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FIXED POINT THEOREMS FOR α-CONTRACTIVE MAPPINGS

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Abstract. In this paper, we prove the existence of the common fixed point with different conditions for $\alpha-\psi$-contractive mappings. And, we generalize the weakly Zamfirescu maps into the modified weakly Zamfirescu maps.

1. Introduction

The Banach contraction principle [3] is the simplest and one of the most versatile elementary results in fixed point theory. B. Samet, C. Vetro and P. Vetro introduced a new concept of $\alpha-\psi$-contractive type mappings and established various fixed point theorems for such mappings in complete metric spaces (see [7]). We prove the existence of the common fixed point for two mappings with different conditions. The concept of contractive maps was generalized by Dugundji and Granas [5]. The class of mappings was introduced by Kannan [6] in 1968. In 2010, this concept has been generalized by Ariza-Ruiza and Jimenez-Melado [1]. In 1972, Chaterjea [4] considered a type of contractive condition similar to that of Kannan, but independ of it. Ariza-Ruiz in [2] generalized it to the weakly Zamfirescu mappings. A class of maps introduced by Zamfirescu [8], and was generalized by Ariza-Ruiz [2]. In this paper, we study the modified weakly Zamfirescu mappings.

Definition 1.1 ([7])**.** Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. T is said to be α-admissible if

$$x, y \in X, \quad \alpha(x, y) \geq 1 \quad \Rightarrow \quad \alpha(Tx, Ty) \geq 1.$$

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Theorem 1.2 ([7]). *Let (X, d) be a complete metric space and $T : X \to X$ be an $\alpha - \psi-$contractive mapping satisfies*

(i) *T is $\alpha-$admissible;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
(iii) *T is continuous.*

Then, T has a fixed point, that is, there exists $x^* \in X$ such that $Tx^* = x^*$.

2. Main result

Denote with Ψ the family of onto, nondecreasing and continuous functions $\psi : [0, \infty) \to [0, \infty)$ such that

$$\sum_{n=1}^{\infty} \psi^n(t) < \infty,$$

for all $t > 0$, where ψ^n is the n-th iterate of ψ.

Lemma 2.1. *For every function $\psi : [0, \infty) \to [0, \infty)$, the following holds.*
If ψ is nondecreasing, then for all $t > 0$, $\lim_{n\to\infty} \psi^n(t) = 0$ implies that

$$\psi(t) < t.$$

Let (X, d) be a metric space and $T : X \to X$ be a given mapping. Also let $\alpha : X \times X \to [0, \infty)$ and $\psi \in \Psi$ are two functions.

Theorem 2.2. *Let (X, d) be a complete metric space and $S, T : X \to X$ be mappings satisfy the following conditions:*

(i) *for $x, y \in X$, $\alpha(x, y) \geq 1 \;\Rightarrow\; \alpha(Sx, Ty) \geq 1$ or $\alpha(Tx, Sy) \geq 1$;*
(ii) *there exists $x_0 \in X$ such that $\alpha(x_o, Sx_0) \geq 1$;*
(iii) *S and T are continuous;*
(iv) *$\alpha(x, y)d(Sx, Ty) \leq \psi(d(x, y))$, $\quad \alpha(y, x)d(Sx, Ty) \leq \psi(d(x, y))$ for all $x, y \in X$.*

Then, S and T have a common fixed point.

Proof. Let $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. Define the sequences $\{x_{2n+1}\}$ and $\{x_{2n}\}$ in X by

$$x_{2n+1} = Sx_{2n}, \quad x_{2n} = Tx_{2n-1} \qquad n = 1, 2, \ldots,$$
$$\alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq 1 \quad \Rightarrow \quad \alpha(x_1, x_2) = \alpha(Sx_0, TSx_0) \geq 1.$$
$$\alpha(x_1, x_2) \geq 1 \quad \Rightarrow \quad \alpha(x_2, x_3) = \alpha(Tx_1, Sx_2) \geq 1.$$

By induction, we get

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \forall n \in \mathbb{N},$$

$$\begin{aligned} d(x_{2n+1}, x_{2n+2}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \alpha(x_{2n}, x_{2n+1}) d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \psi d(x_{2n}, x_{2n+1}), \end{aligned}$$

$$\begin{aligned} d(x_{2n}, x_{2n+1}) &= d(Tx_{2n-1}, Sx_{2n}) \\ &\leq \alpha(x_{2n-1}, x_{2n}) d(Tx_{2n-1}, Sx_{2n}) \\ &\leq \psi(d(x_{2n-1}, x_{2n})), \end{aligned}$$

$$\begin{aligned} d(x_{2n-1}, x_{2n}) &= d(Sx_{2n-2}, Tx_{2n-1}) \\ &\leq \alpha(x_{2n-2}, x_{2n-1}) d(Sx_{2n-2}, Tx_{2n-1}) \\ &\leq \psi d(x_{2n-2}, x_{2n-1}). \end{aligned}$$

By induction , we get $d(x_{2n+1}, x_{2n+2}) \leq \psi^{2n+1} d(x_0, x_1)$ for all $n \in \mathbb{N}$.
In general, we have $d(x_n, x_{n+1}) \leq \psi^n d(x_0, x_1)$ for all n.
Fix $\epsilon > 0$ and let $n(\epsilon) \in \mathbb{N}$ such that

$$\sum_{n \geq n(\epsilon)} \psi^n(d(x_0, x_1)) < \epsilon.$$

Let $m, n \in \mathbb{N}$ with $m > n > n(\epsilon)$. Using the triangular inequality, we obtain

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{m-1} \psi^k(d(x_0, x_1)) \\ &\leq \sum_{n \geq n(\epsilon)} \psi^n(d(x_0, x_1)). \end{aligned}$$

Thus we proved that $\{x_n\}$ is a Cauchy sequence in the metric space (X, d). Since X is complete, $x_n \to x$ for some $x \in X$. We claim that x is the required common fixed point of S and T. To justify our claim, we proceed as follows:

$$\begin{aligned} d(Sx, x_{2n}) &= d(Sx, Tx_{2n-1}) \\ &\leq \alpha(x, x_{2n}) d(Sx, Tx_{2n-1}) \\ &\leq \psi(d(x, x_{2n-1})), \end{aligned}$$

when $n \to \infty$, $d(Sx, x) \leq \psi(d(x, x)) = \psi(0) = 0$ gives $Sx = x$.
Similarly,

$$\begin{aligned} d(Tx, x_{2n+1}) &= d(Tx, Sx_{2n}) \\ &\leq \alpha(x, x_{2n}) d(Tx, Sx_{2n}) \\ &\leq \psi(d(x, x_{2n})), \end{aligned}$$

when $n \to \infty$, $d(Tx, x) \leq \psi(d(x, x)) = \psi(0) = 0$ gives $Tx = x$.
Thus x is the common fixed point of S and T. □

Example 2.3. Let $X = \mathbb{R}$ endowed with the standard metric $d(x, y) = |x - y|$ for all $x, y \in \mathbb{R}$. Define the mappings $S, T : X \to X$ by $Tx = \frac{x}{2}$ and $Sx = \frac{x}{3}$.
Now, we define the mapping $\alpha : X \times X \to [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & y \leq \frac{5x}{6}, x, y \in [0, 1], \\ 0 & \text{otherwise.} \end{cases}$$

Clearly S and T satisfy condition (iv) of the previous theorem with $\psi(t) = \frac{t}{2}$ for all $x, y \in X$. Moreover, there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$. In fact, for $x_0 = 1$, we have

$$\begin{aligned} \alpha(1, S1) &= \alpha\left(1, \frac{1}{2}\right) \\ &= 1. \end{aligned}$$

The maps S and T are continuous and also satisfy condition (i) of the previous theorem. consequently, S and T have a common fixed point. In this example 0 is a common fixed point of S and T.

We say $f : X \to X$ is contractive if there exists $\alpha \in [0, 1]$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha d(x, y).$$

([6]) $f : X \to X$ is Kannan map, if there exists $\alpha \in [0, 1]$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \frac{\alpha}{2}[d(x, f(x)) + d(y, f(y))].$$

([4]) $f : X \to X$ is Chatterjea map, if there exists $\alpha \in [0, 1]$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \frac{\alpha}{2}[d(x, f(y)) + d(y, f(x))].$$

([8]) $f : X \to X$ is Zamfirescu map, if there exists $\alpha \in [0, 1]$ such that, for all $x, y \in X$,

$$d(f(x), f(y)) \leq \alpha M_f(x, y),$$

where

$$M_f(x, y) = \max\left\{d(x, y), \frac{1}{2}[d(x, f(y)) + d(y, f(x))] , \frac{1}{2}[d(x, f(x)) + d(y, f(y))]\right\}.$$

([5]) $f : X \to X$ is a weakly contractive map, if there exists $\alpha : X \times X \to [0, 1]$, satisfies

$$\theta(a, b) := \sup \{\alpha(x, y) : a \leq d(x, y) \leq b\} < 1,$$

for every $0 < a \leq b$, such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \alpha(x, y)d(x, y).$$

([1]) $f : X \to X$ is a weakly Kannan map, if there exists $\alpha : X \times X \to [0, 1]$, satisfies

$$\theta(a, b) := \sup\{\kappa(x, y) : a \leq d(x, y) \leq b\} < 1,$$

for every $0 < a \leq b$, such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \frac{\kappa(x, y)}{2}[d(x, f(x)) + d(y, f(y))].$$

for all $x, y \in D$.

([1]) $f : X \to X$ is a weakly Chatterjea map, if there exists $\alpha : X \times X \to [0, 1]$, satisfies

$$\theta(a, b) := \sup\{\xi(x, y) : a \leq d(x, y) \leq b\} < 1,$$

for all $0 < a \leq b$, such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \frac{\xi(x, y)}{2}[d(x, f(y)) + d(y, f(x))].$$

for all $x, y \in D$.

([8]) $f : X \to X$ is a weakly Zamfirescu map, if there exists $\alpha : X \times X \to [0, 1]$, satisfies

$$\theta(a, b) := \sup\{\alpha(x, y) : a \leq d(x, y) \leq b\} < 1,$$

for every $0 < a \leq b$, such that, for all $x, y \in D$,

$$d(f(x), f(y)) \leq \alpha(x, y)M_f(x, y), \ (Z_w),$$

for all $x, y \in D$, where

$$M_f(x,y) = \max\left\{d(x,y), \frac{1}{2}[d(x,f(y)) + d(y,f(x))] \right.$$
$$\left. , \frac{1}{2}[d(x,f(x)) + d(y,f(y))]\right\}.$$

We define the modified weakly Zamfirescu maps in the following method. Let (X,d) be a metric space and $f : X \to X$ be a given mapping. If there exists a function $\beta : X \times X \to [0,\infty)$ such that

$$\beta(x,y)d(fx,fy) \le \alpha(x,y)M_f(x,y), \ (Z_{mw}),$$

A self-mapping f on a metric space (X,d) is said to be asymptotically regular at $x_0 \in X$ if $\lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_n)) = 0$.

Definition 2.4. Let $f : X \to X$ and $\beta : X \times X \to [0,\infty)$. We say f is β-admissible if

$$x, y \in X, \quad \beta(x,y) \ge 1 \quad \Rightarrow \quad \beta(fx,fy) \ge 1.$$

Theorem 2.5. *Let (X,d) be a metric space. If $f : X \to X$ is a modified weakly Zamfirescu map, such that f satisfies the following conditions:*

(i) *f is β- admissible;*
(ii) *there exists $x_0 \in X$ such that $\beta(x_0, fx_0) \ge 1$,*

then, f is asymptotically regular at each point in X.

Proof. Let $x_0 \in X$ such that $\beta(x_0, fx_0) \ge 1$. Define the picard iterates $x_n = f(x_{n-1}) = f^n(x_0)$ for $n = 1, 2, \ldots$. We first prove that for all $n \ge 1$,

$$d(x_n, x_{n+1}) \le \alpha(x_{n-1}, x_n)d(x_{n-1}, x_n), \tag{2.1}$$

where α is the function associated to f by the condition (Z_{mw}).
We have

$$\beta(x_0,x_1) = \beta(x_0,fx_0) \ge 0 \quad \Rightarrow \quad \beta(fx_0,fx_1) = \beta(x_1,x_2) \ge 1.$$

By induction, we get $\beta(x_{n-1}, x_n) \geq 1$, for all $n \in \mathbb{N}$. Observe that, for all $n \geq 1$

$$\begin{aligned} M_f(x_{n-1}, x_n) &= \max\left\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, f(x_{n-1})) + d(x_n, f(x_n))]\right. \\ &\qquad \left., \frac{1}{2}[d(x_{n-1}, f(x_n)) + d(x_n, f(x_{n-1}))]\right\} \\ &= \max\left\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right. \\ &\qquad \left., \frac{1}{2}[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]\right\} \\ &= \max\left\{d(x_{n-1}, x_n), \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]\right. \\ &\qquad \left., \frac{1}{2}d(x_{n-1}, x_{n+1})\right\}. \end{aligned}$$

To complete the proof of (2.1), we shall appeal to condition (Z_{mw}), and hence, we consider the following three cases:

Case 1. If $M_f(x_{n-1}, x_n) = d(x_{n-1}, x_n)$, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \beta(x_{n-1}, x_n) d(f(x_{n-1}), f(x_n) \\ &\leq \alpha(x_{n-1}, x_n) d(x_{n-1}, x_n). \end{aligned}$$

Case 2. If $M_f(x_{n-1}, x_n) = \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$, we have

$$\begin{aligned} \beta(x_{n-1}, x_n) d(x_n, x_{n+1}) &\leq \alpha(x_{n-1}, x_n) \frac{[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]}{2}, \\ d(x_n, x_{n+1}) &\leq \frac{\alpha(x_{n-1}, x_n)}{2\beta(x_{n-1}, x_n) - \alpha(x_{n-1}, x_n)} d(x_{n-1}, x_n) \\ &\leq \alpha(x_{n-1}, x_n) d(x_{n-1}, x_n). \end{aligned}$$

Case 3. If $M_f(x_{n-1}, x_n) = \frac{1}{2}d(x_{n-1}, x_{n+1})$, we have

$$\begin{aligned} \beta(x_{n-1}, x_n) d(x_n, x_{n+1}) &\leq \frac{\alpha(x_{n-1}, x_n)}{2} \frac{1}{2} d(x_{n-1}, x_{n+1}) \\ &\leq \frac{\alpha(x_{n-1}, x_n)}{2} [d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \end{aligned}$$

i.e.,

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \frac{\alpha(x_{n-1}, x_n)}{2\beta(x_{n-1}, x_n) - \alpha(x_{n-1}, x_n)} d(x_{n-1}, x_n) \\ &\leq \alpha(x_{n-1}, x_n) d(x_{n-1}, x_n). \end{aligned}$$

Thus, relation (2.1) is proved.

As a consequence, we obtain that the sequence $\{d(x_n, x_{n+1})\}$ is non increasing, since $0 \leq \alpha(x_{n-1}, x_n) \leq 1$. Then, it is convergent to the real number

$$d = \inf\{d(x_{n-1}, x_n) : \ n = 1, 2, \ldots\}.$$

It suffices to prove that $d = 0$. Suppose that $d > 0$ and get a contradiction . For any $n \in \mathbb{N}$, we have

$$0 < d \leq d(x_n, x_{n+1}) \leq d(x_0, x_1),$$

and the definition of $\theta = \theta(d, d(x_0, x_1))$, results $\alpha(x_{n-1}, x_n) \leq \theta$. This, together with (2.1) gives

$$d \leq d(x_n, x_{n+1}) \leq \theta^n d(x_0, x_1),$$

for all $n \in \mathbb{N}$, which is impossible since $d > 0$ and $0 \leq \theta < 1$. Therefore, f is asymptotically regular at x_0. □

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