Sahand Communications in Mathematical Analysis (SCMA) Vol. 1 No. 2 (2014), 1-17.
http://scma.maragheh.ac.ir

A UNIFIED THEORETICAL HARMONIC ANALYSIS APPROACH TO THE CYCLIC WAVELET TRANSFORM (CWT) FOR PERIODIC SIGNALS OF PRIME DIMENSIONS

ARASH GHAANI FARASHAHI[1]* AND MOZHGAN MOHAMMAD-POUR[2]

ABSTRACT. The article introduces cyclic dilation groups and finite affine groups for prime integers, and as an application of this theory it presents a unified group theoretical approach for the cyclic wavelet transform (CWT) of prime dimensional periodic signals.

1. INTRODUCTION

In signal processing, time-frequency analysis comprises those techniques that analyze a signal in both the time and frequency domains simultaneously, commonly called time-frequency methods or representations (see [6, 16, 25]). The mathematical motivation for this analysis is that signals and their transform representation are often tightly connected, and they can be understood better by analyzing them jointly, as a two-dimensional object, rather than separately. Commonly used coherent (structured) methods and techniques in time-frequency analysis are Gabor analysis (see [10, 11, 14, 17]) and wavelet analysis (see [1, 2, 21, 22, 32]) as methods of windowed transforms.

The theory of Gabor analysis is based on the modulations and translations of a given window signal (atom). The phase space (time-frequency plane) has a unified group structure (see [8, 9, 17, 24]), which implies concrete discretizations (see [11, 12]). The theory of finite dimensional Gabor analysis from abstract harmonic analysis point of view, is originally well studied in [11, 12, 24, 28, 29].

2010 *Mathematics Subject Classification.* 42C13, 15A60.
Key words and phrases. Cyclic wavelet transform (CWT), Finite affine group, Prime integer, Digital signal processing.
Received: 15 June 2014, Accepted: 5 November 2014.
* Corresponding author.

Wavelet analysis is a time-frequency method in signal processing which is based on the continuous affine group $(0,\infty)\ltimes\mathbb{R}$ as the group of dilations and translation (see [7, 21, 31, 32, 33]). Standard discretization and quantization of the continuous wavelet transform use dyadic dilations and integer translates of the window single (wavelet), see [3, 4, 35].

Signal processing of periodic signals is the basis of digital signal processing. Classical methods for wavelet analysis of periodic signals or signals of finite size rely on embedding the vector space of finite signals in the Hilbert space of all complex valued sequences with finite energy (ℓ_2-norm), see [5, 30, 34]. Traditional wavelet techniques for periodic signals are not on finite dimensional analogous to the continuous setting as is the case in Gabor analysis (see [5, 24, 30]).

In this article, first the dilation group $\mathbb{U}_p$ and the finite affine group $\mathbb{W}_p$ for the cyclic group $\mathbb{Z}_p$ is introduced, where p is a positive prime integer. If $\mathbf{y}\in\mathbb{C}^p$ is a window signal, we define the cyclic wavelet transform (CWT) $\mathcal{W}_{\mathbf{y}}$ as a transform defined on $\mathbb{C}^p$ with complex values which are indexed in the finite affine group $\mathbb{W}_p$. These techniques imply a unified group theoretical based time-frequency (dilation and translation) representations for signals in $\mathbb{C}^p$. The article presents conditions in such away that the transform $\mathcal{W}_{\mathbf{y}}$ will be isometric. It is shown that CWT satisfies various types of inversion formulas, as well. The matrix representation of CWT will be presented, which can be helpful for any programming related to the CWT computational algorithms.

2. Preliminaries and Notations

Throughout this article the standard and traditional harmonic analysis modeling for the linear vector space of all periodic signals should be used. A given one dimensional (1D) finite discrete data or signal $\mathbf{x}$, i.e. a signal of a given length $N\in\mathbb{N}$ denoted by $\mathbf{x}=[\mathbf{x}(0),...,\mathbf{x}(N-1)]$, which is a function defined on the set $\{0,...,N-1\}\subset\mathbb{Z}$. This type of modeling for indexing of finite signals persists us to consider finite signals as functions defined on the group of unit roots of order N, or equivalently as periodic discrete functions (sequences) $\mathbf{x}:\mathbb{Z}\to\mathbb{C}$ with $\mathbf{x}(n+kN)=\mathbf{x}(n)$ for all $0\leq n\leq N-1$, and $k\in\mathbb{Z}$. With above mathematical modeling of one dimensional (1D) finite discrete signals, the notation $\mathbb{C}^N$ precisely stands for the complex linear vector space of all finite signals of size N. To make the formulas in this article more readable, the finite additive (Abelian) group of all integers between zero and some non-zero integer number N by $\mathbb{Z}_N$ or $\langle N\rangle$ shoud be denoted. Roughly speaking, $\mathbb{Z}_N$ contains the set of all equivalence classes of remainders of all integer numbers module N. The set $\mathbb{Z}_N=\{\overline{0},\overline{1},...,\overline{N-1}\}$ or with a simple notation $\mathbb{Z}_N=\{0,1,...,N-1\}$ is a finite cyclic group with respect to the addition module N with the identity element 0 and the additive inverse $N-l$ for the element $l\in\mathbb{Z}_N$. The set of all bijective (i.e. injective and surjective) homomorphisms on $\mathbb{Z}_N$ (i.e. isomorphisms on $\mathbb{Z}_N$) is denoted by $Aut(\mathbb{Z}_N)$ which is a group with respect to composition of automorphisms. For a matrix $\mathbf{X}\in\mathcal{M}_{N\times M}(\mathbb{C})$, the notation : used to denote all elements indexed by a variable. As an example, if $0\le n\le N$ then $\mathbf{X}(n,:)$ is the n-th row of $\mathbf{X}$. This notation coincides with the notations in MATLAB and FORTRAN programming. For a matrix $\mathbf{X}\in\mathcal{M}_{N\times M}(\mathbb{C})$, the Frobenius norm is defined by

$$\|\mathbf{X}\|_{\mathbf{Fr}}=\left(\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}|\mathbf{X}(n,m)|^2\right)^{1/2}. \tag{2.1}$$

The inner product of $\mathbf{x},\mathbf{y}\in\mathbb{C}^N$ is

$$\langle\mathbf{x},\mathbf{y}\rangle=\sum_{k=0}^{N-1}\mathbf{x}(k)\overline{\mathbf{y}(k)},$$

which induces the ℓ^2-norm or the Frobenius norm

$$\|\mathbf{x}\|_2^2=\|\mathbf{x}\|_{\mathrm{Fr}}^2=\langle\mathbf{x},\mathbf{x}\rangle\quad\text{for}\quad\mathbf{x}\in\mathbb{C}^N.$$

Let $\ell,l\in\mathbb{Z}_N$. The translation operator $T_l:\mathbb{C}^N\to\mathbb{C}^N$ is

$$T_l\mathbf{x}(k)=\mathbf{x}(k-l)\quad\text{for}\quad\mathbf{x}\in\mathbb{C}^N\quad\text{and}\quad l,k\in\mathbb{Z}_N.$$

The modulation operator $M_\ell:\mathbb{C}^N\to\mathbb{C}^N$ is given by

$$M_\ell\mathbf{x}(k)=e^{-2\pi i\ell k/N}\mathbf{x}(k)$$

for $\mathbf{x}\in\mathbb{C}^N$ and $l,k\in\mathbb{Z}_N$. The translation and modulation operators on the Hilbert space $\mathbb{C}^N$ are unitary operators in the Frobenius norm. For $\ell,l\in\mathbb{Z}_N$ we have $(T_l)^*=(T_l)^{-1}=T_{N-l}$ and $(M_\ell)^*=(M_\ell)^{-1}=M_{N-\ell}$. The circular convolution of $\mathbf{x},\mathbf{y}\in\mathbb{C}^N$ is defined by

$$\mathbf{x}*\mathbf{y}(k)=\frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\mathbf{x}(l)\mathbf{y}(k-l)\text{ for }k\in\mathbb{Z}_N.$$

In terms of the translation operators we have

$$\mathbf{x}*\mathbf{y}(k)=\frac{1}{\sqrt{N}}\sum_{l=0}^{N-1}\mathbf{x}(l)T_l\mathbf{y}(k)$$

for $k\in\mathbb{Z}_N$. The circular involution or circular adjoint of $\mathbf{x}\in\mathbb{C}^N$ is given by $\mathbf{x}^*(l)=\overline{\mathbf{x}(-l)}=\overline{\mathbf{x}(N-l)}$. The complex linear space $\mathbb{C}^N$ equipped with the ℓ^1-norm, the circular convolution, and involution is a Banach

$*$-algebra (see [13, 34]). The unitary discrete Fourier transform (DFT) of a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ is defined by

$$\widehat{\mathbf{x}}(\ell) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}(k)\overline{\mathbf{w}_\ell(k)},$$

for all $\ell \in \mathbb{Z}_N$ where for all $\ell, k \in \mathbb{Z}_N$ we have $\mathbf{w}_\ell(k) = e^{2\pi i \ell k/N}$. The set $\{\mathbf{w}_\ell : \ell \in \mathbb{Z}_N\}$ is precisely the additive group of all pure frequencies (characters) $\widehat{\mathbb{Z}_N}$ (i.e homomorphisms or characters into the circle group $\mathbb{T}$) on the additive group $\mathbb{Z}_N$. More precisely, the map $\ell \mapsto \mathbf{w}_\ell$ is a group isomorphism between $\mathbb{Z}_N$ and $\widehat{\mathbb{Z}_N}$ (see [13]). Therefore, $\mathbf{w}_{\ell+\ell'} = \mathbf{w}_\ell \mathbf{w}_{\ell'}$ and $\overline{\mathbf{w}_\ell} = \mathbf{w}_{N-\ell}$ for all $\ell, \ell' \in \mathbb{Z}_N$. Thus DFT of a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ at the frequency $\ell \in \mathbb{Z}_N$ is

$$\begin{aligned} \widehat{\mathbf{x}}(\ell) &= \mathcal{F}_N(\mathbf{x})(\ell) \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}(k)\overline{\mathbf{w}_\ell(k)} \\ &= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathbf{x}(k) e^{-2\pi i \ell k/N}. \end{aligned} \tag{2.2}$$

The discrete Fourier transform (DFT) is a unitary transform in the Frobenius norm, i.e. for all $\mathbf{x} \in \mathbb{C}^N$ satisfies the Parseval formula $\|\mathcal{F}_N(\mathbf{x})\|_{\mathbf{Fr}} = \|\mathbf{x}\|_{\mathbf{Fr}}$, which equivalently implies (see [13, 34])

$$\sum_{\ell=0}^{N-1} |\widehat{\mathbf{x}}(\ell)|^2 = \sum_{k=0}^{N-1} |\mathbf{x}(k)|^2.$$

The Polarization identity implies the Plancherel formula

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{C}^N} = \langle \widehat{\mathbf{x}}, \widehat{\mathbf{y}} \rangle_{\mathbb{C}^N}$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$, which equivalently implies

$$\sum_{\ell=0}^{N-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{\mathbf{y}}(\ell)} = \sum_{k=0}^{N-1} \mathbf{x}(k)\overline{\mathbf{y}(k)}.$$

The unitary DFT (2.2) satisfies (see [13, 19, 20, 34]);

$$\widehat{T_l\mathbf{x}} = M_l\widehat{\mathbf{x}}, \quad \widehat{M_l\mathbf{x}} = T_{N-l}\widehat{\mathbf{x}}, \quad \widehat{\mathbf{x}^*} = \overline{\widehat{\mathbf{x}}}, \quad \widehat{\mathbf{x} * \mathbf{y}} = \widehat{\mathbf{x}}.\widehat{\mathbf{y}},$$

for $\mathbf{x}, \mathbf{y} \in \mathbb{C}^N$ and $l \in \mathbb{Z}_N$. The inversion discrete Fourier formula (IDFT) for a 1D discrete signal $\mathbf{x} \in \mathbb{C}^N$ is

$$\mathbf{x}(\ell) = \sum_{k=0}^{N-1} \widehat{\mathbf{x}}(k)\mathbf{w}_\ell(k) = \sum_{k=0}^{N-1} \widehat{\mathbf{x}}(k)e^{2\pi i\ell k/N}, \quad 0 \le \ell \le N-1. \tag{2.3}$$

3. Prime Dimensional Cyclic Wavelet Transform (CWT)

This section contains two parts. We start by an introduction of cyclic dilation operators and finite affine groups. The article studies basic and interesting properties of cyclic dilation operators. In second part theory of cyclic wavelet transform (CWT) for periodic signals of prime dimension is presented. Throughout this section the study assumes that p is a positive prime integer.

3.1. **Dilation operators and finite affine groups.** The set

$$\mathbb{U}_p := Aut(\mathbb{Z}_p) = \mathbb{Z}_p - \{0\} = \{1, ..., p-1\}, \tag{3.1}$$

is a finite multiplicative Abelian group (see [18, 27]) of order $p-1$ with respect to the multiplication module p with the identity element 1 ([18, 27]). The multiplicative inverse for $m \in \mathbb{U}_p$ (i.e. an element $m_p \in \mathbb{U}_p$ with $mm_p \overset{p}{\equiv} m_p m \overset{p}{\equiv} 1$) is m_p which satisfies $m_p m + np = 1$ for some $n \in \mathbb{Z}$, which can be done by Bezout lemma ([27]).

For prime integer p and $m \in \mathbb{U}_p$, the cyclic dilation operator $D_m : \mathbb{C}^p \to \mathbb{C}^p$ is defined by

$$D_m\mathbf{x}(k) := \mathbf{x}(|m_p k|_p) \quad \text{for} \quad \mathbf{x} = [\mathbf{x}(0), ..., \mathbf{x}(p-1)] \in \mathbb{C}^p,$$

where m_p is the multiplicative inverse of m in $\mathbb{U}_p$. The notation $|.|_p$ stands for the remainder module p. Throughout the article, for simplicity mk is used instead of $|mk|_p$. Thus the dilation operator will be

$$D_m\mathbf{x}(k) = \mathbf{x}(m_p k) \quad \text{for} \quad \mathbf{x} \in \mathbb{C}^p.$$

The following results give us generic properties of dilation operators. First we express algebraic properties of dilation operators.

Proposition 3.1. *Let p be a positive prime integer and $m \in \mathbb{U}_p$. Then,*

(i) *The cyclic dilation operator $D_m : \mathbb{C}^p \to \mathbb{C}^p$ is an $*$-homomorphism.*
(ii) *For $l \in \mathbb{Z}_p$ we have $D_m T_l = T_{ml} D_m$.*

Proof. (i) Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$ and $m \in \mathbb{U}_p$. Then for all $k \in \mathbb{Z}_p$ we have

$$\begin{aligned} D_m(\mathbf{x} * \mathbf{y})(k) &= \mathbf{x} * \mathbf{y}(m_p k) \\ &= \sum_{l=0}^{p-1} \mathbf{x}(l)\mathbf{y}(m_p k - l). \end{aligned}$$

Replacing l with $m_p l$ we get

$$\begin{aligned} \sum_{l=0}^{p-1} \mathbf{x}(l)\mathbf{y}(m_p k - l) &= \sum_{l=0}^{p-1} \mathbf{x}(m_p l)\mathbf{y}(m_p k - m_p l) \\ &= \sum_{l=0}^{p-1} \mathbf{x}(m_p l)\mathbf{y}(m_p(k - l)) \\ &= \sum_{l=0}^{p-1} \mathbf{x}(m_p l) D_m \mathbf{y}(k - l) \\ &= \sum_{l=0}^{p-1} D_m \mathbf{x}(l) D_m \mathbf{y}(k - l) \\ &= (D_m \mathbf{x}) * (D_m \mathbf{y})(k), \end{aligned}$$

which implies $D_m(\mathbf{x} * \mathbf{y}) = (D_m \mathbf{x}) * (D_m \mathbf{y})$. As well as, we can write

$$\begin{aligned} (D_m \mathbf{x})^*(l) &= \overline{D_m \mathbf{x}(p - l)} \\ &= \overline{\mathbf{x}(m_p(p - l))} \\ &= \overline{\mathbf{x}(m_p p - m_p l)} \\ &= \overline{\mathbf{x}(p - m_p l)} \\ &= \mathbf{x}^*(m_p l) \\ &= D_m \mathbf{x}^*(l), \end{aligned}$$

which implies $(D_m \mathbf{x})^* = D_m \mathbf{x}^*$.

(ii) The proof is straightforward. □

Next proposition states analytic and geometric properties of dilation operators.

Proposition 3.2. *Let p be a positive prime integer. Then,*

(i) *Dilation operators are unitary operators in the Frobenius norm.*
(ii) *For $m \in \mathbb{U}_p$ we have $(D_m)^* = (D_m)^{-1} = D_{m_p}$.*
(iii) *For $m \in \mathbb{U}_p$ and $\mathbf{x} \in \mathbb{C}^p$ we have $\widehat{D_m \mathbf{x}} = D_{m_p} \widehat{\mathbf{x}}$.*

Proof. (i) Let $m \in \mathbb{U}_p$ and $\mathbf{x} \in \mathbb{C}^p$. Then

$$\begin{aligned} \|D_m\mathbf{x}\|_{\mathrm{Fr}}^2 &= \sum_{k=0}^{p-1} |D_m\mathbf{x}(k)|^2 \\ &= \sum_{k=0}^{p-1} |\mathbf{x}(m_p k)|^2. \end{aligned}$$

Replacing k with mk we get

$$\begin{aligned} \sum_{k=0}^{p-1} |\mathbf{x}(m_p k)|^2 &= \sum_{k=0}^{p-1} |\mathbf{x}(k)|^2 \\ &= \|\mathbf{x}\|_{\mathrm{Fr}}^2. \end{aligned}$$

(ii) The proof is straightforward.
(iii) Let $m \in \mathbb{U}_p$ and $\mathbf{x} \in \mathbb{C}^p$. For $\ell \in \mathbb{Z}_p$ we have

$$\begin{aligned} \widehat{D_m\mathbf{x}}(\ell) &= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} D_m\mathbf{x}(k) e^{-2\pi i \ell k/p} \\ &= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \mathbf{x}(m_p k) e^{-2\pi i \ell k/p}. \end{aligned}$$

Replacing k with mk we achieve

$$\begin{aligned} \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \mathbf{x}(m_p k) e^{-2\pi i \ell k/p} &= \frac{1}{\sqrt{p}} \sum_{k=0}^{p-1} \mathbf{x}(k) e^{-2\pi i \ell m k/p} \\ &= D_{m_p}\widehat{\mathbf{x}}(\ell). \end{aligned}$$

□

For $m \in \mathbb{U}_p$, let $\widetilde{m} : \mathbb{Z}_p \to \mathbb{Z}_p$ be given by

$$\widetilde{m}(l) := ml \quad \text{for} \quad 0 \leq l \leq p-1. \tag{3.2}$$

The map $\widetilde{m}$ is a bijection (i.e. surjective and injective) for $m \in \mathbb{U}_p$. If $m \in \mathbb{U}_p$ and $0 \leq l, l' \leq p-1$, then we have

$$m(l + l') = ml + ml',$$

$$m(p - l) = mp - ml \overset{p}{\equiv} p - ml,$$

which implies that $\widetilde{m}$ is a bijection homomorphism (i.e. automorphism) of the additive group $\mathbb{Z}_p$ and so we get $\widetilde{m} \in Aut(\mathbb{Z}_p)$ for $m \in \mathbb{U}_p$ is obtained.

The homomorphism $m \mapsto \widetilde{m}$ induces the semi-direct product group $\mathbb{W}_p := \mathbb{U}_p \ltimes \mathbb{Z}_p$, which has the underlying set $\{(m,l) : m \in \mathbb{U}_p, l \in \mathbb{Z}_p\}$, and it is equipped with the following group operations;

$$(m,l) \ltimes (n,k) := (mn, l+mk) \quad \text{for} \quad (m,l),(n,k) \in \mathbb{W}_p, \tag{3.3}$$

$$(m,l)^{-1} := (m_p, m_p.(p-l)) \quad \text{for} \quad (m,l) \in \mathbb{W}_p, \tag{3.4}$$

where m_p is the multiplicative inverse of m in $\mathbb{U}_p$ (i.e $m_p.m$ and $m.m_p$ are 1 module p). Then $\mathbb{W}_p$ is a finite non-Abelian group of order $(p-1)\times p$. We call $\mathbb{W}_p$ as the **finite affine group** on p integers or the **wavelet group** associated to the integer p or simply we call it as the p-**wavelet group**. Let $L(\mathbb{W}_p)$ be the complex linear space of all complex valued functions on the finite group $\mathbb{W}_p$. The linear space $L(\mathbb{W}_p)$ is precisely $\mathcal{M}_{(p-1)\times p}(\mathbb{C})$ (i.e. the complex linear space of all $(p-1)\times p$-matrices $\mathbf{X}$ with complex entries).

Proposition 3.3. *Let p be a positive prime integer and $\mathbb{W}_p = \mathbb{U}_p \ltimes \mathbb{Z}_p$. Then, $\mathbb{W}_p$ is a non-Abelian group of order $p(p-1)$ which contains $\mathbb{Z}_p$ as a normal Abelian subgroup and $\mathbb{U}_p$ as a non-normal Abelian subgroup.*

Proof. The proof is straightforward. □

3.2. **Cyclic Wavelet Transform.** Let $\mathbf{y} \in \mathbb{C}^p$ be a given window signal. The cyclic wavelet transform (**CWT**) of a signal $\mathbf{x} \in \mathbb{C}^p$ with respect to the window signal $\mathbf{y}$ ($\mathbf{y}$-wavelet transform) is defined by

$$\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) := \sum_{k=0}^{p-1} \mathbf{x}(k)\overline{\mathbf{y}}(m_p(k-l)) \quad \text{for} \quad (m,l) \in \mathbb{W}_p. \tag{3.5}$$

The map $\mathcal{W}_{\mathbf{y}} : \mathbb{C}^p \to \mathcal{M}_{(p-1)\times p}(\mathbb{C})$ given by $\mathbf{x} \mapsto \mathcal{W}_{\mathbf{y}}\mathbf{x}$ is linear. From (3.5) we can see that

$$\begin{aligned}\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) &= \sum_{k=0}^{p-1} \mathbf{x}(k)[T_l D_m \overline{\mathbf{y}}](k) \\ &= \langle \mathbf{x}, T_l D_m \mathbf{y}\rangle, \quad \text{for} \quad (m,l) \in \mathbb{W}_p.\end{aligned} \tag{3.6}$$

Using (3.6) we also have

$$\begin{aligned}\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) &= \langle \mathbf{x}, T_l D_m \mathbf{y}\rangle \\ &= \langle T_{p-l}\mathbf{x}, D_m \mathbf{y}\rangle, \quad \text{for} \quad (m,l) \in \mathbb{W}_p.\end{aligned} \tag{3.7}$$

Remark 3.4. For a window signal $\mathbf{y} \in \mathbb{C}^p$, we define

$$\mathbf{D_y} := \begin{bmatrix} D_1\overline{\mathbf{y}} \\ D_2\overline{\mathbf{y}} \\ . \\ . \\ . \\ D_{p-1}\overline{\mathbf{y}} \end{bmatrix}, \quad \mathbf{T_x} := [T_{p-1}\mathbf{x}, T_{p-2}\mathbf{x}, ..., T_0\mathbf{x}],$$

where for $1 \leq l \leq p$ and $1 \leq m \leq p-1$ we have

$$D_m\overline{\mathbf{y}} = [\overline{\mathbf{y}}(0), \overline{\mathbf{y}}(m_p), ..., \overline{\mathbf{y}}(m_p(p-1))],$$
$$T_{p-l}\mathbf{x} = [\mathbf{x}(l-p), \mathbf{x}(1+l-p), ..., \mathbf{x}(l-1)]^T.$$

Then for $1 \leq m \leq p-1$, $1 \leq l \leq p$ and $1 \leq k \leq p$ we have

$$\mathbf{T_x}(k,l) = (\mathbf{T_x})_{kl} = T_{p-l}\mathbf{x}(k-1),$$
$$\mathbf{D_y}(m,k) = (\mathbf{D_y})_{mk} = D_m\overline{\mathbf{y}}(k-1).$$

Now for $(m,l) \in \mathbb{W}_p$ we can write

$$\begin{aligned}
\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) &= \langle T_{p-l}\mathbf{x}, D_m\mathbf{y}\rangle \\
&= \sum_{k=0}^{p-1} T_{p-l}\mathbf{x}(k)\overline{D_m\mathbf{y}(k)} \\
&= \sum_{k=0}^{p-1} T_{p-l}\mathbf{x}(k)D_m\overline{\mathbf{y}}(k) \\
&= \sum_{k=1}^{p} T_{p-l}\mathbf{x}(k-1)D_m\overline{\mathbf{y}}(k-1) \\
&= \sum_{k=1}^{p} (\mathbf{D_y})_{mk}(\mathbf{T_x})_{kl} \\
&= (\mathbf{D_y})_{m\cdot}(\mathbf{T_x})^l \\
&= (\mathbf{D_y}.\mathbf{T_x})_{ml},
\end{aligned}$$

where $(\mathbf{D_y})_m = \mathbf{D_y}(m,:)$ (resp. $(\mathbf{T_x})^l = \mathbf{T_x}(:,l)$) is the m-th row of the matrix $\mathbf{D_y}$ (resp. l-th column of the matrix $\mathbf{T_x}$) and . is the standard product of matrices. Thus we achieve

$$\mathcal{W}_{\mathbf{y}}\mathbf{x} = \mathbf{D_y}.\mathbf{T_x}.$$

The following proposition gives us a Fourier representation and a circular convolution representation for the wavelet transform defined in (3.5).

Proposition 3.5. *Let* $\mathbf{x}, \mathbf{y} \in \mathbb{C}^p$. *For* $(m,l) \in \mathbb{W}_p$ *we have*

(i) $\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) = \sqrt{p}\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-l)$.
(ii) $\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) = \mathbf{x} * D_m\mathbf{y}^*(l)$.

Proof. Let $(m,l) \in \mathbb{W}_p$. (i) Using the Plancherel formula we have

$$\begin{aligned}
\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) &= \sum_{k=0}^{p-1} \mathbf{x}(k)\overline{T_l D_m \mathbf{y}(k)} \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{T_l D_m \mathbf{y}}(\ell)} \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{M_l \widehat{D_m \mathbf{y}}(\ell)} \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{D_m \mathbf{y}}(\ell)}\mathbf{w}_l(\ell) \\
&= \sum_{\ell=0}^{p-1} \left(\widehat{\mathbf{x}}.\overline{\widehat{D_m \mathbf{y}}}\right)(\ell)\overline{\mathbf{w}_{p-l}(\ell)} \\
&= \sqrt{p}\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-l).
\end{aligned}$$

(ii) Using (i) we can write

$$\begin{aligned}
\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) &= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)\overline{\widehat{D_m \mathbf{y}}(\ell)}\mathbf{w}_l(\ell) \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)(\widehat{D_m \mathbf{y}})^*(\ell)\mathbf{w}_l(\ell) \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x}}(\ell)(\widehat{D_m \mathbf{y}^*})(\ell)\mathbf{w}_l(\ell) \\
&= \sum_{\ell=0}^{p-1} \widehat{\mathbf{x} * D_m \mathbf{y}^*}(\ell)\mathbf{w}_l(\ell) \\
&= \mathbf{x} * D_m\mathbf{y}^*(l).
\end{aligned}$$

□

Theorem 3.6. *Let* $\mathbf{y} \in \mathbb{C}^p$ *be a window signal and* $\mathbf{x} \in \mathbb{C}^p$. *Then,*

(3.8)
$$\|\mathcal{W}_{\mathbf{y}}\mathbf{x}\|_{\mathrm{Fr}}^2 = p\left((p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + \left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)\right).$$

Proof. Let $m \in \mathbb{U}_p$. Using Proposition (3.5) we have

$$\begin{aligned}\sum_{l=0}^{p-1} |\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l)|^2 &= p\sum_{l=0}^{p-1} \left|\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(p-l)\right|^2 \\ &= p\sum_{l=0}^{p-1} \left|\mathcal{F}_p(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(l)\right|^2 \\ &= p\sum_{\ell=0}^{p-1} \left|(\widehat{\mathbf{x}}.\overline{\widehat{D_m\mathbf{y}}})(\ell)\right|^2 \\ &= p\sum_{\ell=0}^{p-1} \left|\widehat{\mathbf{x}}(\ell).\overline{\widehat{D_m\mathbf{y}}(\ell)}\right|^2.\end{aligned}$$

Therefore we achieve

$$\begin{aligned}\sum_{m=1}^{p-1}\sum_{l=0}^{p-1} |\mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l)|^2 &= p\sum_{m=1}^{p-1}\sum_{\ell=0}^{p-1} \left|\widehat{\mathbf{x}}(\ell).\overline{\widehat{D_m\mathbf{y}}(\ell)}\right|^2 \\ &= p\sum_{m=1}^{p-1}\sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\left|\overline{\widehat{D_m\mathbf{y}}(\ell)}\right|^2 \\ &= p\sum_{\ell=0}^{p-1}\sum_{m=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2|\widehat{D_m\mathbf{y}}(\ell)|^2 \\ &= p\sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{D_m\mathbf{y}}(\ell)|^2\right) \\ &= p\sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|D_{m_p}\widehat{\mathbf{y}}(\ell)|^2\right) \\ &= p\sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right).\end{aligned}$$

Now we can write

$$\begin{aligned}\sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right) &= |\widehat{\mathbf{x}}(0)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(0)|^2\right) \\ &\quad + \sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m\ell)|^2\right).\end{aligned}$$

Replacing m with $m\ell_p$ we have

$$\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m\ell)|^2 = \sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2,$$

which implies

$$\begin{aligned}\sum_{\ell=0}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 \left(\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m\ell)|^2\right) &= (p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 \\ &\quad + \sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2 \left(\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m\ell)|^2\right) \\ &= (p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 \\ &\quad + \left(\sum_{\ell=1}^{p-1} |\widehat{\mathbf{x}}(\ell)|^2\right)\left(\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2\right).\end{aligned}$$

Thus we achieve (3.8). □

Now the following interesting result is proven, which shows when the windowed transform (3.5) is an isometric transform.

Corollary 3.7. *Let $\mathbf{y} \in \mathbb{C}^p$ be a non-zero window signal with $\widehat{\mathbf{y}}(0) \neq 0$. The cyclic wavelet transform $\mathcal{W}_{\mathbf{y}} : \mathbb{C}^p \to \mathcal{M}_{(p-1)\times p}(\mathbb{C})$ is an isometric transform (up to a normalization) if and only if the window signal $\mathbf{y}$ satisfies the following p-admissibility condition;*

$$\|\mathbf{y}\|_{\mathbf{Fr}} = \sqrt{p}|\widehat{\mathbf{y}}(0)|. \tag{3.9}$$

Proof. If the condition (3.9) holds, then we get

$$\sum_{\ell=0}^{p-1} |\widehat{\mathbf{y}}(\ell)|^2 = p|\widehat{\mathbf{y}}(0)|^2.$$

Thus

$$(p-1)|\widehat{\mathbf{y}}(0)|^2 = \sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2.$$

Using (3.8) we achieve

$$\begin{aligned}\|\mathcal{W}\|_{\mathrm{Fr}}^2 &= p\left((p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2 + \left(\sum_{l=1}^{p-1}|\widehat{\mathbf{x}}(l)|^2\right)\left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)\right)\\ &= p\left((p-1)|\widehat{\mathbf{y}}(0)|^2\left(|\widehat{\mathbf{x}}(0)|^2 + \sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\right)\\ &= p(p-1)|\widehat{\mathbf{y}}(0)|^2\left(\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right)\\ &= p(p-1)|\widehat{\mathbf{y}}(0)|^2\|\widehat{\mathbf{x}}\|_{\mathrm{Fr}}^2\\ &= p(p-1)|\widehat{\mathbf{y}}(0)|^2\|\mathbf{x}\|_{\mathrm{Fr}}^2.\end{aligned}$$

Conversely, let for some positive real number α we have

$$\|\mathcal{W}_{\mathbf{y}}\mathbf{x}\|_{\mathrm{Fr}}^2 = \alpha\|\mathbf{x}\|_{\mathrm{Fr}}^2, \quad \text{for} \quad \mathbf{x}\in\mathbb{C}^p.$$

Thus

$$\begin{aligned}\|\mathcal{W}_{\mathbf{y}}\mathbf{x}\|_{\mathrm{Fr}}^2 &= \alpha\left(\sum_{k=0}^{p-1}|\mathbf{x}(k)|^2\right)\\ &= \alpha\left(\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right).\end{aligned} \tag{3.10}$$

Using (3.8) we have

$$\begin{aligned}\alpha/p\left(\sum_{\ell=0}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right) &= (p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2\\ &\quad + \left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right).\end{aligned} \tag{3.11}$$

Splitting the left side of (3.11) we get

$$\begin{aligned}\alpha/p|\widehat{\mathbf{x}}(0)|^2 + \alpha/p\left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|\right) &= (p-1)|\widehat{\mathbf{y}}(0)|^2|\widehat{\mathbf{x}}(0)|^2\\ &\quad + \left(\sum_{m=1}^{p-1}|\widehat{\mathbf{y}}(m)|^2\right)\left(\sum_{\ell=1}^{p-1}|\widehat{\mathbf{x}}(\ell)|^2\right),\end{aligned}$$

for $\mathbf{x}\in\mathbb{C}^p$. Let $\mathbf{x}\in\mathbb{C}^p$ be such that $\widehat{\mathbf{x}}(0) = 1$ and $\widehat{\mathbf{x}}(\ell) = 0$ for $1\le\ell\le p-1$. Then

$$\alpha/p = (p-1)|\widehat{\mathbf{y}}(0)|^2. \tag{3.12}$$

Similarly, if we assume $\mathbf{x} \in \mathbb{C}^p$ with $\widehat{\mathbf{x}}(0) = 0$ and $\widehat{\mathbf{x}}(\ell) = 1$ for $1 \leq \ell \leq p-1$. Then

$$(p-1)\alpha/p = (p-1)\left(\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2\right),$$

which implies

$$\alpha/p = \sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2. \tag{3.13}$$

The equations (3.12) and (3.13) imply

$$(p-1)|\widehat{\mathbf{y}}(0)|^2 = \sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2,$$

which is equivalent to (3.9). □

A non-zero window signal $\mathbf{y} \in \mathbb{C}^p$ is called a p**-wavelet** if and only if $\mathbf{y}$ satisfies

$$\|\mathbf{y}\|_{\mathbf{Fr}} = \sqrt{p}|\widehat{\mathbf{y}}(0)|. \tag{3.14}$$

It is evident to see that each p-wavelet satisfies $\widehat{\mathbf{y}}(0) \neq 0$. From (3.9) we deduce that a window signal $\mathbf{y} \in \mathbb{C}^p$ is a p-wavelet if and only if

$$\left|\sum_{k=0}^{p-1} \mathbf{y}(k)\right|^2 = \sum_{k=0}^{p-1} |\mathbf{y}(k)|^2. \tag{3.15}$$

If $\mathbf{y}$ is a p-wavelet, we call

$$\alpha_{\mathbf{y}} := (p-1)|\sum_{k=0}^{p-1} \mathbf{y}(k)|^2 = p(p-1)|\widehat{\mathbf{y}}(0)|^2 = p\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2,$$

as the p-wavelet constant.

Corollary 3.8. *Let $\mathbf{y} \in \mathbb{C}^p$ be a p-wavelet. For $\mathbf{x}, \mathbf{z} \in \mathbb{C}^p$ we have*

$$\sum_{m=1}^{p-1}\sum_{l=0}^{p-1} \mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l)\overline{\mathcal{W}_{\mathbf{y}}\mathbf{z}}(m,l) = \alpha_{\mathbf{y}} \sum_{l=0}^{p-1} \mathbf{x}(l)\overline{\mathbf{z}(l)}. \tag{3.16}$$

The following results state an inversion formula for the windowed transform given in (3.5).

Corollary 3.9. *Let $\mathbf{y} \in \mathbb{C}^p$ be a p-wavelet. For $\mathbf{x} \in \mathbb{C}^p$ we have the following reconstruction formula;*

$$\mathbf{x}(k) = \frac{1}{\alpha_{\mathbf{y}}} \sum_{m=1}^{p-1}\sum_{l=0}^{p-1} \mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) T_l D_m \mathbf{y}(k), \quad \textit{for} \quad 0 \leq k \leq p-1. \tag{3.17}$$

Corollary 3.10. *Let* $\mathbf{y} \in \mathbb{C}^p$ *be a* p*-wavelet and* $\mathbf{X} \in \mathcal{M}_{(p-1)\times p}(\mathbb{C})$ *be a* $\mathbf{y}$*-wavelet matrix (i.e. any matrix in the range of the* $\mathbf{y}$*-wavelet transform). The signal* $\mathbf{x} \in \mathbb{C}^p$ *given by*

$$\mathbf{x}(k) = \frac{1}{\alpha_{\mathbf{y}}} \sum_{m=1}^{p-1} \sum_{l=0}^{p-1} \mathbf{X}(m,l) T_l D_m \mathbf{y}(k) \quad \text{for} \quad 0 \leq k \leq p-1, \tag{3.18}$$

has the $\mathbf{y}$*-wavelet transform* $\mathbf{X}$*.*

Remark 3.11. For a positive prime integer p, we define

$$\mathcal{B}_p := \left\{ \mathbf{x} \in \mathbb{C}^p : \widehat{\mathbf{x}}(0) = \sum_{k=0}^{p-1} \mathbf{x}(k) = 0 \right\}. \tag{3.19}$$

The set $\mathcal{B}_p$ is a complex linear subspace of $\mathbb{C}^p$ with dimension $p-1$. Using (3.8), for a window signal $\mathbf{y} \in \mathbb{C}^p$ and $\mathbf{x} \in \mathcal{B}_p$ we have

$$\|\mathcal{W}_{\mathbf{y}}\mathbf{x}\|_{\mathrm{Fr}}^2 = p \left(\sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2 \right) \|\mathbf{x}\|_{\mathrm{Fr}}^2. \tag{3.20}$$

For a window signal $\mathbf{y} \in \mathbb{C}^p$, we define

$$\begin{aligned} \beta_{\mathbf{y}} &:= p \sum_{m=1}^{p-1} |\widehat{\mathbf{y}}(m)|^2 \\ &= p \left(\|\mathbf{y}\|_{\mathrm{Fr}}^2 - |\widehat{\mathbf{y}}(0)|^2 \right) \\ &= p\|\mathbf{y}\|_{\mathrm{Fr}}^2 - |\sum_{k=0}^{p-1} \mathbf{y}(k)|^2. \end{aligned} \tag{3.21}$$

Then for any window signal $\mathbf{y} \in \mathbb{C}^p$ with $\beta_{\mathbf{y}} \neq 0$ and $\mathbf{x}, \mathbf{z} \in \mathcal{B}_p$ we have

$$\sum_{m=1}^{p-1} \sum_{l=0}^{p-1} \mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) \overline{\mathcal{W}_{\mathbf{y}\mathbf{z}}(m,l)} = \beta_{\mathbf{y}} \sum_{l=0}^{p-1} \mathbf{x}(l) \overline{\mathbf{z}(l)}. \tag{3.22}$$

If $\mathbf{y} \in \mathbb{C}^p$ is a window signal with $\beta_{\mathbf{y}} \neq 0$, then each $\mathbf{x} \in \mathcal{B}_p$ satisfies the following reconstruction formula;

$$\mathbf{x}(k) = \frac{1}{\beta_{\mathbf{y}}} \sum_{m=1}^{p-1} \sum_{l=0}^{p-1} \mathcal{W}_{\mathbf{y}}\mathbf{x}(m,l) T_l D_m \mathbf{y}(k) \quad \text{for} \quad 0 \leq k \leq p-1. \tag{3.23}$$

Remark 3.12. It should be mentioned that when p is a prime integer $\mathbb{F} := \mathbb{Z}_p$ is a finite field, see [23]. In this case

$$\mathbb{F}^* = \mathbb{F} - \{0\} = \mathbb{Z}_p - \{0\} = \mathbb{U}_p,$$

and hence the finite affine group $\mathbb{W}_p$ is precisely the group $\mathbb{F}^* \ltimes \mathbb{F}$. Thus Corollaries 3.7 and 3.9 coincides with the direct consequences of results in [26]. The advantage of our approach which is deeply originated from the automorphism group of the finite cyclic group $\mathbb{Z}_p$ is the direct and unified equation (3.8) which gives a direct formulation for the Frobenius norm of the cyclic wavelet transform given by (3.5).

Acknowledgment. The author would like to thank the referees for their valuable comments and remarks.

References

1. S. T. Ali, J. P. Antoine and J. P. Gazeau, *Coherent States, Wavelets and Their Generalizations*, Springer-Verlag, New York, 2000.
2. J. P. Antoine, R. Murenzi and P. Vendergheynst, S. T. Ali, *Two Dimensional Wavelets and Their Relatives*, Cambridge University Press, Cambridge, 2003.
3. J. J. Benedetto and S. Li, *The theory of multiresolution analysis frames and applications to filter bands*, Appl. Comput. Harmon. Anal., 5 (1998) 389-427.
4. J. J. Benedetto and O. M. Treiber, *Wavelet frames: multiresolution analysis and extension principles.* Wavelet transforms and time-frequency signal analysis, 336, Appl. Numer. Harmon. Anal., Birkhuser Boston, Boston, MA, 2001.
5. G. Caire, R. L. Grossman and H. Vincent Poor, *Wavelet transforms associated with finite cyclic Groups*, IEEE Transaction On Information Theory, Vol. 39, No. 4, July 1993.
6. L. Cohen, *Time-Frequency Analysis*, Prentice-Hall, New York, 1995.
7. I. Daubechies, *Ten Lectures on Wavelets*, Society for Industrial and Applied Mathematics, 1992.
8. H. G. Feichtinger and G. Zimmermann, *A Banach space of test functions for Gabor analysis*, Gabor analysis and algorithms, 123-170, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA 1998.
9. H. G. Feichtinger and W. Kozek, *Quantization of TF lattice-invariant operators on elementary LCA groups*, Gabor analysis and algorithms, 233-266, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA 1998.
10. H. G. Feichtinger and T. Strohmer, *Advances in Gabor Analysis*, Series: Applied and Numerical Harmonic Analysis, Birkhuser 2002.
11. H. Feichtinger and F. Luef, *Gabor analysis and time-frequency methods,* Encyclopedia of Applied and Computational Mathematics, 2012.
12. H. G. Feichtinger, T. Strohmer, and O. Christensen, *A group-theoretical approach to Gabor analysis,* Opt. Eng., 34 (1697) 1704, 1995.
13. G. B. Folland, *A Course in Abstract Harmonic Analysis*, CRC press, 1995.
14. D. Gabor, *Theory of communication*, JIEEE, **93**(26), Part III, 1964, 429-457.
15. A. Grossmann, J. Morlet and T. Paul, *Transforms associated to square integrable group representations I. general results*, J. Math. Phys., 26 (10) (1985) 2473-2479.
16. K. Gröchenig, *Foundation of Time-Frequency Analysis*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA 2001.
17. K. Gröchenig, *Aspects of Gabor analysis on locally compact abelian groups*, Gabor analysis and Algorithms, 211-231, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, MA 1998.
18. G. H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, New York, 1979.
19. E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis*, Vol. 1, 1963.

20. R. Reiter and J. D. Stegeman, *Classical Harmonic Analysis*, 2nd Ed, Oxford University Press, New York, 2000.
21. S. Mallat, *A Wavelet Tour of Signal Processing*, 2nd Edition, Academic Press, 1999.
22. S. Mallat, *Multiresolution approximations and wavelet orthonormal bases of* $L^2(\mathbb{R})$, Trans. Amer. Math. Soc., 315 (1989), 69-87.
23. G. L. Mullen and D. Panario, *Handbook of Finite Fields*, Series: Discrete Mathematics and Its Applications, Chapman and Hall/CRC, 2013.
24. G. Pfander, *Gabor Frames in Finite Dimensions.* In G. E. Pfander, P. G. Casazza, and G. Kutyniok, editors, Finite Frames, Applied and Numerical Harmonic Analysis, 193-239. Birkhauser Boston, 2013.
25. P. Flandrin, *Time-Frequency/Time-Scale Analysis*, Wavelet Analysis and its Applications, Vol. 10 Academic Press, San Diego, 1999.
26. K. Flornes, A. Grossmann, M. Holschneider and B. Torrésani, *Wavelets on discrete fields*, Appl. Comput. Harmon. Anal., 1 (1994) 137-146.
27. H. Riesel, *Prime Numbers and Computer Methods for Factorization* (second edition), Boston: Birkhuser, (1994).
28. T. Strohmer., *Numerical algorithms for discrete Gabor expansions*, Gabor Analysis and Algorithms: Theory and Applications, pages 267–294. Birkhäuser Boston, Boston, 1998.
29. T. Strohmer. *A Unified Approach to Numerical Algorithms for Discrete Gabor Expansions*, In Proc. SampTA - Sampling Theory and Applications, Aveiro/Portugal, pages 297–302, 1997.
30. S. Sarkar, H. Vincent Poor, *Cyclic Wavelet Transforms for Arbitrary Finite Data Lengths*, Signal Processing, 80 (2000) 2541-2552.
31. G. Strang and T. Nguyen, *Wavelets and Filter Banks*, Wellesley-Cambridge Press, Wellesley, MA, 1996.
32. M. Vetterli and J. Kovacevic, *Wavelets and Subband Coding*, Prentice Hall, 1995.
33. M.V. Wickerhauser, *Lectures on Wavelet Packet Algorithms*, Technical Report, Washington University, Department of Mathematics, 1992.
34. M.W. Wong, *Discrete Fourier Analysis*, Pseudo-differential Operators Theory and applications Vol. 5, Birkhauser 2010.
35. Z. Zhang. *Periodic wavelet frames.* Adv. Comput. Math., 22 (2) (2005) 165-180.

---

[1] Numerical Harmonic Analysis Group (NuHAG), Faculty of Mathematics, University of Vienna, Vienna, Austria.

E-mail address: arash.ghaani.farashahi@univie.ac.at

[2] Faculty of Mathematics, Ferdowsi University of Mashhad, Mashhad, Iran.

E-mail address: mozhgan.mohammadpour@stu.um.ac.ir