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ON A SEQUENCE RELATED TO THE COPRIME INTEGERS

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ABSTRACT. The asymptotic behaviour of the sequence with general term $P_n = (\varphi(1) + \varphi(2) + \cdots + \varphi(n))/(1 + 2 + \cdots + n)$, is studied which appears in the studying of coprime integers, and an explicit bound for the difference $P_n - 6/\pi^2$ is found.

1. INTRODUCTION

In this note, we are motivated by the well-known fact asserting that, if two positive integers a and b are chosen at random, the probability that they are coprime is $6/\pi^2 \approxeq 0.61$. Assume that two arbitrary integers a and b from the set $\{1, 2, \ldots, n\}$ are chosen. We let

$$P_n := \text{the probability that } a \text{ and } b \text{ are coprime, and } 1 \leq b \leq a \leq n.$$

The above mentioned assertion is indeed the validity of the following limit relation

$$\lim_{n\to\infty} P_n = \frac{6}{\pi^2}.$$

Our aim in writing this note, is to detect some details of this limit relation. Some known facts about P_n were pictured and reviewed, and finally, the following explicit bounds for P_n were found.

Theorem 1.1. *For any integer $n \geq 2$ we have*

$$\left| P_n - \frac{6}{\pi^2} \right| < 3.82 \frac{\log n}{n}.$$

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2. Behaviour of P_n

To get a formula for P_n, we consider a triangular array, consisting all of pairs (a, b) with $1 \le b \le a \le n$. The pair (a, b) was replaced by the number $\gcd(a, b)$, and it is noted that the number of appearing 1 on the k^{th} line is equal to $\varphi(k)$, where as usual φ refers to the Euler function. Hence,

$$P_n = \frac{\varphi(1) + \varphi(2) + \varphi(3) + \cdots + \varphi(n)}{1 + 2 + 3 + \cdots + n},$$

is obtained.

As Table 1 and Figure 1 show, the sequence $\{P_n\}$ has asymptotically oscillation behaviour. One may justify such behaviour by considering the well-known sequence

$$a_n = \frac{1}{n} \sum_{m \le n} \varphi(m),$$

which simply is related by P_n with $P_n = 2a_n/(n+1)$.

Table 1. Some values of the sequence P_n, and the sign of the difference $P_n - 6/\pi^2$.

n	P_n	sgn	n	P_n	sgn
1000	0.6077762238	−	2000	0.6079900050	+
3000	0.6078391647	−	4000	0.6077983004	−
5000	0.6079150570	−	6000	0.6078522468	−
7000	0.6078783031	−	8000	0.6079171979	−
9000	0.6078711748	−	10000	0.6078889311	−

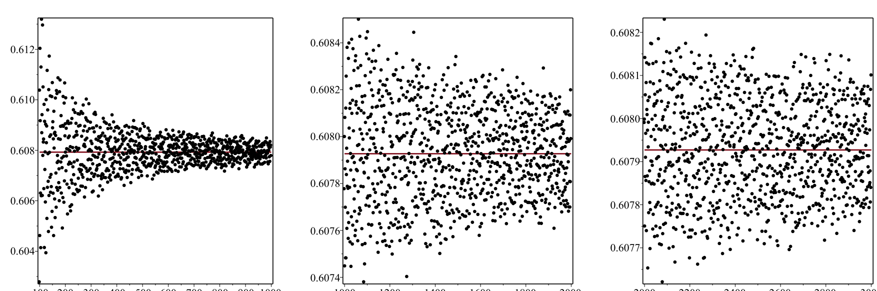

Figure 1. Graph of the pointset (n, P_n) for $100 \le n \le 3000$, and horizontal line at height $6/\pi^2$.

To justify asymptotic behaviour of the sequence defined by P_n, we consider the following best known approximation, due to Walfisz [5], where he proved that

$$a_n = \frac{3}{\pi^2}n + R(n), \quad \text{where} \quad R(n) \ll (\log n)^{\frac{2}{3}}(\log\log n)^{\frac{4}{3}}. \tag{2.1}$$

This implies that

$$P_n = \frac{6}{\pi^2} + E(n), \tag{2.2}$$

where

$$E(n) = \frac{2}{n+1}\left(R(n) - \frac{3}{\pi^2}\right) \ll \frac{(\log n)^{\frac{2}{3}}(\log\log n)^{\frac{4}{3}}}{n}.$$

To justify oscillation behaviour of P_n, we consider a result due to Erdös and Shapiro [2], which asserts that the inequalities

$$R(n) > c\log\log\log\log n \quad \text{and} \quad R(n) < -c\log\log\log\log n,$$

each are valid for infinitely many integers n for some $c > 0$. Hence, for some $c > 0$, the inequalities

$$E(n) > c\frac{\log\log\log\log n}{n} \quad \text{and} \quad E(n) < -c\frac{\log\log\log\log n}{n},$$

each are valid for infinitely many integers n (but surly very large enough). In this note, some explicit bounds for the remainder term $R(n)$ is obtained. Indeed, we show the following, which implies the assertion of Theorem 1.1, too.

Theorem 2.1. *For any integer $n \geq 2$ we have*

$$\left|\frac{1}{n}\sum_{m\leq n}\varphi(m) - \frac{3}{\pi^2}n\right| < 1.91\log n.$$

3. Proof of results

We apply the Euler–Maclaurin summation formula (see for example [4]) to get the following lemma.

Lemma 3.1. *For any positive integer $n \geq 1$ we have*

$$\sum_{d\leq n}\frac{1}{d} = \log n + \gamma + J_n, \tag{3.1}$$

where $\gamma \approxeq 0.577215664901$ is the Euler's constant, and

$$|J_n| \leq \frac{3n+1}{6n^2}.$$

Also, we have

$$\sum_{d>n} \frac{1}{d^2} = \frac{1}{n} - \frac{1}{2n^2} + \frac{K_n}{3n^3}, \tag{3.2}$$

where $0 \le K_n \le 1$.

Proof of the Theorem 2.1. We recall the notion of saw function, which is defined by $\psi(x) = \{x\} - 1/2$, where $\{x\}$ denotes the fractional part of real number x. We start from the known relation $\varphi(m) = m \sum_{d|m} \mu(d)/d$, and we utilize $\sum_{d=1}^{\infty} \mu(d)/d^2 = 6/\pi^2$, to write

$$\begin{aligned}
\sum_{m \le n} \varphi(m) &= \sum_{m \le n} \sum_{d|m} \mu(d) \frac{m}{d} \\
&= \sum_{d \le n} \mu(d) \sum_{q \le \frac{n}{d}} q \\
&= \frac{1}{2} \sum_{d \le n} \mu(d) \left[\frac{n}{d}\right] \left(\left[\frac{n}{d}\right] + 1\right) n \\
&= \frac{1}{2} \sum_{d \le n} \mu(d) \left(\left(\frac{n}{d} - \psi\left(\frac{n}{d}\right)\right)^2 - \frac{1}{4}\right) \\
&= \frac{3}{\pi^2} n^2 - nS(n) - S'(n) + S''(n),
\end{aligned}$$

where

$$\begin{aligned}
S(n) &= \sum_{d \le n} \frac{\mu(d)}{d} \, \psi\left(\frac{n}{d}\right), \\
S'(n) &= \frac{n^2}{2} \sum_{d > n} \frac{\mu(d)}{d^2}, \\
S''(n) &= \frac{1}{2} \sum_{d \le n} \mu(d) \left(\psi\left(\frac{n}{d}\right)^2 - \frac{1}{4}\right).
\end{aligned} \tag{3.3}$$

Hence, for any positive integer $n \ge 1$

$$|nR(n)| = \left| \sum_{m \le n} \varphi(m) - \frac{3}{\pi^2} n^2 \right| \le n|S(n)| + |S'(n)| + |S''(n)|,$$

is obtained.

We have $|S''(n)| \le n/8$. Also, by using the relation (3.2) we get

$$\left|S'(n)\right| \le \frac{n}{2} - \frac{1}{4} + \frac{1}{6n}.$$

Thus, for any positive integer $n \geq 1$

$$|nR(n)| \leq n\,|S(n)| + \frac{5}{8}n - \frac{1}{4} + \frac{1}{6n}, \tag{3.4}$$

is obtained.

By applying the relation (3.1), for any positive integer $n \geq 1$ we deduce

$$|S(n)| \leq \frac{1}{2}\left(\log n + \gamma + \frac{3n+1}{6n^2}\right).$$

We combine the above bounds to get

$$|nR(n)| \leq \frac{1}{2}n\log n + \left(\frac{\gamma}{2} + \frac{5}{8}\right)n + \frac{1}{4n},$$

for any positive integer $n \geq 1$. Finally, we note that the function

$$f : [2,\infty) \to [0,\infty)$$

defined by

$$f(n) = \frac{\left(\frac{\gamma}{2} + \frac{5}{8}\right)n + \frac{1}{4n}}{n\log n},$$

is strictly decreasing, and so, we have $f(n) \leq f(2)$ for any integer $n \geq 2$. Hence, for any integer $n \geq 2$ we obtain

$$\left|\sum_{m\leq n}\varphi(m) - \frac{3}{\pi^2}n^2\right| \leq \eta\, n\log n,$$

where

$$\eta := \frac{1}{2} + f(2) = \frac{1}{2} + \frac{\gamma + \frac{11}{8}}{2\log 2} \approxeq 1.9082259292495959178 < 1.91.$$

This completes proof of Theorem 2.1. □

Proof of Theorem 1.1. We consider (2.2), to imply

$$|E(n)| \leq \frac{2}{n+1}\left(|R(n)| + \frac{3}{\pi^2}\right) \leq \frac{2}{n+1}\left(\eta\log n + \frac{3}{\pi^2}\right),$$

for any integer $n \geq 2$. The function $g : [2,\infty) \to [0,\infty)$ defined by

$$g(n) = \frac{\frac{2}{n+1}\left(\eta\log n + \frac{3}{\pi^2}\right)}{\frac{\log n}{n}},$$

is strictly increasing, and we have

$$\ell := \lim_{n\to\infty} g(n) = \frac{8\log 2 + 8\gamma + 11}{8\log 2} \approxeq 3.8164518584991918358 < 3.82.$$

This completes the proof of Theorem 1.1. □

4. Further remarks

The relation (3.4) implies that

$$\begin{aligned} |R(n)| &= \left| \frac{1}{n} \sum_{m \le n} \varphi(m) - \frac{3}{\pi^2} n \right| \\ &< |S(n)| + \frac{5}{8}, \end{aligned}$$

for any positive integer $n \ge 1$. This inequality shows that the main term in the upper bound for $|R(n)|$ comes from

$$|S(n)| = \left| \sum_{d \le n} \frac{\mu(d)}{d} \ \psi \left(\frac{n}{d} \right) \right|.$$

Our approximation for $|S(n)|$ is of order $\log n$. The result of Walfisz, which we mentioned in (2.1) asserts that this order can be reduced up to $(\log n)^{\frac{2}{3}} (\log \log n)^{\frac{4}{3}}$. Naturally, we ask the following question.

Question. What can be said about the asymptotic size of $|S(n)|$ under assumption that the Riemann Hypothesis is true?

To attack the above question, the following information seems to be useful. It is known, due to de la Vallée Poussin (see [3]), that

$$\sum_{d \le n} \left\{ \frac{n}{d} \right\} = (1 - \gamma) n + O(\sqrt{n}).$$

This relation implies that

$$\sum_{d \le n} \psi \left(\frac{n}{d} \right) = \left(\frac{1}{2} - \gamma \right) n + O(\sqrt{n}),$$

and by using the partial summation formula, we get

$$\sum_{d \le n} \frac{1}{d} \psi \left(\frac{n}{d} \right) = \left(\frac{1}{2} - \gamma \right) (1 + \log n) + O \left(\frac{1}{\sqrt{n}} \right).$$

On the other hand, it is known (see for example [1]) that the Riemann hypothesis is equivalent to the estimate

$$\sum_{d \le n} \mu(d) = O(n^{\frac{1}{2} + \epsilon}), \quad \text{for any } \epsilon > 0.$$

One may obtain a reasonable combination of the last two relations, to answer the above proposed question.

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