 we deduce that the differential equation (3.6) has a solution $q \in H(\mathbb{U})$, with $q(0) = H(0) = 1$.
Let us put

$$\Gamma(v,w) = v + \frac{w}{v+1} + \frac{\gamma}{2}. \tag{3.7}$$

From (3.1), (3.6) and (3.7), we obtain

$$\mathrm{Re}\left(\Gamma(q(z), zq'(z))\right) > 0, \quad (z \in \mathbb{U}).$$

Now we should prove that

$$\mathrm{Re}\,\{\Gamma(is,t)\} \le 0, \quad (s \in \mathbb{R}; t \le -\frac{1}{2}(1+s^2)).$$

Indeed, from (3.7), and the assumption $0 < \gamma \le 1$, we have

$$\begin{aligned} \mathrm{Re}\,\{\Gamma(is,t)\} &= \mathrm{Re}\left\{is + \frac{t}{is+1} + \frac{\gamma}{2}\right\} \\ &= \frac{t}{1+s^2} + \frac{\gamma}{2} \le \frac{\gamma-1}{2} \le 0. \end{aligned}$$

Hence by lemma (2.3) we conclude that

$$\mathrm{Re}\,\{q(z)\} > 0 \quad (z \in \mathbb{U}),$$

So the function $U_1(z)$ is convex in $\mathbb{U}$. By lemma(2.4) and above inequality, $L(z;t)$ is a subordination chain, thus according to lemma(2.6) and since $U_1 \in A$ and U_1 is convex (univalent) in $\mathbb{U}$, the differential equation $\chi\left(U_1(z), zU_1'(z)\right) = u_1(z)$, has the univalent solution U_1. Furthermore, we conclude that $u_1(z) \prec u_2(z)$ implies $U_1(z) \prec U_2(z)$, and since U_1 is a univalent solution of the differential equation $\chi\left(U_1(z), zU_1'(z)\right) = u_1(z)$, hence it is the best subordinant of the given differential superordination.

□

Theorem 3.2. Let $\gamma \in \mathbb{C}$ with , $0 < \gamma \le 1$, and let $f_1, f_2 \in \mathcal{K}_\gamma^g$, such that $\frac{f_1(z)}{z} \neq 0$, $\frac{f_2(z)}{z} \neq 0$ for $z \in \mathbb{U}$. Suppose that

$$\mathrm{Re}\left[1 + \frac{zu_1''(z)}{u_1'(z)}\right] > -\frac{\gamma}{2},\ z \in \mathbb{U}, \tag{3.8}$$

where $u_1(z) = g(z)^{\gamma-1} f_1(z) g'(z)$. Then

$$g(z)^{\gamma-1} f_2(z) g'(z) \prec g(z)^{\gamma-1} f_1(z) g'(z)$$

implies

$$g(z)^{\gamma}\left(\frac{\Upsilon_{g,\gamma}[f_2](z)}{z}\right) \prec g(z)^{\gamma}\left(\frac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right),$$

and the function $g(z)^{\gamma}\left(\frac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right)$ is the best dominant of the given subordination.

Proof. The first part of the proof is similar to that of Theorem 3.1, by letting $F_1 = \Upsilon_{g,\gamma}[f_1]$, $F_2 = \Upsilon_{g,\gamma}[f_2]$, $u_1(z) = g(z)^{\gamma-1}f_1(z)g'(z)$, $u_2(z) = g(z)^{\gamma-1}f_2(z)g'(z)$, $U_1(z) = g(z)^{\gamma}\left(\frac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right)$ and $U_2(z) = g(z)^{\gamma}\left(\frac{\Upsilon_{g,\gamma}[f_2](z)}{z}\right)$, we should prove that $u_2(z) \prec u_1(z)$ implies $U_2(z) \prec U_1(z)$.

Because $f_1, f_2 \in \mathcal{K}_{\gamma}^{g}$, then $u_1, u_2 \in A$ and by Lemma 2.2 we have $F_1(z)/z \neq 0$ and $F_2(z)/z \neq 0$, $z \in \mathbb{U}$, hence $U_1, U_2 \in H(\mathbb{U})$ and moreover $U_1, U_2 \in A$.

If we differentiate the relation $F_1(z) = \Upsilon_{g,\gamma}[f_1](z)$ we have

$$g(z)^{\gamma}\frac{F_1(z)}{z}\left(\frac{zF_1'(z)}{F_1(z)} + \gamma\frac{zg'(z)}{g(z)}\right) = (\gamma+1)f_1(z)g(z)^{\gamma-1}g'(z) \tag{3.9}$$

Since $U_1(z) = g(z)^{\gamma}\frac{F_1(z)}{z}$, by differentiating this relation we obtain

$$\frac{zF_1'(z)}{F_1(z)} + \gamma\frac{zg'(z)}{g(z)} = \frac{zU_1'(z)}{U_1(z)} + 1,$$

by replacing in (3.9) we obtain

$$u_1(z) = \frac{1}{1+\gamma}U_1(z) + \frac{1}{1+\gamma}zU_1'(z) = \chi(U_1(z), zU_1'(z)). \tag{3.10}$$

If we suppose

$$L(z;t) = \frac{1}{1+\gamma}U_1(z) + \frac{t}{1+\gamma}zU_1'(z), \tag{3.11}$$

obviously $L(z;0) = u_1(z)$. Denoting $L(z;t) = a_1(t)z + \dots$, then

$$a_1(t) = \frac{\partial L(0;t)}{\partial z} = \left(\frac{2+t}{1+\gamma}\right)U_1'(0) = \frac{2+t}{1+\gamma},$$

hence $\lim_{t\to+\infty}|a_1(t)| = +\infty$, since $\gamma > 0$ we obtain $a_1(t) \neq 0$, $\forall t \geq 0$.

Now, by Lemma 2.4 we will prove that $L(z;t)$ is a subordination chain. From (3.11), we can obtain

$$\operatorname{Re}\left[z\frac{\partial L/\partial z}{\partial L/\partial t}\right] = \operatorname{Re}\left[2 + \frac{zU_1''(z)}{U_1'(z)}\right] + t\operatorname{Re}\left[1 + \frac{zU_1''(z)}{U_1'(z)}\right].$$

Now denote $q(z) = 1 + \frac{zU_1''(z)}{U_1'(z)}$, if we differentiate (3.10) we have

$$u_1'(z) = \frac{1}{1+\gamma}U_1'(z) + \frac{1}{1+\gamma}\left[U_1'(z) + zU_1''(z)\right],$$

and by computing the logarithmical derivative of the above equality we have

$$q(z)+\frac{zq'(z)}{1+q(z)}=1+\frac{zu_1''(z)}{u_1'(z)}\equiv H(z). \tag{3.12}$$

By a same process as in proof of Theorem (3.1) we conclude that

$$\mathrm{Re}\,\{q(z)\}>0 \quad (z\in\mathbb{U}),$$

So the function $U_1(z)$ is convex in $\mathbb{U}$. By lemma(2.4) and above inequality, $L(z;t)$ is a subordination chain. By definition of subordination chain we have,

$$L(z;0)\prec L(z;t), \quad (z\in\mathbb{U}; 0\le t<+\infty),$$

this implies that

$$L(\zeta;t)\notin L(\mathbb{U},0)=u_1(\mathbb{U}), \quad (\zeta\in\partial\mathbb{U}, 0\le t<+\infty)$$

We have $u_2(z)\prec u_1(z)$. Now suppose that U_2 is not subordinate to U_1 then by lemma (2.7), there exist points $z_0\in\mathbb{U}$ and $\zeta_0\in\partial\mathbb{U}$ such that

$$U_2(z_0)=U_1(\zeta_0) \quad \text{and} \quad z_0U_2'(z_0)=(1+t)\zeta_0U_1'(\zeta_0), \quad (0\le t<+\infty).$$

Hence we have

$$\begin{aligned} L(\zeta_0;t) &= \frac{1}{1+\gamma}U_1(\zeta_0)+\frac{1+t}{1+\gamma}\zeta_0U_1'(\zeta_0)\\ &= \frac{1}{1+\gamma}U_2(z_0)+\frac{1}{1+\gamma}\left(z_0U_2(z_0)\right)\\ &= u_2(z_0)\in u_1(\mathbb{U}), \end{aligned}$$

by virtue of the subordination condition $u_2\prec u_1$. This contradicts the above observation that $L(\zeta_0;t)\notin u_1(\mathbb{U})$, therefore $u_2\prec u_1$ must implies $U_2\prec U_1$. Considering $U_1(z)=U_2(z)$, we see that the function F_1 is the best domonant. □

Theorem 3.3. Let $\gamma\in\mathbb{C}$ with $0<\gamma\le 1$, and let $f_1,f_2\in\mathcal{K}_\gamma^g$, such that $f_k(z)/z\neq 0$ for $z\in\mathbb{U}$ and $k=1,2$. Suppose that the next two conditions are satisfied

$$\mathrm{Re}\left[1+\frac{zu_k''(z)}{u_k'(z)}\right]>\frac{-\gamma}{2},\ z\in\mathbb{U},\ \text{for } k=1,2, \tag{3.13}$$

where $u_k(z)=g(z)^{\gamma-1}f_k(z)g'(z)$ and $k=1,2$.

Let $f\in\mathcal{K}_\gamma^g$ such that $g(z)^{\gamma-1}f(z)g'(z)$ is univalent in $\mathbb{U}$ and

$$g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f](z)}{z}\right)\in\ Q.$$

Then

$$g(z)^{\gamma-1}f_1(z)g'(z)\prec g(z)^{\gamma-1}f(z)g'(z)\prec g(z)^{\gamma-1}f_2(z)g'(z)$$

implies

$$g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right)\prec g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f](z)}{z}\right)\prec g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_2](z)}{z}\right).$$

Moreover, the functions $g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_1](z)}{z}\right)$ and $g(z)^\gamma\left(\frac{\Upsilon_{g,\gamma}[f_2](z)}{z}\right)$ are respectively the best subordinant and the best dominant.

Let consider $g(z)=\exp(az)$ and $f_k(z)=\exp(\lambda_k(z))$, with $a,\lambda_1,\lambda_2\in\mathbb{C}$. Then we can obtain the next special case of the Theorem 3.3.

Corollary 3.4. Let $\gamma\in\mathbb{C}$ with $0<\gamma\leq 1$. Let $\lambda_1,\lambda_2\in\mathbb{C}$, and for $a\in\mathbb{C}$ suppose that

$$|a\gamma+\lambda_k|\leq r_0=\frac{\gamma}{2}+1. \tag{3.14}$$

Let $f\in\mathcal{K}_\gamma^{\exp(az)}$ such that $a\exp(a\gamma z)f(z)$ is univalent in $\mathbb{U}$ and

$$\exp(a\gamma z)\left(\frac{\Upsilon_{\exp(az),\gamma}[f](z)}{z}\right)\in\ Q.$$

Then

$$a\exp(a\gamma+\lambda_1)z\prec a\exp(a\gamma z)f(z)\prec a\exp(a\gamma+\lambda_2)z$$

implies

$$\begin{aligned}\exp(a\gamma z)\left(\frac{\Upsilon_{\exp(az),\gamma}\left[\exp(\lambda_1 z)\right](z)}{z}\right)\quad&\prec\quad\exp(a\gamma z)\left(\frac{\Upsilon_{\exp(az),\gamma}[f](z)}{z}\right)\\&\prec\quad\exp(a\gamma z)\left(\frac{\Upsilon_{\exp(az),\gamma}\left[\exp(\lambda_2 z)\right](z)}{z}\right).\end{aligned}$$

Moreover, the functions $\exp(a\gamma z)\left(\frac{\Upsilon_{\exp(az),\gamma}\left[\exp(\lambda_1 z)\right](z)}{z}\right)$ and $\exp(a\gamma z)\left(\frac{\Upsilon_{\exp(az),\gamma}\left[\exp(\lambda_2 z)\right](z)}{z}\right)$ are respectively the best subordinant and the best dominant.

By using lemma (2.8) we can obtain the following corollaries.

Corollary 3.5. Let $\alpha=0.294\ldots$ and $\delta=1.2464\ldots$ Moreover let $\omega=4.3162\ldots$, for $|\lambda|\leq\frac{\delta}{\omega+1}=0.2345\ldots$ let f be univalent and $f\in R$ such that $\frac{f(z)}{z}\exp(\lambda z)$ is univalent in $\mathbb{U}$ and $\frac{F(z)}{z}\in\ Q$ then

$$\frac{f_1(z)}{z}\exp(\lambda z)\prec\frac{f(z)}{z}\exp(\lambda z)\prec\frac{f_2(z)}{z}\exp(\lambda z)$$

implies

$$\frac{F_1(z)}{z}\prec\frac{F(z)}{z}\prec\frac{F_2(z)}{z},$$

where $F(z)=\int_0^z \frac{f(t)}{t}\exp(\lambda t)dt$.
Moreover, the functions $\frac{F_1(z)}{z}$ and $\frac{F_2(z)}{z}$ are respectively the best subordinant and the best dominant.

Corollary 3.6. Let $\alpha=0.294\ldots$ and $\delta=1.2464\ldots$ Moreover let $\omega=4.3162\ldots$, for $|\lambda|\leq\frac{\delta}{\delta+\omega+1}=0.19\ldots$ let f be univalent and $f\in R$ such that $\frac{f(z)}{z}(1+\lambda z)$ is univalent in $\mathbb{U}$ and $\frac{F(z)}{z}\in\ Q$ then

$$\frac{f_1(z)}{z}(1+\lambda z)\prec\frac{f(z)}{z}(1+\lambda z)\prec\frac{f_2(z)}{z}(1+\lambda z)$$

implies

$$\frac{F_1(z)}{z}\prec\frac{F(z)}{z}\prec\frac{F_2(z)}{z},$$

where $F(z)=\int_0^z \frac{f(t)}{t}\exp(\lambda t)dt$.
Moreover, the functions $\frac{F_1(z)}{z}$ and $\frac{F_2(z)}{z}$ are respectively the best subordinant and the best dominant.

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