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NONSTANDARD FINITE DIFFERENCE SCHEMES FOR DIFFERENTIAL EQUATIONS

MOHAMMAD MEHDIZADEH KHALSARAEI[1*] AND FAYYAZ KHODADOSTI[2]

Abstract. In this paper, the reorganization of the denominator of the discrete derivative and nonlocal approximation of nonlinear terms are used in the design of nonstandard finite difference schemes (NSFDs). Numerical examples confirming the efficiency of schemes, for some differential equations are provided. In order to illustrate the accuracy of the new NSFDs, the numerical results are compared with standard methods.

1. Introduction

Analysis has been the most dominant part of mathematics, and also differential equation is the heart of analysis. A major difficulty in the study of differential equations is, in general, the lack of exact analytical solutions that cannot be solved by a straight forward formula. One way to proceed is to use numerical integration techniques to obtain useful information on the possible solution behaviors. A popular and important one is based on the use of finite differences to construct discrete models of the differential equations of interest. Almost all of the standard procedures yields schemes which are convergent with restriction on the stepsize. The preservation of the qualitative properties of the considered differential equation with respect to these schemes is of great interest in finite difference methods of solving differential equations. The major consequence of this result is that such scheme does not allow numerical instabilities to occur. Mickens [7] gave a novel approach for developing new finite difference schemes for differential equations. His approach

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* Corresponding author.

consists of the renormalization of denominator of the discrete derivative and the nonlocal approximation of nonlinear terms in the date.

Now a brief summery of the NSFDs for some ordinary differential equations (ODEs) and partial differential equations (PDEs) is given. The numerical solution, using NSFD schemes, of initial value problems for first-order and systems of ODEs were dealt with, which can be written in the form

$$\frac{d}{dt}u(t) = F(u(t)), \quad (t \geq 0), \quad u(0) = u_0, \tag{1.1}$$

where u may be a single function or a vector of functions of length k mapping $[t_0, T) \to \mathcal{C}^k$ and the corresponding F a single function or a vector of functions of length k mapping $([t_0, T), \mathcal{C}^k) \to \mathcal{C}^k$. Discretization of the continuous differential equation, or beginning instead with a difference equation, we define $t_n = t_0 + n\Delta t$, where Δt is a positive step size, and says that the discretized version of the function u at time t_n is

$$u_n \approx u(t_n). \tag{1.2}$$

Then the discretized version of Eq. (1) becomes

$$\mathcal{D}_{\Delta t} u_n = \mathcal{F}_n(F, u_n), \tag{1.3}$$

where $\mathcal{D}_{\Delta t} u_n$ represents the discretized version of $\frac{d}{dt}u(t)$ and $\mathcal{F}_n(F, u_n)$ approximates $F(u(t_n))$ at time t_n.

2. Construction of numerical schemes

For the construction of the numerical schemes the rules 1 and 2 of the non-standard modeling rules will be used as in Mickens [7].

Rule 1:

The denominator function for the discrete derivatives must be expressed in terms of more complicated function φ of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of Δt in the denominator with the condition that

$$\varphi(\Delta t) = \Delta t + O(\Delta t^2) \quad as \quad 0 < \Delta t \to 0. \tag{2.1}$$

Rule 2:

The non-linear terms must in general be modelled (approximated) non-locally on the computational grid or lattice in many different ways, for instance, the non-linear terms u^2 and u^3 can be modelled as follows as in Anguelov and Lubuma [6]:

$$u^2 \approx a u_k^2 + b u_k u_{k+1}, \quad a + b = 1, \quad a, b \in R, \tag{2.2}$$

TABLE 1. Errors for $u\prime = 1 + u(t)^2, \quad u(0) = 1.$

h	$E(t=1)$	E_{max}	CPU
$\frac{1}{32}$	7.18×10^{-3}	13.9181	0.000001
$\frac{1}{64}$	1.79×10^{-3}	3.6385	0.000001
$\frac{1}{128}$	4.48×10^{-4}	1.2008	0.000001
$\frac{1}{256}$	1.12×10^{-4}	67.1306	0.000001
$\frac{1}{512}$	$2.80\ \times10^{-5}$	16.9879	0.000001
$\frac{1}{1024}$	7.00×10^{-6}	4.26056	0.000001
$\frac{1}{2048}$	1.75×10^{-6}	1.06579	0.000001

$$u^3 \approx au_k^3 + (1-a)u_k^2 u_{k+1}, \quad a \in R. \tag{2.3}$$

In general for any linear combination of the expressions listed above with the sum of the coefficient equal to 1 approximates u^2 or u^3 the error being of order $O(\Delta t)$ for sufficiently smooth u. In this way the terms may be approximated by an expression which contains certain number of free parameters. To illustrate the efficiency of the nonstandard finite difference method, some examples are presented.

2.1. **Application to ODEs.** To verify the desired stability properties of the NSFDs, several initial value-problems of different nature have been integrated. All the numerical experiments were performed in a MATLAB compiler on personal computer.

A singular IVP

As first example we consider first-order ordinary differential equation:

$$\frac{du}{dt} = 1 + u^2, \quad u(0) = 1, \quad t \in [0, b], \tag{2.4}$$

with $b > \frac{\pi}{4}$. Theoretical solution is $u(t) = \tan(t + \frac{\pi}{4})$ which has a simple pole at $t = \frac{\pi}{4}$. The solution becomes unbounded in the neighborhood of the singularity at $t = \pi/4 \approx 0.785398163397448$. Finite difference methods based on local polynomial interpolation behave poorly if the solution has singularities. Using (2.2) for the approximation of the nonlinear term we have:

$$\frac{u_{k+1} - u_k}{\Delta t} = 1 + u_k u_{k+1}. \tag{2.5}$$

In Table 1 the results for our method is presented where we may observe the ability of the method to cross the singularity. In Figure 1 the numerical solution after joining the point (t_n, u_n) is shown for $\Delta t = \frac{1}{128}$.

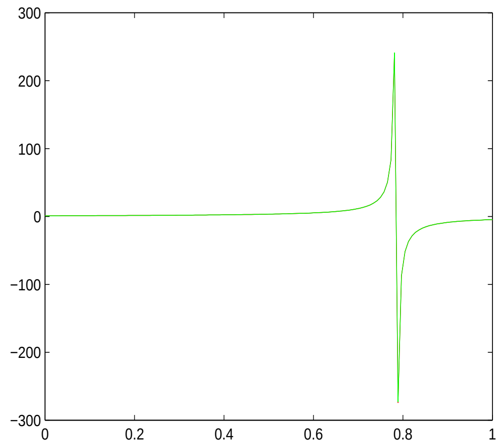


FIGURE 1. Numerical solution for equation (2.4) with $\Delta t = \frac{1}{128}$.

A singularly perturbed IVP

Now we consider a singularly perturbed IVP given by [9]:

$$u' = -2\kappa u^2(t), \quad u(0) = 10, \tag{2.6}$$

which has the exact solution

$$u_e(t) = \frac{10}{1 + 20\kappa t}.$$

Here, consider $\kappa = 500$ so that the solution drops quickly from its initial value of 10 to very small values. In fact, this problem exhibits an initial layer, the thickness of which is on the order of $\epsilon = \frac{1}{k}$. Table 2 shows the error

$$E_{\max} = \max_{0 \leq x \leq 1} |u_{(t_n)} - u_n|$$

obtained with our NSFD method

$$u_{k+1} = u_k - h(2\kappa u_k u_{k+1}),$$

when the integration is performed on the interval $[0, 1]$. Comparing with the error obtained from full linearization (FL) method [9] it is observed that the new method performs very well.

Predator-prey model

The general Rosenzweig-MacArthur predator-prey model [4, p. 182]

TABLE 2. Errors for $u' = -2\kappa u(t)^2, u(0) = 10, u(0) = 10, \kappa = 500$.

h	Error for FL	Error for NSFD	CPU
$\frac{1}{16}$	4.984026	4.3e-019	0.000001
$\frac{1}{32}$	4.968102	8.7e-019	0.000001
$\frac{1}{64}$	4.936407	8.7e-019	0.000001
$\frac{1}{128}$	4.877368	8.7e-019	0.000001

with a logistic growth of the prey population has the following form:

$$\begin{aligned} \frac{du}{dt} &= bu(1-u) - ag(u)uv, \quad u(t_0) = u_0 \geq 0, \\ \frac{dv}{dt} &= g(u)uv - dv, \quad v(t_0) = v_0 \geq 0, \end{aligned} \tag{2.7}$$

where u and v represent the prey and predator population sizes, respectively, $b > 0$ represents the intrinsic growth rate of the prey, $a > 0$ stands for the capturing rate and $d > 0$ is the predator death rate. In (2.7) it is reasonable to assume

$$g(u) \geq 0, \quad g'(u) \leq 0, \quad [ug(u)]' \geq 0, \tag{2.8}$$

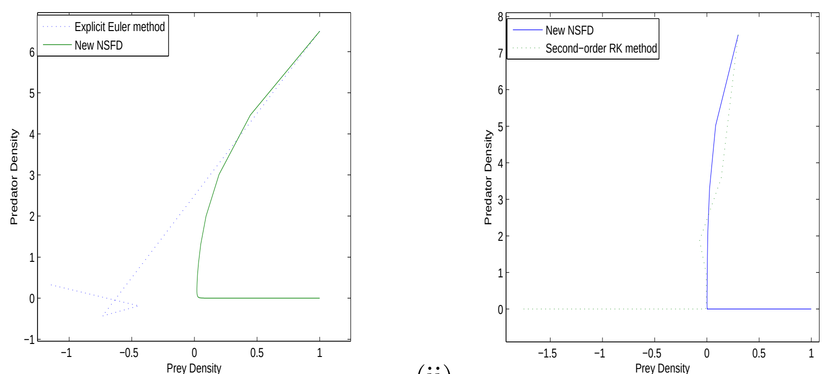
and that $ug(u)$ is bounded as $u \to \infty$. For more details see [4]. The new non-standard scheme to solve this system is to take:

$$\begin{aligned} \frac{u_{k+1} - u_k}{\varphi(\Delta t)} &= 2bu_k - (bu_k + b + ag(u_k)v_k)u_{k+1} \\ \frac{v_{k+1} - v_k}{\varphi(\Delta t)} &= (d + g(u_k)u_k)v_k - 2dv_{k+1}. \end{aligned} \tag{2.9}$$

Where the function $\varphi(\Delta t)$ satisfies the condition in (4). To illustrate the advantages of the designed NSFD, we consider the system (2.7) with a Holling-type II predator functional response of the form $ug(u) = u/(c+u)$ [5], which satisfies (2.8). System (2.7) becomes

$$\begin{aligned} \frac{du}{dt} &= bu(1-u) - \frac{auv}{c+u} \\ \frac{dv}{dt} &= \frac{uv}{c+u} - dv. \end{aligned} \tag{2.10}$$

First system (2.10) is examined in the case when the constants are $a = 2.0, b = 1.0, c = 0.5$ and $d = 0.6$, i.e., $g(1) = \frac{2}{3} < d$. In Figure 2 numerical approximations of the solution of system (2.10) have been shown with the proposed NSFD and Euler method . The NSFD preserves the stability, while the approximation obtained by the standard methods diverges. Similar behavior is observed when the standard second-order Runge-Kutta method is used to numerically solve system (2.10).



(i) (ii)

FIGURE 2. (i)$h = 0.2, u_0 = 1, v_0 = 6.5$; (ii) $h = 0.2, u_0 = 1, v_0 = 6.5$;

2.2. **Application to partial differential.** We now present, without giving the technical details, the NSFDs for the partial differential equations. To illustrate efficiency of the nonstandard finite difference method, we have integrated several partial differential equations.

Heat equation

We are seeking a numerical solution of $\partial u/\partial t = \partial^2 u/\partial x^2$ which satisfies

$$u(x, 0) = \sin(\pi x), \quad 0 \leq x \leq 1, \quad u(0, t) = u(1, t) = 0, \quad t \geq 0. \tag{2.11}$$

The solution of which gives the temperature u at a distance x from one end of a thin uniform rod after a time t. The analytical solution of this partial differential equation satisfying (2.11) is

$$u(x, t) = e^{-\pi^2 t} \sin(\pi x).$$

We rewrite the component of heat equation in the form

$$\frac{u_i^{j+1} - u_i^j}{\varphi(k)} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\psi(h)}, \tag{2.12}$$

where, φ and ψ satisfy the conditions in (2.1) and u_i^j represents the value of u at the mesh point (ih, jk), h and k being the space and time step, respectively.

The corresponding standard form [3] is

$$\frac{u_i^{j+1} - u_i^j}{k} = \frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{h^2}. \tag{2.13}$$

t	x	$Error for$ (2.13)	$Error\ for\ NSFD$
0.0025	0.1	3.00000e-005	2.09999e-005
0.0050	0.1	5.89999e-005	4.09999e-005
0.0075	0.1	8.79999e-005	6.09999e-005
0.0100	0.1	1.12999e-004	7.80000e-005
0.0125	0.1	1.39000e-004	9.59999e-005
0.0150	0.1	1.63000e-004	1.13000e-004
0.0175	0.1	1.85999e-004	1.28999e-004
0.0200	0.1	2.07000e-004	1.43000e-004
0.0225	0.1	2.27000e-004	1.58000e-004
0.0250	0.1	2.45999e-004	1.70000e-004

TABLE 3. Absolute errors for NSFD and Euler explicit method with $h = 0.1$, $k = 0.0025$.

Table 3 shows the errors obtained from the proposed NSFD with $\varphi(k) = 1-\exp(-k)$, $\psi(h) = h^2$ and $h = 0.2$, $k = 0.05$. Compared with numerical solution of corresponding standard one, it is observed that the NSFD performs well.

Another non-standard scheme is

$$\frac{u_i^{j+1} - u_i^j}{\varphi(k)} = \frac{1}{2}\Big[\frac{u_{i+1}^j - 2u_i^j + u_{i-1}^j}{\psi(h)} + \frac{u_{i+1}^{j+1} - 2u_i^{j+1} + u_{i-1}^{j+1}}{\psi(h)}\Big], \tag{2.14}$$

where, $\varphi(k) = k$, $\psi(h) = 4\sin^2(\frac{h^2}{2})$. We have employed the method of renormalization of the denominator function for Crank-Nicolson [3] scheme. The results presented in Table 4 indicate that the absolute errors of the NSFD are smaller than that for Crank-Nicolson method with $h = 0.2$, $k = 0.05$.

3. Conclusion

Family of schemes for some differential equations have been conducted. The schemes have been tested numerically in terms of their consistency with the known behavior of the analytic solution of the particular initial value problem. From the numerical experiments it can concluded that the non-standard schemes performed better than the standard schemes.

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t	x	$Error for$ (2.14)	$Error\ for\ NSFD$
0.05	0.2	2.38630e-003	1.77630e-003
0.10	0.2	2.92390e-003	2.17390e-003
0.15	0.2	2.68690e-003	1.99590e-003
0.20	0.2	1.72210e-003	1.62910e-003
0.25	0.2	1.67990e-003	1.24590e-003
0.30	0.2	1.23450e-003	6.86499e-004
0.35	0.2	7.72600e-004	5.12600e-004
0.40	0.2	6.18000e-004	3.70999e-004
0.45	0.2	4.25700e-004	3.14699e-004
0.50	0.2	2.89800e-004	2.13799e-004

TABLE 4. Absolute errors for NSFD and Crank-Nikicolson method with $h = 0.2$, $k = 0.05$.

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[1] DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF MARAGHEH, MARAGHEH, IRAN.
E-mail address: Muhammad.mehdizadeh@gmail.com

[2] DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, UNIVERSITY OF MARAGHEH, MARAGHEH, IRAN.
E-mail address: fayyaz64dr@gmail.com