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SIMULATION STUDY ON COPULAS

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Abstract. There are several theorical results about order statistics and copulas in the literature that have been mentioned also by Nelsen [11]. The present study after reviewing some of these results, relies on simulation technique to investigate the mentioned results about order statistics and copulas. The study concentrates on two well known Archimedean Gumbel and Frank families in the case that marginal functions $F(k)$ and $G(k)$ have different distributions.

1. Introduction

In fact, the structure of dependence between random variables is important. In decision support, proper accounting and modeling of these dependences and correlations are essential in deriving reliable valuations. Copulas are a way of studying scale-free measures of dependence and also are a tool to build families of bivariate distributions with given margins, hence they are of interest to statisticians [4, 12].

A copula is a function $C : [0, 1]^2 \to [0, 1]$ which satisfies:

(a) for every u, v in [0,1], $C(u, 0) = 0 = C(0, v)$ and $C(u, 1) = u$ and $C(1, v) = v$;
(b) for every u_1, u_2, v_1, v_2 in [0,1] such that $u_1 \leq u_2$, and $v_1 \leq v_2$, $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$.

The importance of copulas in statistics is described in Sklar's theorem [13]: Let X and Y be random variables with joint distribution function H and marginal distribution functions F and G, respectively. Then there exists a copula C such that, $H(x, y) = C(F(x), G(y))$, for all x, y in $\mathbb{R}$. If F and G are continuous, then C is unique. Otherwise, the

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copula C is uniquely determined on $\mathrm{Ran}(F) \times \mathrm{Ran}(G)$. Conversely, if C is a copula and F and G are distribution functions then the function H is a joint distribution function with margins F and G. As a result of the Sklar Theorem, copulas link joint distribution functions to their one-dimensional margins.

In the literature we can see wide applications of copulas in several branches of science. Friend et al. [5], Clemen and Reilly [3] and some other people used copulas in finance. Najjari and Ünsal [10] applied copulas in meteorological data. Çelebioğlu [2], used copulas in modeling of students grades. Najjari et al.[9], applied copulas to model river flow rate. Al-Harthy et al. [1], used copulas to model dependence in petroleum decision making. Many other authors worked on copulas like, Genest and MacKay [6, 7], Hua and Joe [8] among them.

Table 1. Frank and Gumbel families with the first derivatives.

Family	$C_\theta(u,v) =$	$c_u = \frac{\partial C}{\partial u}(u,v)$
Frank	$-\frac{1}{\theta} log\left(1 + \frac{(e^{-\theta u}-1)(e^{-\theta v}-1)}{e^{-\theta}-1}\right)$	$\frac{e^{\theta}(-1+e^{\theta v})}{-e^{\theta}+e^{(\theta+\theta u)}-e^{(\theta u+\theta v)}+e^{(\theta+\theta v)}}$
Gumbel	$\exp\left(-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/\theta}\right)$	$\frac{-log(u)^{(-1+\theta)}[(-log(u))^{\theta}+(-log(v))^{\theta}]^{(-1+1/\theta)}}{u\ exp([(-log(u))^{\theta}+(-log(v))^{\theta}]^{1/\theta})}$

In the Frank Family, $\theta \in (-\infty, \infty) - \{0\}$ and for the Gumbel family $\theta \in [1, \infty)$.

One of the most important classes of copulas is known as Archimedean copulas (AC). These copulas are very easy to construct, many parametric families belong to this class and they have a great variety of different dependence structures. In addition, the Archimedean representation allows us to reduce the study of a multivariate copula to a single univariate function. Basic properties of AC are presented below. More information could be found in Nelsen [11].

Let φ be a continuous, strictly decreasing function from I to $[0, \infty]$ such that $\varphi(1) = 0$. The pseudo-inverse of φ is the function $\varphi^{[-1]}$ given by

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{(-1)}(t), & 0 \le t \le \varphi(0) \\ 0, & \varphi(0) \le t \le \infty. \end{cases} \tag{1.1}$$

Copulas of the form $C(u,v) = \varphi^{[-1]}(\varphi(u) + \varphi(v))$, for every u, v in I are called AC and the function φ is called a generator of the copula. If $\varphi[0] = \infty$ we say that φ is a strict generator. In this case, $\varphi^{[-1]} = \varphi^{(-1)}$ and $C(u,v) = \varphi^{(-1)}(\varphi(u) + \varphi(v))$ is said to be a strict Archimedean copula. The study used two well known Archimedean families: Gumbel and Frank families.

There are several theorical results about order statistics and copulas in the literature that has also been mentioned by Nelsen [11]. Some of these results are shown in the next section. The aim of the present study is to investigate accuracy of the mentioned results about order statistics and copulas by simulations.

The paper is constructed as follows: in Section 2 the theorical results about order statistics and copulas are discussed. Section 3, consists of simulation study. Section 4 summarizes the conclusion of the study.

2. Order statistics and copulas

Let X and Y be continuous random variables with copula C and univariate distribution functions F and G, respectively. Let $\min(X, Y)$ and $\max(X, Y)$ be order statistics of random variables X and Y. It has been shown by Nelsen [11] that,

$$P[\min(X,Y) \le k] = F(k) + G(k) - C(F(k), G(k)) \tag{2.1}$$

and

$$P[\max(X,Y) \le k] = C(F(k), G(k)). \tag{2.2}$$

In the case $F = G$, by (2.1) and (2.2) we have

$$P[\min(X,Y) \le k] = 2F(k) - \delta_C(F(k)) \tag{2.3}$$

and

$$P[\max(X,Y) \le k] = \delta_C(F(k)). \tag{2.4}$$

Also the following relations had been shown by Nelsen [11],

$$\begin{aligned} \max(F(k), G(k)) &\le P[\min(X,Y) \le k] \\ &\le \min(F(k) + G(k),\ 1) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \max(F(k) + G(k) - 1,\ 0) &\le P[\max(X,Y) \le k] \\ &\le \min(F(k), G(k)). \end{aligned} \tag{2.6}$$

The study tries to show the accuracy of the relations (2.1), (2.2) and also (2.5), (2.6) by simulations. The study concentrated on the marginal functions F and G that have different distributions. The simulation in our study is based on the mentioned algorithm by Nelsen [11]:

- Generate two independent uniform $(0, 1)$ variates u and t.
- Set $v = c_u^{(-1)}(t)$, where $c_u^{(-1)}$ denotes a quasi-inverse of c_u.
- Set $x = F^{(-1)}(u)$ and $y = G^{(-1)}(v)$.

In using this algorithm, the study applied Frank and Gumbel families (see Table 1). The derivative of Frank family (c_u) is invertible but inverse of c_u in Gumbel family dosen't have a closed form, so in the related calculations algebraically and numerically methods are used.

TABLE 2. Intervals details of the simulations by Frank family for $\max(X, Y)$.

	K	O	$F(k)$	$G(k)$	$C(F(k), G(k))$	E	F_L	F_U
1	-2.3628	2	0.0091	0.0474	0.0006	0.3821	0	0.0091
2	-2.1262	0	0.0167	0.0664	0.0015	0.9336	0	0.0167
3	-1.8896	1	0.0294	0.0907	0.0036	2.0991	0	0.0294
4	-1.653	5	0.0492	0.1212	0.008	4.3409	0	0.0492
5	-1.4164	7	0.0783	0.1583	0.0162	8.2568	0	0.0783
6	-1.1798	15	0.119	0.202	0.0307	14.456	0	0.119
7	-0.9432	28	0.1727	0.2523	0.054	23.333	0	0.1727
8	-0.7066	32	0.2398	0.3086	0.0888	34.815	0	0.2398
9	-0.47	34	0.319	0.3697	0.137	48.194	0	0.319
10	-0.2334	71	0.4076	0.4343	0.1992	62.159	0	0.4076
11	0.0032	68	0.5011	0.5008	0.2742	75.029	0.0019	0.5008
12	0.2398	94	0.5945	0.5672	0.3593	85.11	0.1617	0.5672
13	0.4764	97	0.6829	0.6317	0.4504	91.057	0.3146	0.6317
14	0.713	99	0.7619	0.6928	0.5425	92.14	0.4547	0.6928
15	0.9496	98	0.8287	0.7489	0.6309	88.373	0.5776	0.7489
16	1.1862	67	0.8821	0.799	0.7114	80.486	0.6811	0.799
17	1.4228	78	0.9225	0.8427	0.7811	69.744	0.7652	0.8427
18	1.6594	53	0.9514	0.8796	0.8388	57.651	0.831	0.8796
19	1.896	43	0.971	0.9099	0.8844	45.62	0.8809	0.9099
20	2.1326	30	0.9835	0.9341	0.9191	34.712	0.9176	0.9341
21	2.3692	24	0.9911	0.953	0.9446	25.524	0.9441	0.953
22	2.6058	15	0.9954	0.9673	0.9629	18.228	0.9627	0.9673
23	2.8424	11	0.9978	0.9777	0.9756	12.698	0.9755	0.9777
24	3.079	12	0.999	0.9852	0.9842	8.6558	0.9842	0.9852
25	3.3156	7	0.9995	0.9905	0.99	5.7841	0.99	0.9905
26	3.5522	6	0.9998	0.994	0.9938	3.7906	0.9938	0.994
27	3.7888	1	0.9999	0.9963	0.9962	2.4349	0.9962	0.9963
28	4.0254	1	1	0.9978	0.9978	1.5313	0.9978	0.9978
29	4.262	1	1	0.9987	0.9987	0.9415	0.9987	0.9987

$F_L = \max(F(k) + G(k) - 1, 0)$ and $F_U = \min(F(k), G(k))$.

The main structure of the study is on the following questions: as $\min(X, Y)$ and $\max(X, Y)$ are two different order statistics, so could these random variables be generated by a copula or not?

In inquiring the answer, the study used mentioned algorithm above to generate (u, v) from Frank (or Gumbel) family and then by marginal distributions we get (x, y). Then we tested the following hypotheses:

$$H_0: \quad \min(X, Y) \quad \text{fits to Frank (or Gumbel) family} \tag{2.7}$$

and also we test same hypotheses for $\max(X, Y)$ was tested.

Namely, the study investigated the accuracy of the relations (2.1), (2.2) by simulation. During the test we let marginal functions F and G have distributed as $N(0, 1)$ and $N(0, 2)$ respectively. In the next section, the simulated data by 1000 iterations will be discussed. Matlab software have been used, for conculations.

3. Simulation study

In this section Frank and Gumbel families' parameter values are supposed to be one. We recall that $F \sim N(0, 1)$, $G \sim N(0, 2)$ and also the Monte-Carlo method have been used in simulations for $n = 1000$

TABLE 3. Intervals details of the simulations by Frank family for $\min(X, Y)$.

	K	O	$F(k)$	$G(k)$	$C(F(k), G(k))$	E	F_L^*	F_U^*
1	-4.276	1	0	0.0012	0	0.5167	0.0012	0.0012
2	-4.0559	0	0	0.0021	0	0.8322	0.0021	0.0021
3	-3.8358	4	0.0001	0.0033	0	1.3127	0.0033	0.0034
4	-3.6157	3	0.0002	0.0053	0	2.03	0.0053	0.0055
5	-3.3956	2	0.0003	0.0082	0	3.0813	0.0082	0.0085
6	-3.1755	5	0.0007	0.0124	0	4.5952	0.0124	0.0131
7	-2.9554	4	0.0016	0.0183	0	6.7377	0.0183	0.0199
8	-2.7353	10	0.0031	0.0266	0.0001	9.7133	0.0266	0.0297
9	-2.5152	15	0.006	0.0377	0.0003	13.756	0.0377	0.0437
10	-2.2951	23	0.0109	0.0523	0.0008	19.1	0.0523	0.0632
11	-2.075	31	0.019	0.0712	0.0019	25.925	0.0712	0.0902
12	-1.8549	36	0.0318	0.0949	0.0041	34.27	0.0949	0.1267
13	-1.6348	47	0.0511	0.1239	0.0085	43.939	0.1239	0.175
14	-1.4147	56	0.0787	0.1587	0.0163	54.424	0.1587	0.2374
15	-1.1946	71	0.1162	0.1993	0.0296	64.895	0.1993	0.3155
16	-0.9745	78	0.1651	0.2455	0.0504	74.28	0.2455	0.4106
17	-0.7544	74	0.2255	0.297	0.0809	81.444	0.297	0.5225
18	-0.5343	83	0.2968	0.353	0.1228	85.402	0.353	0.6498
19	-0.3142	95	0.377	0.4123	0.1767	85.519	0.4123	0.7893
20	-0.0941	81	0.4628	0.4737	0.2424	81.647	0.4737	0.9365
21	0.126	70	0.5505	0.5357	0.3179	74.164	0.5505	1
22	0.3461	48	0.6357	0.5969	0.4003	63.924	0.6357	1
23	0.5662	53	0.7147	0.6558	0.4861	52.116	0.7147	1
24	0.7863	40	0.7844	0.7111	0.5711	40.048	0.7844	1
25	1.0064	30	0.8431	0.7619	0.6517	28.903	0.8431	1
26	1.2265	21	0.8902	0.8073	0.7247	19.526	0.8902	1
27	1.4466	6	0.9262	0.847	0.788	12.314	0.9262	1
28	1.6667	4	0.9523	0.8809	0.8408	7.2355	0.9523	1
29	1.8868	9	0.9705	0.9091	0.8832	3.956	0.9705	1

$F_L^* = \max(F(k), G(k))$ and $F_U^* = \min(F(k) + G(k), 1)$.

iteration as follows:

1. In generating random samples:

- Generate two independent uniform $(0, 1)$ variates u and t.
- Set $v = c_u^{(-1)}(t)$, where $c_u^{(-1)}$ denotes a quasi-inverse of c_u.
- Set $x = F^{(-1)}(u)$ and $y = G^{(-1)}(v)$.

2. For $\min(X, Y)$, after sorting the results we let L, U be lower and upper bounds of the sorted date ($\min(X, Y)$) respectively (the same process is applied also for $\max(X, Y)$).

3. By using $\frac{U-L}{29}$ we divide results of the iterations into 29 intervals. So the number of observations that are located on each interval can be determined.

4. Let K be the upper bounds in any interval. By using the mentioned copulas (Frank and Gumbel) and the relations (2.1) and (2.2) cumulative expected frequencies ere obtained (multiplication by 1000). So by subtracting value of i^{th} interval by value of $(i-1)^{\text{th}}$ interval, we get the expected value of the i^{th} interval was obtained (where $i = 2, 3, ..., r$ and r is number of the intervals). Note that, the study connives at inserting the lower and upper bounds of any 29 intervals in Tables 2, 3.

5. We test observed and expected values by using Goodness of fit test (GOF).

The results of the simulation for Frank and Gumbel families are shown in the following two subsections separately.

TABLE 4. Goodness of fit test for Frank copula (left panel $\max(X, Y)$ and right panel $\min(X, Y)$).

	O	E	$\frac{(O_i-E_i)^2}{E_i}$		O	E	$\frac{(O_i-E_i)^2}{E_i}$
$1-2-3$	3	3.4147	0.0504	$1-2-3$	5	2.6616	2.0544
4	5	4.3409	0.1001	4	3	2.03	0.4635
5	7	8.2568	0.1913	5	2	3.0813	0.3795
6	15	14.456	0.0205	6	5	4.5952	0.0357
7	28	23.333	0.9335	7	4	6.7377	1.1124
8	32	34.815	0.2276	8	10	9.7133	0.0085
9	34	48.194	4.1804	9	15	13.756	0.1125
10	71	62.159	1.2575	10	23	19.1	0.7963
11	68	75.029	0.6585	11	31	25.925	0.9935
12	94	85.11	0.9286	12	36	34.27	0.0873
13	97	91.057	0.3879	13	47	43.939	0.2132
14	99	92.14	0.5107	14	56	54.424	0.0456
15	98	88.373	1.0487	15	71	64.895	0.5743
16	67	80.486	2.2597	16	78	74.28	0.1863
17	78	69.744	0.9773	17	74	81.444	0.6804
18	53	57.651	0.3752	18	83	85.402	0.0676
19	43	45.62	0.1505	19	95	85.519	1.0511
20	30	34.712	0.6396	20	81	81.647	0.0051
21	24	25.524	0.091	21	70	74.164	0.2338
22	15	18.228	0.5716	22	48	63.924	3.9668
23	11	12.698	0.2271	23	53	52.116	0.015
24	12	8.6558	1.292	24	40	40.048	0.0001
25	7	5.7841	0.2556	25	30	28.903	0.0416
26	6	3.7906	1.2878	26	21	19.526	0.1113
27-28-29	3	4.9077	0.7415	27	6	12.314	3.2375
				28	4	7.2355	1.4468
				29	9	3.956	6.4312
			χ^2=19.3645				χ^2=24.3512

3.1. **Simulation results for Frank family.** By using Frank family with parameter value equal to 1, we simulated 1000 iterations as the discussed process in Section 3.

In Table 2 for the defined 29 intervals, observed and expected frequencies, marginal functions F and G, copula value and finally Fréchet-Hoeffding bounds are given for $\max(X, Y)$. Table 3, consists of the same information as the Table 2 for $\min(X, Y)$. Consider that in all tables O means observations and E means expected values.

Table 4 consists of Goodness-of-fit test for $\min(X, Y)$ and $\max(X, Y)$ random variables. The question is can they be generated by the Frank copula or not? The left panel in the Table 4 is related with Goodness-of-fit test for $\max(X, Y)$, and the right panel is related with Goodness-of-fit test for $\min(X, Y)$. In the left panel, since more than 20 percent of expected frequencies are less than 5, therefore rows 1,2,3 and also 27,28,29 are joined together respectively, so $df = 24$ and by the 95 percent confidence the critical point is 36.415 and the calculated test statistic value is $\chi^2 = 19.3645$, hence H_0 is not rejected. Namely, $\max(X, Y)$ fits to Frank family.

In the right panel of Table 4, $H_0 :\ \min(X, Y)$ is tested fits to Frank family. As a similar story in the left panel, since more than 20 percent of expected frequencies are less than 5, therefore rows 1, 2, 3 are joined, so $df = 26$ and by the 95 percent confidence the critical point is 38.885 and the calculated test statistic value is $\chi^2 = 24.3512$, hence H_0 is not rejected. Namely, $\min(X, Y)$ also fits to Frank family.

It's worth mentioning that, Table 2 and Table 3 showed that $\min(X, Y)$ and $\max(X, Y)$ satisfy on the Fréchet-Hoeffding bounds, as the relations (2.5) and (2.6). We summarize that both of the $\min(X, Y)$ and $\max(X, Y)$ have the same result for Frank family and they satisfy on (2.1) and (2.2).

3.2. **Simulation results for Gumbel family.** As there is not a closed form in the inverse of the derivative of the Gumbel family, we used numerical methods. We recall that we simulate 1000 iterations as the discussed process in Section 3 by using Gumbel family with parameter value equal to 1.

In this section, the Tables which consists of details about $\max(X, Y)$, $\min(X, Y)$, marginal functions F and G, etc were eliminated. We have summarized results in Table 5.

Indeed Table 5 consists of Goodness-of-fit test for $\min(X, Y)$ and $\max(X, Y)$ random variables which can be generated by the Gumbel copula or not? The left panel in the Table 5 is related with Goodness-of-fit test for $\max(X, Y)$, and the right panel is related with Goodness-of-fit test for $\min(X, Y)$. In the left panel, since more than 20 percent of expected frequencies are less than 5, therefore rows 26-27-28 and 29 are joined together, so $df = 25$ and by the 95 percent confidence the critical point is 37.652 and the calculated test statistic value is $\chi^2 = 15.3003$, the H_0 is not rejected. Namely, $\max(X, Y)$ fits to Gumbel family.

In the right panel of Table 5 $H_0 :\ \min(X, Y)$ was tested fits to Gumbel family. As a similar story in the left panel, since more than 20 percent of expected frequencies are less than 5, therefore rows 27-28-29 are joined together, so $df = 26$ and by the 95 percent confidence the critical point is 38.885 and the calculated test statistic value is $\chi^2 = 18.2053$, hence we can not reject H_0. Namely, $\min(X, Y)$ also fits to Gumbel family.

Table 5. Goodness of fit test for Gumbel copula (left panel max(X, Y) and right panel min(X, Y)).

	O	E	$\frac{(O_i-E_i)^2}{E_i}$		O	E	$\frac{(O_i-E_i)^2}{E_i}$
1	2	1.5971	0.1016	1	1	1.8069	0.3603
2	6	4.4379	0.5498	2	0	5.1586	5.1586
3	7	10.1970	1.0023	3	1	1.9240	0.4438
4	19	19.8390	0.0355	4	3	2.2066	0.2853
5	32	33.3850	0.0575	5	1	1.1250	0.0139
6	54	49.5390	0.4017	6	3	4.8100	0.6811
7	68	65.9310	0.0649	7	3	4.3920	0.4412
8	70	79.8910	1.2246	8	17	18.9900	0.2085
9	89	89.3040	0.0010	9	23	32.6810	2.8678
10	94	93.1400	0.0079	10	59	58.4670	0.0049
11	104	91.5210	1.7015	11	52	49.1950	0.1599
12	77	85.4320	0.8322	12	68	58.4660	1.5547
13	68	76.2960	0.9021	13	60	67.5410	0.8420
14	66	65.5790	0.0027	14	71	75.3360	0.2496
15	52	54.5330	0.1177	15	98	95.5940	0.0606
16	41	44.0670	0.2135	16	92	82.2280	1.1613
17	34	34.7390	0.0157	17	81	79.6960	0.0213
18	27	26.8080	0.0014	18	87	73.2240	2.5917
19	27	20.3110	2.2029	19	77	76.7580	0.0008
20	21	15.1430	2.2654	20	53	52.6670	0.0021
21	9	11.1310	0.4080	21	46	41.3560	0.5215
22	9	8.0750	0.1060	22	35	35.9520	0.0252
23	7	5.7853	0.2550	23	25	22.1470	0.3675
24	5	4.0932	0.2009	24	17	16.1970	0.0398
25	5	2.8585	1.6043	25	9	10.0320	0.1062
26-27-28-29	7	4.7860	1.0242	26	6	6.3887	0.0236
				27-28-29	11	10.6396	0.0122
			χ^2=15.3003				χ^2=18.2053

4. Conclusion

The main structure of this study is on the following questions: as min(X, Y) and max(X, Y) are two different order statistics, so can these random variables be generated by a copula or not?

In inquiring the answer, by generating (u, v) from Frank (or Gumbel) family and then by marginal distributions (x, y) is obtained. Then we test the following hypotheses were tested.

$$H_0 : \ \min(X, Y) \ \text{ fits to Frank (or Gumbel) family} \tag{4.1}$$

and also same hypotheses for max(X, Y). Namely, accuracy of the relations (2.1), (2.2) by simulations was investigated and it is seen that there is not an evidence on rejection of the mentioned hypothesis.

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