ON $\tilde{X}$-FRAMES AND CONJUGATE SYSTEMS IN
BANACH SPACES

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Abstract. The generalization of p-frame in Banach spaces is considered in this paper. The concepts of an $\tilde{X}$-frame and a system conjugate to $\tilde{X}$-frame were introduced. Analogues of the results on the existence of conjugate system were obtained. The stability of $\tilde{X}$-frame having a conjugate system is studied.

1. Introduction

The concept of frame in Hilbert spaces was introduced by R.J.Duffin and A.C.Schaeffer [1] when studying non-harmonic Fourier series. Let us recall that the system $\{f_i\}_{i \in \mathbb{N}}$ in separable Hilbert space $H$ is called a frame if there are the constants $A > 0$ and $B > 0$ such that

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i \rangle|^2 \leq B\|f\|^2, \quad \forall f \in H.$$ 

The constants $A$ and $B$ are called the frame bounds. Refer to [2, 3] for the theory of frames. There are many other works dedicated to frames and their applications (see [4, 5], etc). Frames are widely used in signal processing, data compression, characterization of function spaces and other fields. Every vector in separable Hilbert space can be expanded in a frame. In other words, if $\{f_i\}_{i \in \mathbb{N}}$ forms a frame for $H$ then there exists the system $\{\tilde{f}_i\}_{i \in \mathbb{N}} \subset H$ such that the recovery formula

$$f = \sum_{k=1}^{\infty} \langle f, \tilde{f}_k \rangle f_k$$
holds for every vector $f \in H$. Note that such a system may not be unique. The system $\{\tilde{f}_i\}_{i \in N}$ also forms a frame for $H$ and is called a frame conjugate to $\{f_i\}_{i \in N}$. One of the most important fields in the theory of frames in Hilbert spaces is the stability of frame. More information about this matter can be found in [6, 7].

K. Gröchenig [8] extended the concept of frame to the case of Banach space. He introduced the concept of a separable Banach frame and the one of atomic decomposition in Banach spaces. Banach frames and atomic decomposition, as well as their stability in Banach spaces have been studied in [9-11]. There are other generalizations of frame, Banach frame and atomic decomposition. In [12], the concepts of $p$-frame and $p$-Riesz basis were introduced and their properties studied. Reference [13], dedicated to $p$-frames, also presented a recovery formula. In [14], the concept of $g$-frame was introduced and the analogues of corresponding frame results have been obtained. As generalizations of $g$-frame, the concepts of $pg$-frame and Banach $g$-frame were introduced in [15] and their stability studied. Frame properties of degenerate trigonometric systems in Lebesgue spaces have been studied in [16, 17] in case when the degeneration coefficient does not satisfy the well-known Muckenhoupt condition (with respect to the Muckenhoupt condition see e.g. [18]).

This work is dedicated to the generalization of frame in Banach spaces with respect to the Banach space of sequences. The concept of $\tilde{X}$-frame is introduced which is a generalization of $p$-frame. The equivalent conditions for the existence of a system conjugate to $\tilde{X}$-frame were studied, and we obtain the result for the stability of $\tilde{X}$-frame was obtained. A $g$-frame with respect to the Banach space of sequences is also considered.

2. $\tilde{X}$-frames in Banach spaces

Let $X, Z$ be Banach spaces, $\tilde{X}$ be a Banach space of sequences of vectors $X$ with the coordinate-wise linear operations such that the operator

$$P_k : X \to \tilde{X}, \quad P_k(x) = \{\delta_{ik}x\}_{i \in N},$$

is bounded ($BK$-space for brevity). $\tilde{X}$ will be called a $BC$-space if the equality

$$\lim_{n \to \infty} \left\| \{x_k\}_{k \in N} - \sum_{k=1}^{n} \{\delta_{ik}x_k\}_{i \in N} \right\|_{\tilde{X}} = 0$$

holds for every $\{x_k\}_{k \in N} \in \tilde{X}$.

It is not difficult to show that if $\tilde{X}$ is a $BC$-space, then $\tilde{X}^*$ is isometrically isomorphic to the Banach space
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$\tilde{Y} = \left\{ \{t_k\}_{k \in \mathbb{N}} \subset X^* : t_k = iP_k, i \in \tilde{X}^* \right\}$
equipped with the norm $\|\{t_k\}_{k \in \mathbb{N}}\|_{\tilde{Y}} = \|i\|_{\tilde{X}^*}$. Every $\tilde{i} \in \tilde{X}^*$ is defined by the formula

$$\tilde{i}(\{x_k\}_{k \in \mathbb{N}}) = \sum_{k=1}^{\infty} t_k(x_k).$$

Therefore $\tilde{X}^*$ is identified with $\tilde{Y}$.

By $L(Z, X)$ we denote the space of all bounded linear operators from $Z$ to $X$.

**Definition 2.1.** System $\{g_k\}_{k \in \mathbb{N}} \subset L(Z, X)$ is called $\tilde{X}$-frame in $Z$ if there are the constants $A > 0$ and $B > 0$ such that

$$A \|z\|_Z \leq \|\{g_k(z)\}_{k \in \mathbb{N}}\|_{\tilde{X}} \leq B \|z\|_Z, \quad \forall z \in Z.$$

Constants $A$ and $B$ are called $\tilde{X}$-frame bounds of $\{g_k\}_{k \in \mathbb{N}}$. In case when $\{g_k\}_{k \in \mathbb{N}}$ satisfies the right-hand side inequality, $\{g_k\}_{k \in \mathbb{N}}$ is called $\tilde{X}$-Bessel system in $Z$ with a bound $B$.

Let $\{g_k\}_{k \in \mathbb{N}} \subset L(Z, X)$ be an $\tilde{X}$-frame in $Z$ and $S \in L(\tilde{X}, Z)$. The pair $\{(g_k)_{k \in \mathbb{N}}, S\}$ is called a Banach $g$-frame in $Z$ with respect to $\tilde{X}$ (see [15]) if

$$S(\{g_n(z)\}_{n \in \mathbb{N}}) = z, \quad \forall z \in Z.$$

Let $\{g_k\}_{k \in \mathbb{N}} \subset L(Z, X)$ be an $\tilde{X}$-frame in $Z$. Denote by $T$ the operator $T(z) = \{g_k(z)\}_{k \in \mathbb{N}}, z \in Z$. It is clear that $T$ maps $Z$ isomorphically onto $R(T)$.

**Theorem 2.2.** Let $\tilde{X}$ be a BC-space, and the system $\{\Lambda_k\}_{k \in \mathbb{N}} \subset L(X, Z)$ be given. Then $\{\Lambda_k\}_{k \in \mathbb{N}}$ is $\tilde{X}^*$-Bessel system in $Z^*$ with a bound $B$ if and only if there exists

$$U \in L(\tilde{X}, Z) : \quad U(\tilde{x}) = \sum_{k=1}^{\infty} \Lambda_k(x_k), \quad \tilde{x} = \{x_n\}_{n \in \mathbb{N}}, \quad \|U\| \leq B.$$

**Proof.** Let’s prove the convergence of the series $\sum_{k=1}^{\infty} \Lambda_k(x_k)$ for any $\tilde{x} = \{x_n\}_{n \in \mathbb{N}}$. For $n > m$ we have
\[ \left\| \sum_{k=m}^{n} \lambda_k(x_k) \right\|_{Z} = \sup_{\|f\|=1} \left\| \sum_{k=m}^{n} f(\lambda_k(x_k)) \right\| \]
\[ = \sup_{\|f\|=1} \left\| \sum_{k=m}^{n} \lambda_k^* f(x_k) \right\| \]
\[ \leq B \left\| \sum_{k=m}^{\infty} \{\delta_{ik} x_k\}_{i \in N} \right\|_{X}. \]

Consequently, the series \( \sum_{k=1}^{\infty} \lambda_k(x_k) \) is convergent. It follows that the operator
\[ U(\tilde{x}) = \sum_{k=1}^{\infty} \lambda_k(x_k), \quad \tilde{x} = \{x_n\}_{n \in N} \]
well-defined, and it is clear that \( U \in L(\tilde{X}, Z) \). Conversely, let there exist
\[ U \in L(\tilde{X}, Z), \quad U(\tilde{x}) = \sum_{k=1}^{\infty} \lambda_k(x_k), \quad \tilde{x} = \{x_n\}_{n \in N}. \]
Then
\[ \left\| \{\lambda_k^* f\}_{k \in N} \right\|_{\tilde{X}^*} = \sup_{\|x_k\|=1} \left\| \sum_{k=1}^{\infty} \lambda_k^* f(x_k) \right\| \]
\[ \leq \|U\| \|f\|. \]

Let us provide the analogue of the results on equivalence Banach \( p \)-frames from [13].

**Theorem 2.3.** Let \( \tilde{X} \) be a \( BK \)-space, \( \{g_k\}_{k \in N} \subset L(Z, X) \) be an \( \tilde{X} \)-frame in \( Z \). Then the following conditions are equivalent:

1) \( R(T) \) is complemented in \( \tilde{X} \);
2) the operator \( T^{-1} : R(T) \to Z \) can be extended to the bounded operator
\[ W : \tilde{X} \to Z; \]
3) there exists the bounded operator \( S \in L(\tilde{X}, Z) \) such that \( (\{g_k\}, S) \) forms a Banach \( g \)-frame for \( Z \) with respect to \( \tilde{X} \).

**Proof.** Let’s show the validity of 1) \( \iff \) 2). Let \( P \) be a projection from \( \tilde{X} \) to \( R(T) \). Consider the operator \( W = T^{-1} P \). Operator \( W \) will be the
desired operator. Now let 2) be true. Define the operator \( P = TW \). We have

\[
P^2 = TWTW \\
= TW \\
= P.
\]

For every \( z \in Z \) we obtain

\[
T(z) = TWT(z) \\
= P(T(z)).
\]

Consequently, \( R(P) = R(T) \), and therefore \( R(T) \) is complemented in \( \tilde{X} \).

Let’s show the validity of 1) \( \Leftrightarrow \) 3). As \( S \), the operator \( W \) was taken. It is clear that \( (\{g_i\}, W) \) forms a Banach \( g \)-frame for \( Z \) with respect to \( \tilde{X} \). Conversely, let there exist the bounded operator \( S \in L(\tilde{X}, Z) \) such that \( (\{g_i\}, S) \) forms a Banach \( g \)-frame for \( Z \) with respect to \( \tilde{X} \). Then the operator \( S \) is a bounded continuation of \( T^{-1} \).

\[ \square \]

**Theorem 2.4.** Let \( \tilde{X} \) and \( \tilde{X}^* \) be BC-spaces, \( \{g_k\}_{k \in N} \subset L(Z, X) \) be an \( \tilde{X} \)-frame in \( Z \). Then the following conditions are equivalent:

1) \( R(T) \) is complemented in \( \tilde{X} \);
2) there exists an \( \tilde{X}^* \)-Bessel system in \( Z^* \) system

\[
\{\Lambda_k\}_{k \in N} \subset L(X, Z)
\]

such that

\[
z = \sum_{k=1}^{\infty} \Lambda_k g_k(z) \quad \text{for every} \quad z \in Z;
\]

3) there exists an \( \tilde{X}^* \)-Bessel system in \( Z^* \) system

\[
\{\Lambda_k\}_{k \in N} \subset L(X, Z)
\]

such that

\[
f = \sum_{k=1}^{\infty} f \Lambda_k g_k, \quad \forall f \in Z^*.
\]

**Proof.** Let’s show the validity of 1) \( \Leftrightarrow \) 2). Let \( R(T) \) be complemented in \( \tilde{X} \). Then, by Theorem 2.3, there exists a bounded continuation \( W \) of operator \( T^{-1} \) to the whole \( \tilde{X} \). Define \( \{\Lambda_k\} \) as follows: \( \Lambda_k = WP_k \). It is clear that

\[
\{\Lambda_k\}_{k \in N} \subset L(X, Z).
\]
For arbitrary \( f \in Z^* \), we obtain \( \Lambda^*_k f \in X^* \) and \( \{ \Lambda^*_k f \}_{k \in N} \in \tilde{X}^* \) because 

\[
\sum_{k=1}^{\infty} \Lambda^*_k f(x_k) = fW(\tilde{x}) \text{ for every } \tilde{x} = \{x_n\}_{n \in N} \in \tilde{X}.
\]

We have

\[
\left\| \{ \Lambda^*_k(f) \}_{k \in N} \right\|_{\tilde{X}^*} = \sup_{\|x_n\| = 1} \left| \sum_{k=1}^{\infty} \Lambda^*_k f(x_k) \right| \leq \|W\| \|f\|,
\]

i.e. \( \{ \Lambda_k \} \) is an \( \tilde{X}^* \)-Bessel system in \( Z^* \). Conversely, let 2) be true.

Denote

\[
U(\tilde{x}) = \sum_{k=1}^{\infty} \Lambda_k(x_k), \quad \tilde{x} = \{x_n\}_{n \in N}.
\]

Operator \( U \) is a continuous continuation of \( T^{-1} \). In fact, \( \forall z \in Z \) we obtain

\[
UT(z) = U(\{g_n(z)\}_{n \in N}) = \sum_{k=1}^{\infty} \Lambda_k(g_k(z)) = z.
\]

The rest follows from Theorem 2.3.

Now let’s prove the validity of 2) \( \iff \) 3). Let 2) be true. Take arbitrary \( f \in Z^* \). We have

\[
\left\| f - \sum_{k=1}^{n} f \Lambda_k g_k \right\| = \sup_{\|z\| = 1} \left| f(z) - \sum_{k=1}^{n} \Lambda^*_k f(g_k(z)) \right| \leq B \left\| \sum_{k=n+1}^{\infty} \delta_{ik} \Lambda^*_k f \right\|_{\tilde{X}^*}.
\]
Hence, \( f = \sum_{k=1}^{\infty} f \Lambda_k g_k \). Conversely, suppose that 3) is true. For arbitrary \( z \in Z \) we have

\[
\left\| z - \sum_{k=1}^{n} \Lambda_k (g_k(z)) \right\| = \sup_{\|f\|=1} \left| \sum_{k=1}^{n} \frac{\Lambda_k}{\|f\|} (g_k(z)) \right|
\]

\[
= \sum_{k=n+1}^{\infty} \frac{\Lambda_k}{\|f\|} (g_k(z)) \leq B_1 \sum_{k=n+1}^{\infty} \| \delta_k g_k(z) \|_{X^*},
\]

where \( B_1 \) is a bound of \( \{\Lambda_k\}_{k \in \mathbb{N}} \). Therefore,

\[
z = \sum_{k=1}^{\infty} \Lambda_k g_k(z).
\]

Now we introduce the concept of a conjugate \( X^* \)-frame.

**Definition 2.5.** Let the system \( \{g_k\}_{k \in \mathbb{N}} \subset L(Z, X) \) be \( X \)-Bessel system in \( Z \). System \( \{\Lambda_k\}_{k \in \mathbb{N}} \subset L(X, Z) \), \( X^* \)-Bessel system in \( Z^* \), is called conjugate to \( \{g_k\}_{k \in \mathbb{N}} \) if the following condition is satisfied:

\[
z = \sum_{k=1}^{\infty} \Lambda_k g_k(z), \quad \forall z \in Z.
\]

**Theorem 2.6.** Let \( \tilde{X} \) be a \( BC \)-space, and the system \( \{g_k\}_{k \in \mathbb{N}} \subset L(Z, X) \) be an \( \tilde{X} \)-frame in \( Z \). Then \( \{g_k\}_{k \in \mathbb{N}} \) has a conjugate system if and only if there exists a bounded left inverse operator of \( T \).

**Proof.** Let \( \{\Lambda_k\}_{k \in \mathbb{N}} \) be a conjugate system for \( \{g_k\}_{k \in \mathbb{N}} \). Denote by \( U : \tilde{X} \to Z \) its synthesizing operator:

\[
U(\tilde{x}) = \sum_{k=1}^{\infty} \Lambda_k (x_k).
\]

As for every \( z \in Z \) we have

\[
\{g_k(z)\}_{k \in \mathbb{N}} = \sum_{k=1}^{\infty} \{\delta_k g_k(z)\}_{i \in \mathbb{N}},
\]
the relation \( U(\{\delta_{ik}g_k(z)\}_{k \in \mathbb{N}}) = \Lambda_i(g_i(z)) \) holds. Then we obtain
\[
z = \sum_{k=1}^{\infty} \Lambda_k g_k(z) \\
= \sum_{k=1}^{\infty} U(\{\delta_{ik}g_k(z)\}_{i \in \mathbb{N}}) \\
= U(\{g_k(z)\}_{k \in \mathbb{N}}) \\
= UT(z),
\]
i.e. \( T \) has a bounded left inverse \( U \). Conversely, let \( T \) have a bounded right inverse \( U \). Consider \( \{\Lambda_k\}_{k \in \mathbb{N}} \subset L(X, Z) \) such that \( \Lambda_k = UP_k \). It is clear that \( \{\Lambda_k\}_{k \in \mathbb{N}} \subset L(X, Z) \). Then for every \( \tilde{x} = \{x_k\}_{k \in \mathbb{N}} \in \tilde{X} \) we have
\[
\left\| \sum_{k=1}^{\infty} \Lambda_k(x_k) \right\| = \left\| \sum_{k=1}^{\infty} U(\{\delta_{ik}x_k\}_{i \in \mathbb{N}}) \right\| \\
= \left\| U(\sum_{k=1}^{\infty} \delta_{ik}x_k) \right\| \\
= \|U(\tilde{x})\| \\
\leq \|\| UT \| \| \tilde{x} \|.
\]

Therefore, \( \{\Lambda_k\}_{k \in \mathbb{N}} \) is \( \tilde{X}^* \)-Bessel system in \( Z^* \). Next, for every \( z \in Z \) we obtain
\[
z = UTz \\
= U(\{g_k(z)\}_{k \in \mathbb{N}}) \\
= \sum_{k=1}^{\infty} \Lambda_k g_k(z).
\]

\[\square\]

**Theorem 2.7.** Let \( \tilde{X} \) be a BC-space, the system \( \{g'_k\}_{k \in \mathbb{N}} \subset L(Z, X) \) be an \( \tilde{X} \)-frame in \( Z \) and have a conjugate system \( \{\Lambda_k\}_{k \in \mathbb{N}} \) with the synthesizing operator \( U_1 \), and the system \( \{g''_k\}_{k \in \mathbb{N}} \subset L(Z, X) \) be such that \( \{g''_k(z)\}_{k \in \mathbb{N}} \in \tilde{X}, \forall z \in Z \). Let the operators \( T_1 \) and \( T_2 \) be defined by the equalities \( T_1(z) = \{g'_k(z)\}_{k \in \mathbb{N}} \) and \( T_2(z) = \{g''_k(z)\}_{k \in \mathbb{N}}, z \in Z \), respectively. Assume that there exist the numbers \( \lambda, \mu \geq 0, \beta \in [0, 1) \) such that

1) \( \lambda \|T_1\| + \beta \|T_2\| + \mu < \|U_1\|^{-1}; \)
2) \[
\left\| \{g'_k(z)\}_{n \in \mathbb{N}} - \{g''_k(z)\}_{n \in \mathbb{N}} \right\|_{\tilde{X}} \\
\leq \lambda \left\| \{g'_k(z)\}_{n \in \mathbb{N}} \right\|_{\tilde{X}} + \beta \left\| \{g''_k(z)\}_{n \in \mathbb{N}} \right\|_{\tilde{X}} \\
+ \mu \|z\|_Z, \quad \forall z \in Z.
\]

Then \(\{g''_k\}_{k \in \mathbb{N}}\) forms an \(\tilde{X}\)-frame for \(Z\) and has a conjugate system \(\{\Gamma_k\}_{k \in \mathbb{N}}\) such that \(\{\Lambda_k - \Gamma_k\}_{k \in \mathbb{N}}\) is \(\tilde{X}^*\)-Bessel system in \(Z^*\).

Proof. For every \(z \in Z\) we have

\[
\|(I - U_1T_2)(z)\| = \|(U_1T_1 - U_1T_2)(z)\| \\
= \|U_1(T_1 - T_2)(z)\| \\
\leq \|U_1\| (\lambda \|T_1(z)\|_{\tilde{X}} + \beta \|T_2(z)\|_{\tilde{X}} + \mu \|z\|_Z) \\
\leq \|U_1\| (\lambda \|T_1\| + \beta \|T_2\| + \mu) \|z\|.
\]

By using Neumann Theorem we obtain that \(U_1T_2\) is bounded invertible operator. Let’s show that \(\{g''_k\}_{k \in \mathbb{N}}\) forms an \(\tilde{X}\)-frame for \(Z\). We have

\[
\|T_2(z)\|_{\tilde{X}} \leq \|(T_1 - T_2)(z)\|_{\tilde{X}} + \|T_1(z)\|_{\tilde{X}} \\
\leq (1 + \lambda) \|T_1(z)\|_{\tilde{X}} + \beta \|T_2(z)\| + \mu \|z\|_Z, \quad z \in Z.
\]

Consequently,

\[
\|T_2(z)\|_{\tilde{X}} \leq \frac{(1 + \lambda) \|T_1\| + \mu}{1 - \beta} \|z\|.
\]

We also have

\[
\|z\| = \|(U_1T_2)^{-1}U_1T_2(z)\| \\
\leq \left\| (U_1T_2)^{-1}U_1 \right\| \|T_2(z)\|_{\tilde{X}}.
\]

Let \(\Gamma_k = (U_1T_2)^{-1}\Lambda_k\). It is clear that \(\{\Gamma_k\}_{k \in \mathbb{N}} \subset L(X, Z)\), and \(\{\Gamma_k\}_{k \in \mathbb{N}}\) forms an \(\tilde{X}^*\)-Bessel system in \(Z^*\) because

\[
\sum_{k=1}^{\infty} \Gamma_k(x_k) = (U_1T_2)^{-1}U_1(\{x_k\}_{k \in \mathbb{N}}), \quad \{x_k\}_{k \in \mathbb{N}} \in \tilde{X}.
\]
For every $z \in Z$ we have
\[
z = (U_1 T_2)^{-1}(U_1 T_2)(z) = (U_1 T_2)^{-1} \sum_{k=1}^{\infty} \Lambda_k(g_k''(z)) = \sum_{k=1}^{\infty} \Gamma_k(g_k''(z)),
\]
i.e. $\{\Gamma_k\}_{k \in N}$ is a conjugate system for $\{g_k''\}_{k \in N}$. Finally, for every $\{x_n\}_{n \in N} \in \bar{X}$ we obtain
\[
\left\| \sum_{k=1}^{\infty} (\Lambda_k - \Gamma_k)(x_k) \right\| = \left\| (I - (U_1 T_2)^{-1}) \sum_{k=1}^{\infty} \Lambda_k(x_k) \right\| \\
\leq \left\| (I - (U_1 T_2)^{-1}) \right\| \sum_{k=1}^{\infty} \Lambda_k(x_k) \\
\leq \left\| (I - (U_1 T_2)^{-1}) \right\| \|U_1\| \left\| \{x_k\}_{k \in N} \right\|.
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References


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