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A CLASS OF COMPACT OPERATORS ON HOMOGENEOUS SPACES

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Abstract. Let ϖ be a representation of the homogeneous space G/H, where G be a locally compact group and H be a compact subgroup of G. For an admissible wavelet ζ for ϖ and $\psi \in L^p(G/H)$, $1 \leq p < \infty$, we determine a class of bounded compact operators which are related to continuous wavelet transforms on homogeneous spaces and they are called localization operators.

1. Introduction

The compact operators are the simplest non-trivial class of operators that play an important and fundemental role in operator theory. These operators behave much like operators on finite dimensional vector spaces and for this reason they are relatively easy to analyse. A set in a topological space is called pre-compact if its closure is compact. A linear operator T from a pre-Hilbert space $\mathcal{H}_1$ to a Hilbert space $\mathcal{H}_2$ is compact if it maps the unit ball in $\mathcal{H}_1$ to a pre-compact set in $\mathcal{H}_2$.

Now, we recall remarkable points of the Radon measures on homogeneous spaces. Let G be a locally compact group and H be a closed subgroup of G. We mean G/H as a homogeneous space on which G acts from the left and μ as a Radon measure on it. For $g \in G$ and Borel subset E of G/H, the translation μ_g of μ is defined by $\mu_g(E) = \mu(gE)$ was defined. A measure μ is said to be G-invariant if $\mu_g = \mu$, for all $g \in G$.

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Definition 1.1. A measure μ is said to be strongly quasi invariant provided that a continuous function $\lambda : G \times G/H \to (0,\infty)$ exists which satisfies

$$d\mu_g(kH) = \lambda(g, kH)d\mu(kH),$$

for all $g, k \in G$.

If the functions $\lambda(g, .)$ reduces to constants, then μ is called relatively invariant under G (for a detailed account of homogeneous spaces, the reader is refered to [3]). A rho-function for the pair (G, H) is defined to be a continuous function $\rho : G \to (0,\infty)$ which satisfies

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g) \quad (g \in G, h \in H),$$

where Δ_G, Δ_H are the modular functions on G and H, respectively.

Proposition 1.2. [3] *Any pair (G, H) admits a rho-function and for each rho-function ρ there is a strongly quasi invariant measure μ on G/H such that*

$$\frac{d\mu_g}{d\mu}(kH) = \frac{\rho(gk)}{\rho(k)} \quad (g, k \in G).$$

As it has been shown in [3], every strongly quasi invariant measure on G/H, arises from a rho-function and all such measures are strongly equivalent. That is, there exists a positive continuous function τ on G/H such that $\frac{d\acute{\mu}}{d\mu} = \tau$, in which $\mu, \acute{\mu}$ are strongly quasi invariant measures arising from rho functions $\rho, \acute{\rho}$, respectively.

The present paper studies a class of bounded compact operators which are called localization operators. The localization operators are denoted by $L_{\psi,\zeta}$ where

$$\psi \in L^p(G/H, \mu), \quad 1 \le p \le \infty,$$

and ζ is admissible wavelet in separable Hilbert space $\mathcal{H}$. We investigate some significant properties of localization operators, such as boundedness and compactness.

2. Main results

For the reader's convenience, we recall from [2] the basic concepts in the theory of unitary representations of homogeneous spaces. Consider G/H as a homogeneous space associated with a relatively invariant measure μ which arises from a rho-function ρ. A continuous unitary representation of a homogeneous space G/H is a map ϖ from G/H into the group $U(\mathcal{H})$, all unitary operators on some nonzero Hilbert space

$\mathcal{H}$, for which the function $gH \mapsto \langle \varpi(gH)x, y\rangle$ is continuous , for each $x, y \in \mathcal{H}$ and

$$\varpi(gkH) = \varpi(gH)\varpi(kH), \quad \varpi(g^{-1}H) = \varpi(gH)^*,$$

for each $g, k \in G$ (In the sequel we always mean by a representation, a continuous unitary representation).
Throughout this point of view, H is considered a compact subgroup of locally compact group G.

Definition 2.1. Let ϖ be a representation of homogeneous space G/H. A nonzero element $\zeta \in \mathcal{H}$ is called an *admissible wavelet* if $\|\zeta\| = 1$ and

$$\int_{G/H} \frac{\rho(e)}{\rho(g)} \ |\langle \zeta, \varpi(gH)\zeta\rangle|^2 d\mu(gH) < \infty, \tag{2.1}$$

where μ is a relatively invariant measure on G/H which arises from a rho function ρ. If irreducible representation ϖ satisfies in (2.1), it is said to be square integrable. In this case, we define the wavelet constant c_ζ is defined as

$$c_\zeta := \int_{G/H} \frac{\rho(e)}{\rho(g)} \ |\langle \zeta, \varpi(gH)\zeta\rangle|^2 d\mu(gH). \tag{2.2}$$

Remark 2.2. Note that since H is a compact subgroup of G, $\Delta_G|_H = \Delta_H = 1$. So, for every rho-function ρ, we have $\rho(gh) = \rho(g)$ for all $g \in G$ and $h \in H$. This implies that there is a function $\tilde{\rho}$ on G/H such that for each $g \in G, \tilde{\rho}(gH) = \rho(g)$. Therefore Definition 2.1 is well defined.

Let ϖ be a square integrable representation of G/H on $\mathcal{H}$ and ζ be an admissible wavelet for ϖ. We introduce the localization operator $L_{\psi,\zeta}$ which is related to the continuous wavelet transform on G/H. To this end, we define a continuous wavelet transform on homogeneous space G/H.

Definition 2.3. Let ϖ be a representation of G/H on a Hilbert space $\mathcal{H}$ and ζ be an admissible wavelet for ϖ. The *continuous wavelet transform associated to the admissible wavelet* ζ is defined as the linear operator $W_\zeta : \mathcal{H} \to C(G/H)$ defined by

$$(W_\zeta x)(gH) = \frac{1}{\sqrt{c_\zeta}} (\frac{\rho(e)}{\rho(g)})^{1/2} \langle x, \varpi(gH)\zeta\rangle,$$

for all $x \in \mathcal{H}, g \in G$ where c_ζ is the wavelet constant associated to ζ as in (2.2).

Note that if ϖ is a square integrable representation of G/H on $\mathcal{H}$ and ζ is an admissible wavelet for ϖ, then W_ζ is a bounded linear operator from $\mathcal{H}$ into $L^2(G/H)$.

In [2] it has been shown that the continuous wavelet transform is isometry. In particular, we have the following theorem:

Theorem 2.4. *Let ϖ be a square integrable representation of G/H on $\mathcal{H}$ and ζ be an admissible wavelet for ϖ. Then*

$$\langle x, y\rangle = \langle W_\zeta x, W_\zeta y\rangle, \tag{2.3}$$

for every $x, y \in \mathcal{H}$. In particular, the wavelet transform $W_\zeta : \mathcal{H} \to L^2(G/H)$ is an isometry.

Now, linear operators is defined as follows which are related to continuous wavelet transforms on homogeneous spaces and they are called localization operators.

Definition 2.5. Let $\mathcal{H}$ be a Hilbert space and ϖ be a square integrable representation of G/H on $\mathcal{H}$ with an admissible wavelet ζ. Define the linear operator $L_{\psi,\zeta}$ on $\mathcal{H}$ as:

$$\langle L_{\psi,\zeta} x, y\rangle_{\mathcal{H}} = \langle \psi.W_\zeta x, W_\zeta y\rangle_{L^2(G/H)}, \tag{2.4}$$

for all $\psi \in L^p(G/H)$ and $x, y \in \mathcal{H}$. The linear operator $L_{\psi,\zeta}$ is called the *localization operator*.

In the following theorem, the boundedness of linear operator $L_{\psi,\zeta}$ is proven, where $\psi \in L^p(G/H)$ and ζ is an admissible wavelet for the representation ϖ of homogeneous space G/H.

Theorem 2.6. *Let $\psi \in L^p(G/H)$, for $1 \leq p \leq \infty$. Then there exists a unique bounded linear operator $L_{\psi,\zeta}$ on a separable Hilbert space $\mathcal{H}$ such that*

$$\|L_{\psi,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p, \tag{2.5}$$

where $\|\psi\|_p$ is defined with respect to a G-invariant measure and $L_{\psi,\zeta}$ is given by (2.4) for all $x \in \mathcal{H}$ and all simple functions ψ on G/H for which

$$\mu(\{gH \in G/H; \quad \psi(gH) \neq 0\}) < \infty.$$

Proof. First, it is shown that for $\psi \in L^\infty(G/H)$, the localization operator $L_{\psi,\zeta}$ is bounded. Using Theorem 2.4 and the Schwarz inequality we have,

$$\begin{aligned} |\langle L_{\psi,\zeta} x, y\rangle| &\leq \|\psi\|_\infty (\|W_\zeta x\|_2)^{1/2} (\|W_\zeta y\|_2)^{1/2} \\ &\leq \|\psi\|_\infty \|x\|\|y\|, \end{aligned}$$

for all $x, y \in \mathcal{H}$. Thus $\|L_{\psi,\zeta}\| \leq \|\psi\|_\infty$. Secondly, let $\psi \in L^1(G/H)$ and $\acute{\mu}$ a G-invariant measure on G/H which arises from the rho-function $\acute{\rho} \equiv 1$

[3]. Then,

$$\frac{d\mu}{d\acute{\mu}} = \tau, \quad \rho(g) = \tau(gH),$$

where μ is a relatively invariant measure which arises from ρ. We have

$$\begin{aligned}
|\langle L_{\psi,\zeta}x, y\rangle| &\leq \frac{1}{c_\zeta}\int_{G/H}\frac{\rho(e)}{\rho(g)} \ \ |\psi(gH)|\ |\langle x, \varpi(gH)\zeta\rangle|\ |\langle \varpi(gH)\zeta, y\rangle| d\mu(gH) \\
&\leq \frac{1}{c_\zeta}\int_{G/H}\frac{\tau(eH)}{\tau(gH)} \ \ |\psi(gH)|\ | < x, \varpi(gH)\zeta > | \\
&\quad | < \varpi(gH)\zeta, y > |\tau(gH) d\acute{\mu}(gH) \\
&\leq \frac{1}{c_\zeta}\rho(e)\|\psi\|_1\|x\|\|y\|.
\end{aligned}$$

Therefore, $\|L_{\psi,\zeta}\| \leq \frac{\rho(e)}{c_\zeta}\|\psi\|_1$. Finally, let $\psi \in L^p(G/H)$ such that $1 \leq p \leq \infty$. Consider a unitary operator Γ of $\mathcal{H}$ into $L^2(\mathbb{R}^n)$ and $\psi \in L^1(G/H)$. Then the linear operator $\tilde{L}_{\psi,\zeta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$\tilde{L}_{\psi,\zeta} = \Gamma L_{\psi,\zeta}\Gamma^{-1}, \tag{2.6}$$

is bounded and $\|\tilde{L}_{\psi,\zeta}\| \leq \frac{\rho(e)}{c_\zeta}\|\psi\|_1$. Also, if $\psi \in L^\infty(G/H)$, then $\tilde{L}_{\psi,\zeta}$ on $L^2(\mathbb{R}^n)$ defined as (2.6) is bounded and $\|\tilde{L}_{\psi,\zeta}\| \leq \|\psi\|_\infty$.

Denote by $\mathfrak{A}$, the set of all simple functions ψ on G/H such that

$$\mu(\{gH \in G/H; \psi(gH) \neq 0\}) < \infty.$$

Let $g \in L^2(\mathbb{R}^n)$ and Φ be a linear transformation from $\mathfrak{A}$ into the set of all Lebesgue measurable functions on $\mathbb{R}^n$ defined as $\Phi_g(\psi) = \tilde{L}_{\psi,\zeta}(g)$. Then for all $\psi \in L^1(G/H)$

$$\begin{aligned}
\|\Phi_g(\psi)\|_2 &= \|\tilde{L}_{\psi,\zeta}(g)\|_2 \\
&\leq \|\tilde{L}_{\psi,\zeta}\|\|g\|_2 \\
&\leq \frac{\rho(e)}{c_\zeta}\|\psi\|_1\|g\|_2.
\end{aligned}$$

Similarly for all $\psi \in L^\infty(G/H)$,

$$\|\Phi_g(\psi)\|_2 \leq \|\psi\|_\infty\|g\|_2.$$

By the Reisz Thorin Interpolation Theorem [4],

$$\|\Phi_g(\psi)\|_2 \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p\|g\|_2.,$$

is obtaned. Therefore

$$\|\tilde{L}_{\psi,\zeta}(g)\|_2 \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p\|g\|_2.$$

So,

$$\|\tilde{L}_{\psi,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p,$$

for each $\psi \in \mathfrak{A}$.
Now, let $\psi \in L^p(G/H)$, for all $1 \leq p \leq \infty$. Then there exists a sequence $\{\psi_k\}_{k=1}^\infty$ of functions in $\mathfrak{A}$ such that ψ_k is convergent to ψ in $L^p(G/H)$ as $k \to \infty$. Also, $\{\tilde{L}_{\psi_k,\zeta}\}_{k=1}^\infty$ is a cauchy sequence in $B(L^2(\mathbb{R}^n))$. Indeed,

$$\|\tilde{L}_{\psi_n,\zeta} - \tilde{L}_{\psi_m,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi_n - \psi_m\|_p \to 0.$$

By the completeness of $B(L^2(\mathbb{R}^n))$, there exists a bounded linear operator $\tilde{L}_{\psi,\zeta} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ such that $\tilde{L}_{\psi_k,\zeta}$ converges to $\tilde{L}_{\psi,\zeta}$ in $B(L^2(\mathbb{R}^n))$, in which

$$\|\tilde{L}_{\psi,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p.$$

Thus, the linear operator $L_{\psi,\zeta} : \mathcal{H} \to \mathcal{H}$ where $L_{\psi,\zeta} = \Gamma^{-1}\tilde{L}_{\psi,\zeta}\Gamma$, is a bounded linear operator and

$$\|L_{\psi,\zeta}\| \leq (\frac{\rho(e)}{c_\zeta})^{1/p}\|\psi\|_p.$$

To prove uniqueness, let $\psi \in L^p(G/H)$, $1 < p < \infty$, and suppose that $P_{\psi,\zeta}$ is another bounded linear operator satisfying the conclusions of the theorem. Consider $\Theta_\psi : L^p(G/H) \to B(\mathcal{H})$ be the linear operator defined by

$$\Theta_\psi = L_{\psi,\zeta} - P_{\psi,\zeta}, \quad \psi \in L^p(G/H).$$

Then by (2.5),

$$\|\Theta_\psi\| \leq 2\left(\frac{\rho(e)}{c_\zeta}\right)^{1/p}\|\psi\|_p.$$

Moreover Θ_ψ is equal to the zero operator on $\mathcal{H}$ for all $\psi \in \mathfrak{A}$. Thus,

$$\Theta_\psi : L^p(G/H) \to B(\mathcal{H})$$

is a bounded linear operator which is equal to zero on the dense subspace $\mathfrak{A}$ of $L^p(G/H)$. Therefore $L_{\psi,\zeta} = P_{\psi,\zeta}$ for all $\psi \in L^p(G/H)$. □

Now, it is proven that the localization operator is compact. To this end, it is shown that the locazation operator $L_{\psi,\zeta}$ is in trace class. That is $L_{\psi,\zeta}$ has an orthonormal eigenbasis $\{e_n\}$ with eigenvalues $\{\lambda_n\}$ such that $\sum \lambda_n < \infty$. In this case we set $\mathrm{tr}(L_{\psi,\zeta}) = \sum \lambda_n$. Note that every trace class operator is compact (see more details about trace class in [3, 4]).

Theorem 2.7. *Let $\psi \in L^p(G/H)$, for $1 \leq p < \infty$. Then localization operator $L_{\psi,\zeta}$ on the Hilbert space $\mathcal{H}$ is a compact.*

Proof. Let $\mathfrak{A}$ be the set of all simple functions ψ on G/H such that $\mu(\{gH \in G/H, \quad \psi(gH) \neq 0\}) < \infty$. Let $\{\psi_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathfrak{A}$ such that $\psi_n \to \psi$ in $L^p(G/H)$ as $n \to \infty$. then

$$\|L_{\psi_n,\zeta} - L_{\psi,\zeta}\| \leq \left(\frac{\rho(e)}{c_\zeta}\right)^{1/p} \|\psi_n - \psi\|_p \to 0,$$

as $n \to \infty$. But $\{L_{\psi_n,\zeta}\}$ is in trace class. Indeed, if ψ_n is positive, then $\langle L_{\psi,\zeta}x, x\rangle$ is positive and for an orthonormal basis $\{\zeta_k\}_{k=1}^{\infty}$ we have,

$$\begin{aligned} tr(L_{\psi_n,\zeta}) &= \sum_{k=1}^{\infty} \langle L_{\psi_n,\zeta}\zeta_k, \zeta_k\rangle \\ &= \sum_{k=1}^{\infty} \frac{1}{c_\zeta} \int_{G/H} \frac{\rho(e)}{\rho(g)} \psi_n(gH) |\langle \zeta_k, \varpi(gH)\zeta\rangle|^2 d\mu(gH) \\ &= \sum_{k=1}^{\infty} \frac{1}{c_\zeta} \int_{G/H} \frac{\tau(eH)}{\tau(gH)} \psi_n(gH) |\langle \zeta_k, \varpi(gH)\zeta\rangle|^2 \tau(gH) d\acute{\mu}(gH) \\ &= \frac{1}{c_\zeta} \int_{G/H} \rho(e)\psi_n(gH) \sum_{k=1}^{\infty} |\langle \zeta_k, \varpi(gH)\zeta\rangle|^2 d\acute{\mu}(gH) \\ &= \frac{\rho(e)}{c_\zeta} \int_{G/H} \psi_n(gH) d\acute{\mu}(gH) \\ &= \frac{\rho(e)}{c_\zeta} \|\psi_n\|_1, \end{aligned}$$

where $\acute{\mu}$ is a G-invariant measure on G/H which arises from $\acute{\rho}(g) = 1$. So $\{L_{\psi_n,\zeta}\}$ is a sequence of compact operators and hence $L_{\psi,\zeta}$ is compact operator. □

When $H = \{e\}$, the localization operators on Hilbert space $\mathcal{H}$ for all $F \in L^p(G), 1 \leq P \leq \infty$ and $x, y \in \mathcal{H}$ are defined as follows:

$$\langle L_{F,\zeta}x, y\rangle = \frac{1}{c_\zeta} \int_G F(g) \langle x, \pi(g)\zeta\rangle \langle \pi(g)\zeta, y\rangle d\lambda(g),$$

where π is a square integrable representation of G with admissible wavelet ζ and λ is the Haar measure on G (see [1, 5]).
We conclude with an example concerning localization operators on some homogeneous spaces.

Example 2.8. Consider the Euclidean group $G = SO(2) \times_\tau \mathbb{R}^2$ with group operations

$$(R_1, p_1).(R_2, p_2) = (R_1R_2, R_1p_2 + p_1), \quad (R, p)^{-1} = (R^{-1}, -R^{-1}p),$$

and $\mathcal{H} = L^2(S^1) \simeq L^2[-\pi, \pi]$. In ([3], Section 6.9) has been shown that Euclidean group G is semidirect product with the action $\tau_R : \mathbb{R}^2 \to \mathbb{R}^2$, such that $\tau_R(p) = Rp$. In this setting any $R \in SO(2)$ and $s \in S^1$ are given explicitly by

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$s = \begin{pmatrix} \sin\gamma \\ \cos\gamma \end{pmatrix}.$$

The representation ϖ of G/H, in which $H = \{(0,0,p_2) \in G\}$, is defined as

$$\varpi(\theta, p_1)\zeta(\gamma) = e^{ip_1 \sin\gamma}\zeta(\gamma - \theta),$$

for all $(\theta, p_1) \in G/H, \zeta \in L^2(S^1)$. For an admissible wavelet $\zeta \in L^2(S^1)$ and $\psi \in L^p(G/H), 1 \leq p \leq \infty$, the localization operator $L_{\psi,\zeta} : L^2(S^1) \to L^2(S^1)$ is given by

$$\langle L_{\psi,\zeta} f, g\rangle = \frac{1}{c_\zeta}\int_0^{2\pi}\int_{-\infty}^{\infty} \psi(\theta, p_1)\langle f, \zeta_{\theta,p_1}\rangle\langle \zeta_{\theta,p_1}, g\rangle d\theta dp_1,$$

where $\zeta_{\theta,p_1}(\gamma) = e^{ip_1 \sin\gamma}\zeta(\gamma - \theta)$.

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