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ON $\tilde{X}$-FRAMES AND CONJUGATE SYSTEMS IN BANACH SPACES

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ABSTRACT. The generalization of p-frame in Banach spaces is considered in this paper. The concepts of an $\tilde{X}$-frame and a system conjugate to $\tilde{X}$-frame were introduced. Analogues of the results on the existence of conjugate system were obtained. The stability of $\tilde{X}$-frame having a conjugate system is studied.

1. INTRODUCTION

The concept of frame in Hilbert spaces was introduced by R.J.Duffin and A.C.Schaeffer [1] when studying non-harmonic Fourier series. Let us recall that the system $\{f_i\}_{i\in N}$ in separable Hilbert space H is called a frame if there are the constants $A>0$ and $B>0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \forall f\in H.$$

The constants A and B are called the frame bounds. Refer to [2, 3] for the theory of frames. There are many other works dedicated to frames and their applications (see [4, 5], etc). Frames are widely used in signal processing, data compression, characterization of function spaces and other fields. Every vector in separable Hilbert space can be expanded in a frame. In other words, if $\{f_i\}_{i\in N}$ forms a frame for H then there exists the system $\left\{\tilde{f}_i\right\}_{i\in N}\subset H$ such that the recovery formula

$$f=\sum_{k=1}^{\infty}\left\langle f,\tilde{f}_k\right\rangle f_k$$

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holds for every vector $f \in H$. Note that such a system may not be unique. The system $\left\{\tilde{f}_i\right\}_{i\in N}$ also forms a frame for H and is called a frame conjugate to $\{f_i\}_{i\in N}$. One of the most important fields in the theory of frames in Hilbert spaces is the stability of frame. More information about this matter can be found in [6, 7].

K. Gröchenig [8] extended the concept of frame to the case of Banach space. He introduced the concept of a separable Banach frame and the one of atomic decomposition in Banach spaces. Banach frames and atomic decomposition, as well as their stability in Banach spaces have been studied in [9-11]. There are other generalizations of frame, Banach frame and atomic decomposition. In [12], the concepts of p-frame and p-Riesz basis were introduced and their properties studied. Reference [13], dedicated to p-frames, also presented a recovery formula. In [14], the concept of g-frame was introduced and the analogues of corresponding frame results have been obtained. As generalizations of g-frame, the concepts of pg-frame and Banach g-frame were introduced in [15] and their stability studied. Frame properties of degenerate trigonometric systems in Lebesgue spaces have been studied in [16, 17] in case when the degeneration coefficient does not satisfy the well-known Muckenhoupt condition (with respect to the Muckenhoupt condition see e.g. [18]).

This work is dedicated to the generalization of frame in Banach spaces with respect to the Banach space of sequences. The concept of $\tilde{X}$-frame is introduced which is a generalization of p-frame. The equivalent conditions for the existence of a system conjugate to $\tilde{X}$-frame were studied, and we obtain the result for the stability of $\tilde{X}$-frame was obtained. A g-frame with respect to the Banach space of sequences is also considered.

2. $\tilde{X}$-frames in Banach spaces

Let X, Z be Banach spaces, $\tilde{X}$ be a Banach space of sequences of vectors X with the coordinate-wise linear operations such that the operator

$$P_k : X \to \tilde{X}, \quad P_k(x) = \{\delta_{ik} x\}_{i\in N}$$

is bounded (BK-space for brevity). $\tilde{X}$ will be called a BC-space if the equality

$$\lim_{n\to\infty} \left\| \{x_k\}_{k\in N} - \sum_{k=1}^{n} \{\delta_{ik} x_k\}_{i\in N} \right\|_{\tilde{X}} = 0$$

holds for every $\{x_k\}_{k\in N} \in \tilde{X}$.

It is not difficult to show that if $\tilde{X}$ is a BC-space, then $\tilde{X}^*$ is isometrically isomorphic to the Banach space

$$\tilde{Y} = \left\{ \{t_k\}_{k\in N} \subset X^* : t_k = \tilde{t}P_k, \tilde{t} \in \tilde{X}^* \right\}$$

equipped with the norm $\left\| \{t_k\}_{k\in N} \right\|_{\tilde{Y}} = \left\| \tilde{t} \right\|_{\tilde{X}^*}$. Every $\tilde{t} \in \tilde{X}^*$ is defined by the formula

$$\tilde{t}(\{x_k\}_{k\in N}) = \sum_{k=1}^{\infty} t_k(x_k).$$

Therefore $\tilde{X}^*$ is identified with $\tilde{Y}$.

By $L(Z, X)$ we denote the space of all bounded linear operators from Z to X.

Definition 2.1. System $\{g_k\}_{k\in N} \subset L(Z, X)$ is called $\tilde{X}$-frame in Z if there are the constants $A > 0$ and $B > 0$ such that

$$\begin{aligned} A \left\| z \right\|_Z \leq & \left\| \{g_k(z)\}_{k\in N} \right\|_{\tilde{X}} \\ & \leq B \left\| z \right\|_Z, \quad \forall z \in Z. \end{aligned}$$

Constants A and B are called $\tilde{X}$-frame bounds of $\{g_k\}_{k\in N}$. In case when $\{g_k\}_{k\in N}$ satisfies the right-hand side inequality, $\{g_k\}_{k\in N}$ is called $\tilde{X}$-Bessel system in Z with a bound B.

Let $\{g_k\}_{k\in N} \subset L(Z, X)$ be an $\tilde{X}$-frame in Z and $S \in L(\tilde{X}, Z)$. The pair $\left(\{g_k\}_{k\in N}, S \right)$ is called a Banach g-frame in Z with respect to $\tilde{X}$ (see [15]) if

$$S\left(\{g_n(z)\}_{n\in N} \right) = z, \quad \forall z \in Z.$$

Let $\{g_k\}_{k\in N} \subset L(Z, X)$ be an $\tilde{X}$-frame in Z. Denote by T the operator $T(z) = \{g_k(z)\}_{k\in N}$, $z \in Z$. It is clear that T maps Z isomorphically onto $R(T)$.

Theorem 2.2. *Let $\tilde{X}$ be a BC-space, and the system $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$ be given. Then $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Bessel system in Z^* with a bound B if and only if there exists*

$$U \in L(\tilde{X}, Z): \quad U(\tilde{x}) = \sum_{k=1}^{\infty} \Lambda_k(x_k), \quad \tilde{x} = \{x_n\}_{n\in N}, \quad \|U\| \leq B.$$

Proof. Let's prove the convergence of the series $\sum_{k=1}^{\infty} \Lambda_k(x_k)$ for any $\tilde{x} = \{x_n\}_{n\in N}$. For $n > m$ we have

$$\begin{aligned}\left\|\sum_{k=m}^{n}\Lambda_k(x_k)\right\|_Z &= \sup_{\|f\|=1}\left|\sum_{k=m}^{n}f(\Lambda_k(x_k))\right| \\ &= \sup_{\|f\|=1}\left|\sum_{k=m}^{n}\Lambda_k^* f(x_k)\right| \\ &\le B\left\|\sum_{k=m}^{n}\{\delta_{ik}x_k\}_{i\in N}\right\|_{\tilde{X}}.\end{aligned}$$

Consequently, the series $\sum_{k=1}^{\infty}\Lambda_k(x_k)$ is convergent. It follows that the operator

$$U(\tilde{x})=\sum_{k=1}^{\infty}\Lambda_k(x_k),\quad \tilde{x}=\{x_n\}_{n\in N}$$

well- defined, and it is clear that $U\in L(\tilde{X},Z)$. Conversely, let there exist

$$U\in L(\tilde{X},Z),\quad U(\tilde{x})=\sum_{k=1}^{\infty}\Lambda_k(x_k),\quad \tilde{x}=\{x_n\}_{n\in N}.$$

Then

$$\begin{aligned}\left\|\{\Lambda_k^* f\}_{k\in N}\right\|_{\tilde{X}^*} &= \sup_{\|\{x_k\}\|=1}\left|\sum_{k=1}^{\infty}\Lambda_k^* f(x_k)\right| \\ &\le \|U\|\,\|f\|.\end{aligned}$$

□

Let us provide the analogue of the results on equivalence Banach p-frames from [13].

Theorem 2.3. *Let $\tilde{X}$ be a BK-space, $\{g_k\}_{k\in N}\subset L(Z,X)$ be an $\tilde{X}$-frame in Z. Then the following conditions are equivalent:*

1) *$R(T)$ is complemented in $\tilde{X}$;*
2) *the operator $T^{-1}:R(T)\to Z$ can be extended to the bounded operator*
$$W:\tilde{X}\to Z;$$
3) *there exists the bounded operator $S\in L(\tilde{X},Z)$ such that $(\{g_i\},S)$ forms a Banach g-frame for Z with respect to $\tilde{X}$.*

Proof. Let's show the validity of 1) $\Leftrightarrow$ 2). Let P be a projection from $\tilde{X}$ to $R(T)$. Consider the operator $W=T^{-1}P$. Operator W will be the desired operator. Now let 2) be true. Define the operator $P = TW$. We have

$$\begin{aligned} P^2 &= TWTW \\ &= TW \\ &= P. \end{aligned}$$

For every $z \in Z$ we obtain

$$\begin{aligned} T(z) &= TWT(z) \\ &= P(T(z)). \end{aligned}$$

Consequently, $R(P) = R(T)$, and therefore $R(T)$ is complemented in $\tilde{X}$.

Let's show the validity of 1) $\Leftrightarrow$ 3). As S, the operator W was taken. It is clear that $(\{g_i\}, W)$ forms a Banach g-frame for Z with respect to $\tilde{X}$. Conversely, let there exist the bounded operator $S \in L(\tilde{X}, Z)$ such that $(\{g_i\}, S)$ forms a Banach g-frame for Z with respect to $\tilde{X}$. Then the operator S is a bounded continuation of T^{-1} .

□

Theorem 2.4. *Let $\tilde{X}$ and $\tilde{X}^*$be BC-spaces, $\{g_k\}_{k\in N} \subset L(Z, X)$ be an $\tilde{X}$-frame in Z. Then the following conditions are equivalent:*

1) *$R(T)$ is complemented in $\tilde{X}$;*
2) *there exists an $\tilde{X}^*$-Bessel system in Z^* system*

$$\{\Lambda_k\}_{k\in N} \subset L(X, Z)$$

such that

$$z = \sum_{k=1}^{\infty} \Lambda_k g_k(z) \quad \text{for every} \quad z \in Z;$$

3) *there exists an $\tilde{X}^*$-Bessel system in Z^* system*

$$\{\Lambda_k\}_{k\in N} \subset L(X, Z)$$

such that

$$f = \sum_{k=1}^{\infty} f \Lambda_k g_k, \quad \forall f \in Z^*.$$

Proof. Let's show the validity of 1) $\Leftrightarrow$ 2). Let $R(T)$ be complemented in $\tilde{X}$. Then, by Theorem 2.3, there exists a bounded continuation W of operator T^{-1} to the whole $\tilde{X}$. Define $\{\Lambda_k\}$ as follows: $\Lambda_k = WP_k$. It is clear that

$$\{\Lambda_k\}_{k\in N} \subset L(X, Z).$$

For arbitrary $f \in Z^*$ we obtain $\Lambda_k^* f \in X^*$ and $\{\Lambda_k^* f\}_{k\in N} \in \tilde{X}^*$ because $\sum_{k=1}^{\infty} \Lambda_k^* f(x_k) = fW(\tilde{x})$ for every $\tilde{x} = \{x_n\}_{n\in N} \in \tilde{X}$. We have

$$\begin{aligned} \left\| \{\Lambda_k^*(f)\}_{k\in N} \right\|_{\tilde{X}^*} &= \sup_{\|\{x_n\}\|=1} \left| \sum_{k=1}^{\infty} \Lambda_k^* f(x_k) \right| \\ &= \sup_{\|\{x_n\}\|=1} \left| fW\left(\{x_n\}_{n\in N}\right) \right| \\ &\leq \|W\| \, \|f\| , \end{aligned}$$

i.e. $\{\Lambda_k\}$ is an $\tilde{X}^*$-Bessel system in Z^*. Conversely, let 2) be true. Denote

$$U(\tilde{x}) = \sum_{k=1}^{\infty} \Lambda_k(x_k), \quad \tilde{x} = \{x_n\}_{n\in N} .$$

Operator U is a continuous continuation of T^{-1}. In fact, $\forall z \in Z$ we obtain

$$\begin{aligned} UT(z) &= U(\{g_n(z)\}_{n\in N}) \\ &= \sum_{k=1}^{\infty} \Lambda_k(g_k(z)) \\ &= z. \end{aligned}$$

The rest follows from Theorem 2.3.

Now let's prove the validity of 2) $\Leftrightarrow$ 3). Let 2) be true. Take arbitrary $f \in Z^*$. We have

$$\begin{aligned} \left\| f - \sum_{k=1}^{n} f\Lambda_k g_k \right\| &= \sup_{\|z\|=1} \left| f(z) - \sum_{k=1}^{n} \Lambda_k^* f(g_k(z)) \right| \\ &= \sup_{\|z\|=1} \left| \sum_{k=n+1}^{\infty} \Lambda_k^* f(g_k(z)) \right| \\ &\leq B \left\| \sum_{k=n+1}^{\infty} \{\delta_{ik}\Lambda_k^* f\}_{i\in N} \right\|_{\tilde{X}^*} . \end{aligned}$$

Hence, $f = \sum_{k=1}^{\infty} f\Lambda_k g_k$. Conversely, suppose that 3) is true. For arbitrary $z \in Z$ we have

$$\begin{aligned}\left\| z - \sum_{k=1}^{n} \Lambda_k(g_k(z)) \right\| &= \sup_{\|f\|=1} \left| f(z) - \sum_{k=1}^{n} \Lambda_k^* f(g_k(z)) \right| \\ &= \sup_{\|f\|=1} \left| \sum_{k=n+1}^{\infty} \Lambda_k^* f(g_k(z)) \right| \\ &\leq B_1 \left\| \sum_{k=n+1}^{\infty} \{\delta_{ik} g_k(z)\}_{i\in N} \right\|_{\tilde{X}},\end{aligned}$$

where B_1 is a bound of $\{\Lambda_k\}_{k\in N}$. Therefore,

$$z = \sum_{k=1}^{\infty} \Lambda_k g_k(z).$$

□

Now we introduce the concept of a conjugate $\tilde{X}^*$-frame.

Definition 2.5. Let the system $\{g_k\}_{k\in N} \subset L(Z, X)$ be $\tilde{X}$-Bessel system in Z. System $\{\Lambda_k\}_{k\in N} \subset L(X, Z)$, $\tilde{X}^*$-Bessel system in Z^*, is called conjugate to $\{g_k\}_{k\in N}$ if the following condition is satisfied:

$$z = \sum_{k=1}^{\infty} \Lambda_k g_k(z), \quad \forall z \in Z.$$

Theorem 2.6. *Let $\tilde{X}$ be a BC-space, and the system $\{g_k\}_{k\in N} \subset L(Z, X)$ be an $\tilde{X}$-frame in Z. Then $\{g_k\}_{k\in N}$ has a conjugate system if and only if there exists a bounded left inverse operator of T.*

Proof. Let $\{\Lambda_k\}_{k\in N}$ be a conjugate system for $\{g_k\}_{k\in N}$. Denote by $U : \tilde{X} \to Z$ its synthesizing operator:

$$U(\tilde{x}) = \sum_{k=1}^{\infty} \Lambda_k(x_k).$$

As for every $z \in Z$ we have

$$\{g_k(z)\}_{k\in N} = \sum_{k=1}^{\infty} \{\delta_{ik} g_k(z)\}_{i\in N},$$

the relation $U(\{\delta_{ik}g_k(z)\}_{k\in N}) = \Lambda_i(g_i(z))$ holds. Then we obtain

$$\begin{aligned} z &= \sum_{k=1}^{\infty} \Lambda_k g_k(z) \\ &= \sum_{k=1}^{\infty} U(\{\delta_{ik}g_k(z)\}_{i\in N}) \\ &= U\left(\{g_k(z)\}_{k\in N}\right) \\ &= UT(z), \end{aligned}$$

i.e. T has a bounded left inverse U. Conversely, let T have a bounded right inverse U. Consider $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$ such that $\Lambda_k = UP_k$. It is clear that $\{\Lambda_k\}_{k\in N} \subset L(X,Z)$. Then for every $\tilde{x} = \{x_k\}_{k\in N} \in \tilde{X}$ we have

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \Lambda_k(x_k) \right\| &= \left\| \sum_{k=1}^{\infty} U(\{\delta_{ik}x_k\}_{i\in N}) \right\| \\ &= \left\| U(\sum_{k=1}^{\infty} \{\delta_{ik}x_k\}_{i\in N}) \right\| \\ &= \|U(\tilde{x})\| \\ &\leq \|U\| \, \|\tilde{x}\| \, . \end{aligned}$$

Therefore, $\{\Lambda_k\}_{k\in N}$ is $\tilde{X}^*$-Bessel system in Z^*. Next, for every $z \in Z$ we obtain

$$\begin{aligned} z &= UTz \\ &= U(\{g_k(z)\}_{k\in N}) \\ &= \sum_{k=1}^{\infty} \Lambda_k g_k(z). \end{aligned}$$

□

Theorem 2.7. *Let $\tilde{X}$ be a BC-space, the system $\{g'_k\}_{k\in N} \subset L(Z,X)$ be an $\tilde{X}$-frame in Z and have a conjugate system $\{\Lambda_k\}_{k\in N}$ with the synthesizing operator U_1, and the system $\{g''_k\}\}_{k\in N} \subset L(Z,X)$ be such that $\{g''_k(z)\}_{k\in N} \in \tilde{X}$, $\forall z \in Z$. Let the operators T_1 and T_2 be defined by the equalities $T_1(z) = \{g'_k(z)\}_{k\in N}$ and $T_2(z) = \{g''_k(z)\}_{k\in N}$, $z \in Z$, respectively. Assume that there exist the numbers $\lambda, \mu \geq 0$, $\beta \in [0,1)$ such that*

1) $\lambda \|T_1\| + \beta \|T_2\| + \mu < \|U_1\|^{-1}$;

2)

$$\left\| \left\{ g'_k(z) \right\}_{n\in N} - \left\{ g''_k(z) \right\}_{n\in N} \right\|_{\tilde{X}} \\ \leq \lambda \left\| \left\{ g'_k(z) \right\}_{n\in N} \right\|_{\tilde{X}} + \beta \left\| \left\{ g''_k(z) \right\}_{n\in N} \right\|_{\tilde{X}} \\ + \mu \left\| z \right\|_Z , \quad \forall z \in Z.$$

Then $\{g''_k\}_{k\in N}$ *forms an* $\tilde{X}$*-frame for* Z *and has a conjugate system* $\{\Gamma_k\}_{k\in N}$ *such that* $\{\Lambda_k - \Gamma_k\}_{k\in N}$ *is* $\tilde{X}^*$*-Bessel system in* Z^**.*

Proof. For every $z \in Z$ we have

$$\begin{aligned} \|(I - U_1T_2)(z)\| &= \|(U_1T_1 - U_1T_2)(z)\| \\ &= \|U_1(T_1 - T_2)(z)\| \\ &\leq \|U_1\| \left(\lambda \|T_1(z)\|_{\tilde{X}} + \beta \|T_2(z)\|_{\tilde{X}} + \mu \|z\|_Z\right) \\ &\leq \|U_1\| \left(\lambda \|T_1\| + \beta \|T_2\| + \mu\right) \|z\| . \end{aligned}$$

By using Neumann Theorem we obtain that U_1T_2 is bounded invertible operator. Let's show that $\{g''_k\}_{k\in N}$ forms an $\tilde{X}$-frame for Z. We have

$$\begin{aligned} \|T_2(z)\|_{\tilde{X}} &\leq \|(T_1 - T_2)(z)\|_{\tilde{X}} + \|T_1(z)\|_{\tilde{X}} \\ &\leq (1+\lambda) \|T_1(z)\|_{\tilde{X}} + \beta \|T_2(z)\| + \mu \|z\|_Z , \quad z \in Z. \end{aligned}$$

Consequently,

$$\|T_2(z)\|_{\tilde{X}} \leq \frac{(1+\lambda)\|T_1\| + \mu}{1-\beta} \|z\| .$$

We also have

$$\begin{aligned} \|z\| &= \left\|(U_1T_2)^{-1}U_1T_2(z)\right\| \\ &\leq \left\|(U_1T_2)^{-1} U_1\right\| \|T_2(z)\|_{\tilde{X}} . \end{aligned}$$

Let $\Gamma_k = (U_1T_2)^{-1}\Lambda_k$. It is clear that $\{\Gamma_k\}_{k\in N} \subset L(X, Z)$, and $\{\Gamma_k\}_{k\in N}$ forms an $\tilde{X}^*$-Bessel system in Z^* because

$$\sum_{k=1}^{\infty} \Gamma_k(x_k) = (U_1T_2)^{-1}U_1(\{x_k\}_{k\in N}), \quad \{x_k\}_{k\in N} \in \tilde{X}.$$

For every $z \in Z$ we have

$$\begin{aligned} z &= (U_1T_2)^{-1}(U_1T_2)(z) \\ &= (U_1T_2)^{-1}\sum_{k=1}^{\infty}\Lambda_k(g_k''(z)) \\ &= \sum_{k=1}^{\infty}\Gamma_k(g_k''(z)), \end{aligned}$$

i.e. $\{\Gamma_k\}_{k\in N}$ is a conjugate system for $\{g_k''\}_{k\in N}$. Finally, for every $\{x_n\}_{n\in N} \in \tilde{X}$ we obtain

$$\begin{aligned} \left\|\sum_{k=1}^{\infty}(\Lambda_k - \Gamma_k)(x_k)\right\| &= \left\|(I-(U_1T_2)^{-1})\sum_{k=1}^{\infty}\Lambda_k(x_k)\right\| \\ &\le \left\|(I-(U_1T_2)^{-1}\right\| \left\|\sum_{k=1}^{\infty}\Lambda_k(x_k)\right\| \\ &\le \left\|(I-(U_1T_2)^{-1}\right\| \|U_1\| \left\|\{x_k\}_{k\in N}\right\|. \end{aligned}$$

□

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