Sahand Communications in Mathematical Analysis (SCMA) Vol. 2 No. 1 (2015), 27-49
http://scma.maragheh.ac.ir

ANALYTICAL SOLUTIONS FOR THE FRACTIONAL FISHER'S EQUATION

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ABSTRACT. In this paper, we consider the inhomogeneous time-fractional nonlinear Fisher equation with three known boundary conditions. We first apply a modified Homotopy perturbation method for translating the proposed problem to a set of linear problems. Then we use the separation variables method to solve obtained problems. In examples, we illustrate that by right choice of source term in the modified Homotopy perturbation method, it is possible to get an exact solution.

1. INTRODUCTION

Fractional partial differential equations (FPDEs) have recently aroused considerable interest in mathematics and its applications. Scientists used them to model many physical, biological and chemical processes [16, 18, 19]. Besides, they have applications in sampling, hold algorithms, and signal processing. There are various analytical method for solving nonlinear FPDEs including the Adomian decomposition [2, 4, 9], Homotopy perturbation method [5, 6, 13], Variation iteration method [14, 15], and Homotopy analysis method [10, 11]. Finding exact solutions for FPDEs are often too complicated and required so much calculations. Nowadays, biological models have been the focus of many mathematical scientists. Fisher's equation

$$u_t = u_{xx} + u(1-u) \tag{1.1}$$

2010 *Mathematics Subject Classification.* 34B24, 34B27.
Key words and phrases. Fractional Fisher's equation, Mittag-Leffler, Method of separating variables.
Received: 5 November 2014, Accepted: 7 December 2014.
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was first proposed by Fisher as a model for the propagation of a mutant gene [10]. In this model, $u(x,t)$ is the population density and $u(u-1)$ denotes the logistic form.

In this paper, we consider the inhomogeneous time-fractional Fisher's equation which is examined in [22, 21]:

$$D_t^{\alpha}u(x,t) = u_{xx}(x,t)+u(x,t)(1-u(x,t))+f(x,t), \quad 0<x<L, 0<\alpha\leq 1, \tag{1.2}$$

where the fractional derivative in (1.2) is the Caputo derivative.

We introduce a scheme to solve (1.2) which is a combination of modified homotopy perturbation [5], an especial case of homotopy analysis method, Laplace transform, and separation of variables. This kind of modification of homotopy perturbation method has the capability of transforming nonlinear terms into linear ones while homotopy analysis method and homotopy perturbation method don't possess this characteristic.

2. Preliminaries

In this section, we give some necessary definitions and lemmas about fractional calculus. For some details, you can refer to [18, 20].

Definition 2.1 ([3])**.** A function $f:\mathbb{R}\to\mathbb{R}^+$ is said to be in the space C_ν, with $\nu\in\mathbb{R}$, if it can be written as $f(x)=x^p f_1(x)$ with $p>\nu$, $f_1(x)\in C[0,\infty)$ and it is said to be in the space C_ν^m if $f^{(m)}\in C_\nu$ for $m\in\mathbb{N}\bigcup\{0\}$.

Definition 2.2 ([12])**.** The Riemann-Liouville fractional integral of $f\in C_\nu$ with order $\alpha>0$ and $\nu\geq -1$ is defined as:

$$\begin{aligned} J^{\alpha}f(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}f(\tau)d\tau, \quad \alpha>0, \quad t>0,\\ J^0 f(t) &= f(t). \end{aligned} \tag{2.1}$$

Definition 2.3 ([20])**.** The Riemann-Liouville fractional derivative of $f\in C_{-1}^m$ with order $\alpha>0$ and $m\in\mathbb{N}\bigcup\{0\}$, is defined as:

$$D_t^{\alpha}f(t) = \tfrac{d^m}{dt^m}J^{m-\alpha}f(t), \quad m-1<\alpha\leq m, \quad m\in\mathbb{N}. \tag{2.2}$$

Definition 2.4 ([20])**.** The Caputo fractional derivative of $f\in C_{-1}^m$ with order $\alpha>0$ and $m\in\mathbb{N}\bigcup\{0\}$, is defined as:

$$^{C}D_t^{\alpha}f(t) = \begin{cases} J^{m-\alpha}f^{(m)}(t), & m-1<\alpha\leq m, \quad m\in\mathbb{N}\\ \frac{d^m f(t)}{dt^m}, & \alpha=m. \end{cases} \tag{2.3}$$

Definition 2.5 ([20])**.** A two-parameter Mittag-Leffler function is defined by the following series

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\alpha k + \beta)}. \tag{2.4}$$

Definition 2.6 ([12])**.** A multivariate Mittag-Leffler function is defined as

$$\begin{aligned} E_{(a_1,a_2,\cdots,a_n),b}(z_1, z_2, \cdots, z_n) \\ = \sum_{k=0}^{\infty} \sum_{l_1+l_2+\cdots+l_n=k} \frac{k!}{l_1! \times l_2! \times \cdots \times l_n!} \frac{\prod_{i=1}^{n} z_i^{l_i}}{\Gamma\left(b + \sum_{i=1}^{n} a_i l_i\right)}, \end{aligned} \tag{2.5}$$

where $b > 0,\ l_1, l_2, \ldots,\ l_n \geq 0,\ |z_i| < \infty, i = 1, 2, \ldots, n.$

Definition 2.7. Let us define the Laplace-transform (LT) operator φ on a function $u(x,t),\ \ (t \geq 0)$ by

$$\varphi\{u(x,t); t \mapsto s\} = \int_0^{\infty} e^{-st} u(x,t) dt, \tag{2.6}$$

and denote it by $\varphi\{u(x,t); t \mapsto s\} = L(u(x,t))$, where s is the LT parameter. For our purpose here, we shall take s to be real and positive. As a consequence, the LT of Mittag-Leffler function takes the following form

$$\begin{aligned} L(E_{\alpha,\beta}(t)) &= \int_0^{\infty} e^{-st} E_{\alpha,\beta}(t) dt \\ &= \sum_{k=0}^{\infty} \frac{1}{s^{k+1}\Gamma(\alpha k + \beta)}. \end{aligned} \tag{2.7}$$

Lemma 2.8 (see [12])**.** *Let* $\mu > \mu_1 > \mu_2 > \ldots > \mu_n \geq 0,\ m_i - 1 < \mu_i \leq m_i,\ m_i \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\},\ d_i \in \mathbb{R}, i = 1, 2, \ldots, n.$ *Consider the initial value problem*

$$\begin{cases} (D^{\mu} y)(x) - \displaystyle\sum_{i=1}^{n} \lambda_i (D^{\mu_i} y)(x) = g(x), \\ y^{(k)}(0) = c_k \in \mathbb{R}, \qquad k = 0, 1, \ldots, m-1,\ \ m - 1 < \mu \leq m, \end{cases}$$

where the function $g(x)$ *is assumed to lie in* C_{-1}*, if* $\mu \in \mathbb{N}$*, in* C^1_{-1}*, if* $\mu \notin \mathbb{N}$ *and the unknown function* $y(x)$ *is to be determined in the space*

C^m_{-1}. *This has solution*

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), \quad x \geq 0,$$

where

$$y_g(x) = \int_0^x t^{\mu-1} E_{(\cdot),\mu}(t) g(x-t) dt$$

and

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^{n} d_i x^{k+\mu-\mu_i} E_{(\cdot),k+1+\mu-\mu_i}(x), \quad k = 0, 1, \ldots, m-1,$$

fulfills the initial conditions

$$u_k^{(l)}(0) = \delta_{kl}, \quad k, l = 0, 1, \ldots, m-1.$$

The function

$$E_{(\cdot),\sigma}(x) = E_{(\mu-\mu_1,\ldots,\mu-\mu_n),\sigma}(d_1 x^{\mu-\mu_1}, \ldots, d_n x^{\mu-\mu_n}),$$

is a particular case of the multivariate Mittag-Leffler function (see [12]*) and the natural numbers* $l_k, k = 0, 1, \ldots, m-1$*, are determined from the condition*

$$\begin{cases} m_{l_k} \geq k+1, \\ m_{l_k+1} \leq k. \end{cases}$$

In the case $m_i \leq k, i = 1, 2, \ldots, n$*, we set* $l_k := 0$*, and if* $m_i \geq k+1, i = 1, 2, \ldots, n$*, then* $l_k := n$*.*

3. Modified homotopy perturbation method (MHPM)

The homotopy perturbation method is one of the most effective methods for solving nonlinear problems. Several modifications of this method is presented. In this paper, we used a modified one for solving the fractional Fisher's equation as follows:

$$\begin{aligned} D_t^\alpha u(x,t) = u_{xx}(x,t) + u(x,t) + ph(u(x,t)) \\ + f_1(x,t) + pf_2(x,t), \end{aligned} \tag{3.1}$$

where $f_1(x,t) + f_2(x,t) = f(x,t)$, $f(x,t)$ is the source term of Eq. (1.2), and p is an embedding parameter that varies from zero to one. For more details see [5]. By choosing proper functions f_1 and f_2, we can improve the success of our method.

As mentioned in the introduction section, this kind of modification of homotopy perturbation method has the capability of transforming nonlinear terms into linear ones while homotopy analysis method and homotopy perturbation method don't possess this characteristic. For converting the nonlinear problems to linear ones, we can also use a special case of modified homotopy analysis method.

4. Inhomogeneous fractional Fisher's equation

4.1. **Dirichlet boundary condition.** In this subsection, we determine the solution of the fractional nonlinear Fisher's equation

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + u(x,t) - u^2(x,t) + f(x,t), \tag{4.1}$$

with the initial and Dirichlet boundary conditions

$$\begin{aligned} &u(x,0) = \phi(x), \quad 0 \le x \le L, \\ &u(0,t) = \mu_1(t), \quad u(L,t) = \mu_2(t), \quad t \ge 0. \end{aligned} \tag{4.2}$$

In order to solve the problem with inhomogeneous boundary conditions, first transform it into a problem with homogeneous boundary conditions. For this purpose let

$$u(x,t) = W(x,t) + V(x,t),$$

where $W(x,t)$ is a new unknown function and

$$V(x,t) = \frac{\mu_2(t) - \mu_1(t)}{L} x + \mu_1(t), \tag{4.3}$$

satisfies the boundary conditions as

$$V(0,t) = \mu_1(t), \quad V(L,t) = \mu_2(t). \tag{4.4}$$

Furthermore, the function $W(x,t)$ satisfies in problem with homogeneous boundary conditions as follows:

$$\begin{cases} D_t^\alpha W(x,t) = W_{xx}(x,t) + W(x,t) + h(W+V) + \tilde{f}(x,t), \\ \quad W(x,0) = g(x), \qquad 0 \le x \le L, \\ \quad W(0,t) = 0, \qquad\quad W(L,t) = 0, \end{cases} \tag{4.5}$$

where

$$\begin{aligned} \tilde{f}(x,t) = f(x,t) &+ \frac{x}{L}\left[D_t^\alpha \mu_1(t) - D_t^\alpha \mu_2(t)\right] - D_t^\alpha \mu_1(t) \\ &+ \frac{\mu_2(t) - \mu_1(t)}{L} x + \mu_1(t), \end{aligned} \tag{4.6}$$

and

$$g(x) = \phi(x) - \frac{x}{L}[\mu_2(0) - \mu_1(0)] - \mu_1(0). \tag{4.7}$$

For solving (4.5) we use MHPM

$$D_t^\alpha W(x,t) = W_{xx}(x,t) + W(x,t) + ph(W+V) + \tilde{f}_1(x,t) + p\tilde{f}_2(x,t). \tag{4.8}$$

By assuming $W(x,t) = \sum_{i=0}^{\infty} W_i p^i$, and substituting it in (4.8), we obtain

$$p^0 : \begin{cases} D_t^{\alpha} W_0(x,t) = \frac{\partial^2 W_0(x,t)}{\partial x^2} + W_0(x,t) + \tilde{f}_1(x,t), \\ W_0(x,0) = g(x) \qquad 0 \le x \le L, \\ W_0(0,\mathrm{t}) = 0, \qquad W_0(\mathrm{L},\mathrm{t}) = 0, \end{cases} \tag{4.9}$$

$$p^1 : \begin{cases} D_t^{\alpha} W_1(x,t) = \frac{\partial^2 W_1(x,t)}{\partial x^2} + W_1(x,t) + \tilde{f}_2(x,t) + A_0, \\ W_1(x,0) = 0, \qquad 0 \le x \le L, \\ W_1(0,t) = 0, \qquad W_1(L,t) = 0, \end{cases} \tag{4.10}$$

$$\vdots$$

$$p^k : \begin{cases} D_t^{2\alpha} W_k(x,t) = \frac{\partial^2 W_k(x,t)}{\partial x^2} + W_k(x,t) + A_{k-1}, \\ W_k(x,0) = 0, \qquad 0 \le x \le L, \\ W_k(0,t) = 0, \qquad W_k(L,t) = 0, \end{cases} \tag{4.11}$$

$$\vdots$$

where $\tilde{f}_1(x,t) + \tilde{f}_2(x,t) = \tilde{f}(x,t)$ and $\tilde{f}_1(x,t)$ must be satisfied in initial and boundary conditions (4.9) [5] and $A_k,\ k = 0,\ 1, \ldots$ are Adomian polynomials [1] and are obtained

$$A_k = \frac{d^k}{dp^k} h\left(\sum_{i=0}^{\infty} W_i p^i + V\right)\Bigg|_{p=0}, \quad k = 0, 1, \ldots. \tag{4.12}$$

We solve the corresponding homogeneous equation in (4.9) by the method of separation of variables. By assuming $W_0(x,t) = X_0(x)T_0(t)$ and substituting it in (4.9), we obtain an ordinary linear differential equation for $X_0(x)$ as

$$X_0''(x) + \lambda^2 X_0(x) = 0, \quad X_0(0) = X_0(L) = 0, \tag{4.13}$$

and a fractional ordinary linear differential equation for $T_0(t)$ as follows:

$$D_t^{\alpha} T_0 - T_0 + \lambda^2 T_0 = 0. \tag{4.14}$$

The Sturm-Liouville problem given by (4.13) has eigenvalues

$$\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, \ldots, \tag{4.15}$$

and corresponding eigenfunctions are

$$(X_0)_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots. \tag{4.16}$$

Now we seek a solution of the inhomogeneous problem in (4.9) of the form

$$W_0(x,t)=\sum_{n=1}^{\infty}(B_0)_n(t)\sin\left(\frac{n\pi x}{L}\right). \tag{4.17}$$

We assumed that the series can be differentiated term by term. In order to determine $(B_0)_n(t)$, we expanded $\tilde{f}_1(x,t)$ as a Fourier series by the eigenfunctions $\sin(\frac{n\pi x}{L})$ as follows:

$$\tilde{f}_1(x,t)=\sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)\sin\left(\frac{n\pi x}{L}\right), \tag{4.18}$$

then

$$(\tilde{f}_1)_n(t)=\frac{2}{L}\int_0^L \tilde{f}_1(x,t)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{4.19}$$

Substituting (4.17) and (4.18) into (4.9) yields

$$\begin{aligned}\sum_{n=1}^{\infty}D^{\alpha}(B_0)_n(t)\sin\left(\frac{n\pi x}{L}\right)+\left(\frac{n^2\pi^2}{L^2}-1\right)\sum_{n=1}^{\infty}D^{\alpha}(B_0)_n(t)\sin\left(\frac{n\pi x}{L}\right)\\ =\sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)\sin\left(\frac{n\pi x}{L}\right).\end{aligned} \tag{4.20}$$

By orthogonality properties of $\sin(\frac{n\pi x}{L})$, we get

$$D_t^{\alpha}(B_0)_n(t)+\left(\frac{n^2\pi^2}{L}-1\right)(B_0)_n(t)=(\tilde{f}_1)_n(t). \tag{4.21}$$

Since $W_0(x,t)$ satisfies the initial conditions in (4.9), we have

$$\sum_{n=1}^{\infty}(B_0)_n(0)\sin\left(\frac{n\pi x}{L}\right)=g(x), \tag{4.22}$$

which yields

$$(B_0)_n(0)=\frac{2}{L}\int_0^X g_1(x)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{4.23}$$

For each value of n, (4.22) and (4.23) make up a fractional initial value problem.

According to Lemma 2.8, the fractional initial value problem with $\mu = \alpha,\ \mu_1 = 0,\ \ m_1 = 0,\ \lambda_1 = 1 - \frac{n^2\pi^2}{L},\ \ m = 1$, has the solution

$$(B_0)_n(t) = \int_0^t \tau^\alpha E_{(\alpha,\alpha)}(\lambda_1 \tau^\alpha)(\tilde{f}_1)_n(t-\tau)d\tau + (B_0)_n(0)\Big[1 + \lambda_1 E_{(\alpha,\alpha+1)}(\lambda_1 t^\alpha)\Big]. \tag{4.24}$$

Hence we get the solution of the initial boundary value problem (4.9) in the form

$$W_0(x,t) = \sum_{n=1}^{\infty} (B_0)_n(t) \sin\left(\frac{n\pi x}{L}\right). \tag{4.25}$$

In a similar way, we can get $W_k,\ k = 1, \dots$ from (4.10) and (4.11). Since in calculating W_{k+1} the value of A_k is known from pervious stages, then all of problems in (4.9)-(4.11) are linear and hence solving them is simpler than main problems. Note that the success of these methods relies mainly on the proper choice of the functions $\tilde{f}_1$ and $\tilde{f}_2$. Furthermore, this proper choice of $\tilde{f}_1$ and $\tilde{f}_2$ may provide the solution only in one iteration of MHPM.

4.2. **Neumann boundary condition.** Now, we obtain the solution of the inhomogeneous fractional Fisher's equation (1.2) with the initial and Neumann boundary conditions as follows:

$$\begin{aligned} &u(x,0) = \phi(x), \qquad 0 \le x \le L,\\ &u_x(0,t) = \mu_1(t), \quad u_x(L,t) = \mu_2(t), \quad t \ge 0, \end{aligned} \tag{4.26}$$

in which $\phi(x),\ \mu_1(t),\ \mu_2(t)$ are as defined in subsection 4.1.

For solving the problem with inhomogeneous boundary conditions, as before, we transform it into a problem with homogeneous boundary conditions. Thus we suppose that

$$u(x,t) = \tilde{W}(x,t) + \tilde{V}(x,t),$$

where $\tilde{W}(x,t)$ is an unknown function and

$$\tilde{V}(x,t) = \frac{\mu_2(t) - \mu_1(t)}{2L} x^2 + \mu_1(t)x, \tag{4.27}$$

which satisfies the following boundary conditions:

$$\tilde{V}_x(0,t) = \mu_1(t), \qquad \tilde{V}_x(L,t) = \mu_2(t). \tag{4.28}$$

Furthermore, $\tilde{W}(x,t)$ satisfies in problem with homogeneous boundary conditions as:

(4.29)
$$\begin{cases} D_t^\alpha \tilde{W}(x,t) = \frac{\partial^2 \tilde{W}(x,t)}{\partial x^2} + \tilde{W}(x,t) + h(\tilde{W}(x,t) + \tilde{V}(x,t)) = \tilde{f}(x,t), \\ \tilde{W}(x,0) = g(x), \qquad 0 \le x \le L, \\ \tilde{W}_x(0,t) = 0, \qquad \tilde{W}_x(L,t) = 0, \quad t \ge 0, \end{cases}$$

in which $\tilde{f}(x,t)$ and $g(x)$ are as the form as below:

(4.30)
$$\begin{aligned} \tilde{f}(x,t) &= f(x,t) + \frac{x^2}{2L}\left(D_t^\alpha \mu_1(t) - D_t^\alpha \mu_2(t)\right) - D_t^\alpha \mu_1(t) + \frac{\mu_2(t) - \mu_1(t)}{L} \\ &\quad - \frac{\mu_2(t) - \mu_1(t)}{2L} x^2 + \mu_1(t)x, \\ g(x) &= \phi(x) - \frac{x^2}{2L}[\mu_2(0) - \mu_1(0)] - \mu_1(0)x. \end{aligned}$$

Now, for solving (4.29) by MHPM, we have

$$\begin{aligned} D_t^\alpha \tilde{W}(x,t) = \tilde{W}_{xx}(x,t) + \tilde{W}(x,t) + ph(\tilde{W}(x,t) + \tilde{V}(x,t)) \\ + \tilde{f}_1(x,t) + p\tilde{f}_2(x,t). \end{aligned} \tag{4.31}$$

If we assume $\tilde{W}(x,t) = \sum_{i=0}^{\infty} \tilde{W}_i p^i$, and substitute it in (4.29), we obtain

$$p^0 : \begin{cases} D_t^\alpha \widetilde{W_0}(x,t) = (\widetilde{W_0})_{xx}(x,t) + \widetilde{W_0}(x,t) + \tilde{f}_1(x,t), \\ \tilde{W}_0(x,0) = g(x), \qquad 0 \le x \le L, \\ (\tilde{W}_0)_x(0,t) = 0, \qquad (\tilde{W}_0)_x(L,t) = 0, \end{cases} \tag{4.32}$$

$$p^1 : \begin{cases} D_t^\alpha \widetilde{W_1}(x,t) = (\widetilde{W_{1xx}})(x,t) + \widetilde{W_1}(x,t) + \tilde{f}_2(x,t) + A_0 \\ \tilde{W}_1(x,0) = 0, \qquad 0 \le x \le L, \\ (\tilde{W}_1)_x(0,t) = 0, \qquad (\tilde{W}_1)_x(L,t) = 0, \end{cases} \tag{4.33}$$

$$\vdots$$

$$p^k : \begin{cases} D_t^\alpha \widetilde{W_k}(x,t) = (\widetilde{W_k})_{xx}(x,t) + \widetilde{W_k}(x,t) + A_{k-1}, \\ \tilde{W}_k(x,0) = 0, \\ (\tilde{W}_k)_x(0,t) = 0, \qquad (\tilde{W}_k)_x(L,t) = 0, \end{cases} \tag{4.34}$$

and so on, in which $\tilde{f}_1(x,t) + \tilde{f}_2(x,t) = \tilde{f}(x,t)$ and $A_k,\ k = 0,\ 1, \ldots$ are Adomian polynomials defined in (4.12).

For solving the corresponding homogeneous equation in (4.32) by the method of separation of variables, we assume that $\tilde{W}_0(x,t) = X_0(x)T_0(t)$ and substitue it in (4.29), we obtain a linear ODE for $X_0(x)$ and a linear

FDE for $T_0(t)$ as

$$\begin{aligned} & X_0''(x) + \lambda X(x) = 0, \qquad\qquad X(0) = X(L) = 0, \\ (4.35) \qquad & D_t^{\alpha} T_0(t) + (\lambda - 1) T_0(t) = 0. \end{aligned}$$

The Sturm-Liouville problem, which is given by (4.35), has eigenvalues and corresponding eigenfunctions as:

$$\begin{aligned} & \lambda_n = \frac{n^2\pi^2}{L^2}, \qquad\qquad n = 1, 2, \ldots, \\ (4.36) \qquad & (X_0)_n(x) = \cos\left(\frac{n\pi x}{L}\right), \qquad n = 1, 2, \ldots \; . \end{aligned}$$

Now we are going to seek a solution of the inhomogeneous problem in (4.29) which takes the form

$$\tilde{W}_0(x,t) = \sum_{n=1}^{\infty} (B_0)_n(t) \cos\left(\frac{n\pi}{L} x\right). \tag{4.37}$$

For determining $(B_0)_n(t)$, by expanding $\tilde{f}_1(x,t)$ as a Fourier series by the eigenfunctions $\cos(\frac{n\pi}{L}x)$ we have:

$$\tilde{f}_1(x,t) = \sum_{n=1}^{\infty} (\tilde{f}_1)_n(t) \cos\left(\frac{n\pi}{L} x\right), \tag{4.38}$$

in which the Fourier coefficients are as the following form:

$$(\tilde{f}_1)_n(t) = \frac{2}{L} \int_0^L \tilde{f}_1(x,t) \cos\left(\frac{n\pi}{L} x\right) dx. \tag{4.39}$$

Then substituting (4.37), (4.38) into (4.29) implies

$$D_t^{\alpha} (B_0)_n(t) + \left(-1 + \frac{n^2\pi^2}{L^2}\right) (B_0)_n(t) = (\tilde{f}_1)_n(t). \tag{4.40}$$

Since $\tilde{W}(x,t)$ fulfills the initial conditions in (4.29), we have

$$\sum_{n=1}^{\infty} (B_0)_n(0) \cos\left(\frac{n\pi}{L} x\right) = g(x), \tag{4.41}$$

which yields

$$(B_0)_n(0) = \frac{2}{L} \int_0^L g(x) \cos\left(\frac{n\pi}{L} x\right) dx. \tag{4.42}$$

Therefore, Lemma 2.8 implies that the fractional initial value problem has the solution as follows:

(4.43)

$$
\begin{aligned}
u(x,t) &= \sum_{n=1}^{\infty}(B_0)_n(t)\cos\left(\frac{n\pi x}{L}\right) \\
&= \mu_1(t)x + \frac{\mu_2(t)-\mu_1(t)}{2L}x^2 \\
&\quad + \sum_{n=1}^{\infty}\cos\left(\frac{n\pi}{L}x\right)\left[\int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}\left(\left(1-\frac{n^2\pi^2}{L^2}\right)\tau^{\alpha}(\tilde{f}_1)_n(t-\tau)\right)d\tau\right].
\end{aligned}
$$

4.3. **Robin boundary condition.** In this subsection, we try to solve (4.1) with the initial and Robin boundary conditions as

$$
\begin{cases}
u(x,0) = \phi(x), & 0 \le x \le L, \\
u(0,t) + \alpha_1 u_x(0,t) = \mu_1(t), & t \ge 0, \\
u(L,t) + \beta_1 u_x(L,t) = \mu_2(t), & t \ge 0,
\end{cases}
$$

where α_1, β_1 are nonzero constants. To solve this problem, we translate the inhomogeneous bondary conditions to the homogenouse ones. So, suppose that

$$u(x,t) = \bar{W}(x,t) + \bar{V}(x,t),$$

where $\bar{W}(x,t)$ is a new unknown function and

$$\bar{V}(x,t) = \frac{\mu_1(t)-\mu_2(t)}{\alpha_1-\beta_1-L}x - \frac{(L+\beta_1)\mu_1(t)-\alpha_1\mu_2(t)}{\alpha_1-\beta_1-L}. \tag{4.44}$$

Therefore, we will have

$$
\begin{cases}
\bar{V}(0,t) + \alpha_1\bar{V}_x(0,t) = \mu_1(t), \\
\bar{V}(L,t) + \beta_1\bar{V}(L,t) = \mu_2(t).
\end{cases}
$$

The function $\bar{W}(x,t)$ is the solution of nonlinear problem with homogeneous boundary conditions:

$$
\begin{cases}
D_t^{\alpha}\bar{W}(x,t) = \bar{W}_{xx}(x,t) + \bar{W}(x,t) + h(\bar{W}+\bar{V}) + \tilde{f}(x,t), \\
\quad \bar{W}(x,0) = g(x), \qquad\qquad\qquad 0 \le x \le L, \\
\quad \bar{W}(0,t) + \alpha_1\bar{W}_x(0,t) = 0, \\
\quad \bar{W}(L,t) + \beta_1\bar{W}(L,t) = 0,
\end{cases} \tag{4.45}
$$

where

$$\tilde{f}(x,t) = f(x,t) - D_t^{\alpha}\bar{V}(x,t). \tag{4.46}$$

For solving (4.45), again we use MHPM. By assuming

$$\bar{W}(x,t) = \sum_{i=0}^{\infty}\bar{W}_i p^i,$$

and substituting it in (4.45), we obtain

$$p^0 : \begin{cases} D_t^{\alpha}\bar{W}_0(x,t) = \frac{\partial^2 \bar{W}_0(x,t)}{\partial x^2} + \bar{W}_0(x,t) + \tilde{f}_1(x,t), \\ \bar{W}_0(x,0) = g(x), & 0 \le x \le L, \\ \bar{W}_0(0,t) + \alpha_1(\bar{W}_0)_x(0,t) = \mu_1(t), \\ \bar{W}_0(L,t) + \beta_1(\bar{W}_0)_x(L,t) = \mu_2(t), & t \ge 0, \end{cases} \tag{4.47}$$

$$p^1 : \begin{cases} D_t^{\alpha}\bar{W}_1(x,t) = \frac{\partial^2 \bar{W}_1(x,t)}{\partial x^2} + \bar{W}_1(x,t) + \tilde{f}_2(x,t) + A_0, \\ \bar{W}_1(x,0) = 0, & 0 \le x \le L, \\ \bar{W}_1(0,t) + \alpha_1(\bar{W}_1)_x(0,t) = \mu_1(t), \\ \bar{W}_1(L,t) + \beta_1(\bar{W}_1)_x(L,t) = \mu_2(t), & t \ge 0, \end{cases} \tag{4.48}$$

$$\vdots$$

$$p^k : \begin{cases} D_t^{2\alpha}\bar{W}_k(x,t) = \frac{\partial^2 \bar{W}_k(x,t)}{\partial x^2} + \bar{W}_k(x,t) + A_{k-1}, \\ \bar{W}_k(x,0) = 0, & 0 \le x \le L, \\ \bar{W}_k(0,t) + \alpha_1(\bar{W}_k)_x(0,t) = \mu_1(t), \\ \bar{W}_k(L,t) + \beta_1(\bar{W}_k)_x(L,t) = \mu_2(t), & t \ge 0, \end{cases} \tag{4.49}$$

$$\vdots$$

where $\tilde{f}_1(x,t) + \tilde{f}_2(x,t) = \tilde{f}(x,t)$ and $A_k,\ k = 0,\ 1, \dots$ are Adomian polynomials and are obtained from (4.12). We solve the corresponding homogeneous equation in (4.47) by the method of separation of variables. By assuming $\bar{W}_0(x,t) = X_0(x)T_0(t)$ and substituting it in (4.9), we obtain an ordinary linear differential equation for $X_0(x)$:

$$\begin{aligned} &X_0''(x) + \lambda^2 X_0(x) = 0, \\ &X_0(0) + \alpha_1 X_0'(0) = 0, \\ &X_0(L) + \beta_1 X_0'(L) = 0. \end{aligned} \tag{4.50}$$

and a fractional ordinary linear differential equation for $T_0(t)$ as (4.14). The Sturm-Liouville problem given by (4.50) has eigenvalues λ_n^2 and corresponding eigenfunctions are

$$(X_0)_n(x) = -\alpha_1\lambda_n \cos(\lambda_n x) + \sin(\lambda_n x), \quad n = 1, 2, \dots. \tag{4.51}$$

Now we seek a solution for the nonhomogeneous problem in (4.47) of the form

$$\bar{W}_0(x,t) = \sum_{n=1}^{\infty} (B_0)_n(t)(X_0)_n(x). \tag{4.52}$$

Like previous section, in order to determine $(B_0)_n(t)$, we expand $\tilde{f}_1(x,t)$ as a Fourier series by the eigenfunctions $(X_0)_n(x)$ as follows

$$\tilde{f}_1(x,t) = \sum_{n=1}^{\infty} (\tilde{f}_1)_n(t)(X_0)_n(x). \tag{4.53}$$

We know that

$$(\tilde{f}_1)_n(t) = \frac{2}{L}\int_0^L \tilde{f}_1(x,t)(X_0)_n(x)dx. \tag{4.54}$$

By substituting (4.52) and (4.53) into (4.47) we have

$$\begin{aligned}\sum_{n=1}^{\infty} D^{\alpha}(B_0)_n(t)(X_0)_n(x) + (\lambda_n^2+1)\sum_{n=1}^{\infty}(B_0)_n(t)(X_0)_n(x) \\ = \sum_{n=1}^{\infty}(\tilde{f}_1)_n(t)(X_0)_n(x).\end{aligned} \tag{4.55}$$

We know that eigenfunctions of Stumr-Lioville equations are orthogonal, so

$$D_t^{\alpha}(B_0)_n(t) + (\lambda_n^2 - 1)(B_0)_n(t) = (\tilde{f}_1)_n(t). \tag{4.56}$$

Since $\bar{W}_0(x,t)$ satisfies the initial conditions in (4.47), we have

$$\sum_{n=1}^{\infty}(B_0)_n(0)(X_0)_n(x) = g(x), \tag{4.57}$$

which yields

$$(B_0)_n(0) = \frac{2}{L}\int_0^L g(x)(X_0)_n(x)dx. \tag{4.58}$$

According to Lemma 2.8, the fractional initial value problem with $\mu = \alpha,\ \mu_1 = 0,\ \ m_1 = 0,\ \ \ \lambda_1 = -(1+\lambda_n^2),\ \ \ m = 1$, has the solution

$$\begin{aligned}(B_0)_n(t) = \int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}(\lambda_1\tau^{\alpha})(\tilde{f}_1)_n(t-\tau)d\tau \\ + (B_0)_n(0)\Big[1 + \lambda_1 t^{\alpha}E_{(\alpha,\alpha+1)}(\lambda_1 t^{\alpha})\Big].\end{aligned} \tag{4.59}$$

Hence we get the solution of the initial boundary value problem (4.47) in the form

$$\bar{W}_0(x,t) = \sum_{n=1}^{\infty}(B_0)_n(t)\left(-\alpha_1\lambda_n\cos(\lambda_n x) + \sin(\lambda_n x)\right). \tag{4.60}$$

By applying the same way we can get $\bar{W}_k(x,t)$.

5. Examples

In this section, we consider three examples with different initial and boundary conditions and source term. We show that the solution obtained above agree with those established in these examples.

Example 5.1. Consider the fractional nonlinear Fisher's equation (4.1) with the initial and Dirichlet boundary conditions

$$\begin{aligned}&\mathrm{u(x,0)} = 2, \quad 0 \le x \le 1, \\ &\mathrm{u(0,t)} = 2, \quad u(1,t) = t^2 + 2, \quad t \ge 0,\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}f(x,t) = \sin(3\pi x)\left(t^6\sin(3\pi x) + 2t^5 x + (9\pi^2+3)t^3 + \frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha}\right) \\ + x\left(t^4 x + 3t^2 - \frac{\Gamma(3)}{\Gamma(3-\alpha)}t^{2-\alpha}\right) + 2.\end{aligned}$$

In order to solve this problem, we first transform it into a problems homogeneous boundary conditions as

$$u(x,t) = W(x,t) + V(x,t) \tag{5.2}$$
$$= W(x,t) + t^2 x + 2,$$

$$\begin{cases} D_t^{\alpha} W(x,t) = \frac{\partial^2 W(x,t)}{\partial x^2} + W(x,t) + (W+V)^2 + \tilde{f}(x,t), \\ W(x,0) =, \qquad 0 \le x \le 1, \\ W(0,t) = 0, \qquad W(1,t) = 0, \end{cases} \tag{5.3}$$

where

$$\tilde{f}(x,t) = \sin(3\pi x)\left(t^6 \sin(3\pi x) + 2t^5 x + (9\pi^2 + 3)t^3 + \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha}\right) + t^4 x^2 + 4t^2 + 4.$$

$$\tilde{f}(x,t) = \sin(3\pi x)\left(t^6 \sin(3\pi x) + 2t^5 x + (9\pi^2 + 3)t^3 + \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha}\right) + t^4 x^2 + 4t^2 + 4.$$

For solving (5.3) we apply MHPM

$$D_t^{\alpha} W(x,t) = \frac{\partial^2 W(x,t)}{\partial x^2} + W(x,t) - p(W+V)^2 + \tilde{f}_1(x,t) + p\tilde{f}_2(x,t). \tag{5.4}$$

By assuming $W(x,t) = \sum_{i=0}^{\infty} W_i p^i$, and substituting it in (5.4), we obtain

$$p^0 : \begin{cases} D_t^{\alpha} W_0(x,t) = W_0(x,t) + \frac{\partial^2 W_0(x,t)}{\partial x^2} + \tilde{f}_1(x,t), \\ W_0(x,0) = 2, \qquad 0 \le x \le 1, \\ W_0(0,t) = 0, \qquad W_0(L,t) = 0, \end{cases} \tag{5.5}$$

$$p^1 : \begin{cases} D_t^{\alpha} W_1(x,t) = W_1(x,t) + \frac{\partial^2 W_1(x,t)}{\partial x^2} + \tilde{f}_2(x,t) + A_0, \\ W_1(x,0) = 0, \qquad 0 \le x \le 1, \\ W_1(0,t) = 0, \qquad W_1(L,t) = 0, \end{cases} \tag{5.6}$$

$$\vdots$$

$$p^k : \begin{cases} D_t^{\alpha} W_k(x,t) = W_k(x,t) + \frac{\partial^2 W_k(x,t)}{\partial x^2} + A_{k-1}, \\ W_k(x,0) = 0, \qquad 0 \le x \le 1, \\ W_k(0,t) = 0, \qquad W_k(L,t) = 0, \end{cases} \tag{5.7}$$

$$\vdots$$

where $\tilde{f}_1(x,t) + \tilde{f}_2(x,t) = \tilde{f}(x,t)$ and

$$\tilde{f}_1(x,t) = \sin(3\pi x)\left(\frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} + t^3(9\pi^2 - 1)\right) \tag{5.8}$$

and

$$\tilde{f}_2(x,t) = \left(t^3 \sin(3\pi x) + t^2 x + 2\right)^2. \tag{5.9}$$

With similar calculation, in subsection 4.1, we obtain a Sturm-Liouville problem and an ordinary linear differential equation respect to x and t respectively. The eigenvalues and eigenfunctions of the Sturm-Liouville problem are

$$\lambda_n = n^2\pi^2, \qquad (X_0)_n(x) = \sin(n\pi x), \quad n = 1, 2, \ldots. \tag{5.10}$$

Furthermore, we have

$$(f_1)_n(t) = \frac{2}{1}\int_0^1 \tilde{f}_1(x,t)\sin(n\pi x)dx = \begin{cases} H(t), & n = 3 \\ 0, & n \neq 3 \end{cases}$$

with

$$H(t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} + t^3(9\pi^2 - 1). \tag{5.11}$$

So

$$(B_0)_n(t) = \begin{cases} \int_0^t E_{\alpha,\alpha}\left((1-n^2\pi^2)\tau^\alpha\right) H(t-\tau)d\tau, & n = 3 \\ 0, & n \neq 3. \end{cases} \tag{5.12}$$

To evaluate $(B_0)_3(t)$, we take laplace transform from both side of (5.12):

$$\begin{aligned} L\left[(B_0)_3(t)\right] &= L\left[\int_0^t \tau^{\alpha-1}E_{(\alpha,\alpha)}((1-9\pi^2)\tau^\alpha)H(t-\tau)d\tau\right] \\ &= L\left[\tau^{\alpha-1}E_{(\alpha,\alpha)}\left((1-9\pi^2)\tau^\alpha\right)\right]L\left[H(t)\right] \\ &= L\left[\sum_{k=0}^{\infty}\frac{1-9\pi^2}{\Gamma(\alpha k+\alpha)}t^{\alpha(k+1)-1}\right]L\left[H(t)\right] \\ &= \left(\sum_{k=0}^{\infty}\frac{(1-9\pi^2)^k}{s^\alpha(k+1)}\right)L\left[H(t)\right] \\ &= \left(\frac{1}{s^\alpha}\sum_{k=0}^{\infty}(\frac{1-9\pi^2}{s^\alpha})^k\right)L\left[H(t)\right] \\ &= \frac{1}{s^\alpha}\frac{1}{1-\frac{1-9\pi^2}{s^\alpha}}\left(\frac{6}{s^{4-\alpha}} + \frac{9\pi^2-1}{s^4}\right) \\ &= \frac{1}{s^\alpha-(1-9\pi^2)}\frac{s^\alpha+9\pi^2-1}{s^4} \\ &= \frac{6}{s^4}. \end{aligned} \tag{5.13}$$

From (5.12) and (5.13), we get

$$(B_0)_n(t) = \begin{cases} t^3, & n = 3, \\ 0, & n \neq 3. \end{cases}$$

Therefore, the solution for (5.18) is

$$\tilde{W}_0(x,t) = t^3\sin(3\pi x).$$

Again by arguments in section 4.1, we have

$$(W)_i(x,t) = 0, \ i = 1, \ 2, \ldots .$$

Then the exact solution for the fractional Fisher's equation given in Example 5.1 is

$$u(x,t) = t^3 \sin(3\pi x) + t^2 x + 2.$$

Example 5.2. Once again, we consider the fractional Fisher's equation

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + u(x,t)\left(1 - u(x,t)\right) + f(x,t), \tag{5.14}$$

with the initial and Neumann boundary conditions as

$$\begin{aligned} &u(x,0) = \cos(5\pi x), \\ &u_x(0,t) = t^3, \qquad u_x(1,t) = 2t^4 + t^3, \quad t \geq 0, \end{aligned} \tag{5.15}$$

and

$$\begin{aligned} f(x,t) = \cos(5\pi x)\Big[&\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha} + 25\pi^2(t^\gamma+1) - (t^\gamma+1) \\ &+ (t^{\gamma+1})^2\cos(5\pi x) + 2t^3(t^{\gamma+1})x + 2t^4(t^{\gamma+1})x^2\Big] \\ &+ \frac{\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} x^2 + \frac{\Gamma(4)}{\Gamma(4-\alpha)} t^{3-\alpha} x - 2t^4 - x^2 t^4 - t^3 x \\ &+ t^6 x^2 + t^8 x^4 + 2t^7 x^3. \end{aligned}$$

Now, if we assume that

$$u(x,t) = \tilde{W}(x,t) + \tilde{V}(x,t) = \tilde{W}(x,t) + t^4 x^2 + t^3 x,$$

we get

$$\begin{cases} D_t^\alpha \tilde{W}(x,t) = \frac{\partial^2 \tilde{W}(x,t)}{\partial x^2} + \tilde{W}(x,t) - (\tilde{W}(x,t) + t^4x^2 + t^3x)^2 + \tilde{f}(x,t), \\ \tilde{W}(x,0) = \cos(5\pi x), \\ \tilde{W}_x(0,t) = 0, \quad \tilde{W}_x(1,t) = 0, \qquad t \geq 0, \end{cases} \tag{5.16}$$

in which

$$\begin{aligned} \tilde{f}(x,t) = \cos(5\pi x)&\left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha} + (25\pi^2 - 1)(t^\gamma + 1)\right] \\ &+ \left[(t^\gamma+1)\cos(5\pi x) + t^4x^2 + t^3x\right]^2. \end{aligned}$$

To solve (5.16) we use MHPM

$$D_t^\alpha \tilde{W}(x,t) = \frac{\partial^2 \tilde{W}(x,t)}{\partial x^2} + \tilde{W}(x,t) + ph(\tilde{W} + \tilde{V}) + \tilde{f}_1(x,t) + p\tilde{f}_2(x,t). \tag{5.17}$$

Therefore if we assume $\tilde{W}(x,t) = \sum_{i=0}^{\infty} \tilde{W}_i p^i$, and substitute it in (5.17), we derive

$$p^0 : \begin{cases} D_t^{\alpha} \tilde{W}_0(x,t) = \frac{\partial^2 \tilde{W}_0(x,t)}{\partial x^2} + \tilde{W}_0(x,t) + \tilde{f}_1(x,t), \\ \tilde{W}_0(x,0) = \cos(3\pi x), \qquad 0 \le x \le L, \\ (\tilde{W}_0)_x(0,t) = 0, \qquad (\tilde{W}_0)_x(L,t) = 0, \end{cases} \tag{5.18}$$

$$p^1 : \begin{cases} D_t^{\alpha} \tilde{W}_1(x,t) = \frac{\partial^2 \tilde{W}_1(x,t)}{\partial x^2} + \tilde{W}_1(x,t) + \tilde{f}_2(x,t) + A_0, \\ \tilde{W}_1(x,0) = 0, \qquad 0 \le x \le L, \\ (\tilde{W}_1)_x(0,t) = 0, \qquad (\tilde{W}_1)_x(L,t) = 0, \end{cases} \tag{5.19}$$

$$\vdots$$

$$p^k : \begin{cases} D_t^{\alpha} \tilde{W}_k(x,t) = \frac{\partial^2 \tilde{W}_k(x,t)}{\partial x^2} + \tilde{W}_k(x,t) + A_{k-1}, \\ \tilde{W}_k(x,0) = 0, \qquad 0 \le x \le L, \\ (\tilde{W}_k)_x(0,t) = 0, \qquad (\tilde{W}_k)_x(L,t) = 0, \end{cases} \tag{5.20}$$

in which we have $\tilde{f}_1(x,t) + \tilde{f}_2(x,t) = \tilde{f}(x,t)$ and $\tilde{f}_1(x,t)$ must be satisfied in initial and boundary conditions (5.18) [8]. We choose

$$\tilde{f}_1(x,t) = \cos(5\pi x)\left[\frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha} + (25\pi^2 - 1)(t^{\gamma}+1)\right],$$
$$\tilde{f}_2(x,t) = \left[(t^{\gamma}+1)\cos(5\pi x) + t^4 x^2 + t^3 x\right]^2.$$

With a similar manner as Example 5.1, we first apply separation method for the corresponding homogeneous equation in (5.18). We obtain the eigenvalue and eigenfunctions of the Sturm-Liouville problem as

$$\lambda_n = n^2\pi^2, \qquad (X_0)_n(x) = \cos(n\pi x), \quad n = 1, 2, \ldots. \tag{5.21}$$

By supposing that

$$\tilde{W}_0(x,t) = \sum_{n=1}^{\infty} (B_0)_n(t)\cos(n\pi x), \tag{5.22}$$

and substituting in (5.18) we derive

$$D_t^{\alpha}(B_0)_n(t) + \left[(n\pi)^2 - 1\right](B_0)_n(t) = (\tilde{f}_1)_n(t), \tag{5.23}$$

so, same as Example 5.1, we have

$$H(t) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} t^{\gamma-\alpha} + (25\pi^2 - 1)(t^{\gamma}+1).$$

Since $\tilde{W}_0(x,t)$ satisfies the initial conditions in (5.18), we have

$$\sum_{n=1}^{\infty} (B_0)_n(0)\cos(n\pi x) = \cos(5\pi x), \tag{5.24}$$

which gives

$$(B_0)_n(0) = \frac{2}{1}\int_0^1 \cos(5\pi x)\cos(n\pi x)dx = \begin{cases} 1, & n = 5, \\ 0, & n \neq 5. \end{cases} \tag{5.25}$$

where

$$(f_1)_n(t) = \frac{2}{1}\int_0^1 \tilde{f}_1(x,t)\cos(n\pi x)dx = \begin{cases} H(t), & n = 5, \\ 0, & n \neq 5. \end{cases}$$

Furthermore Lemma 2.8 implies that

$$\begin{aligned}(B_0)_n(t) &= \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left((1-25\pi^2)\tau^\alpha\right)(\tilde{f}_1)_n(t-\tau)d\tau \\ &+ \cos(5\pi x)((B_0)_n)_0(t) \\ &= \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left((1-25\pi^2)\tau^\alpha\right)\begin{cases} H(t-\tau), & n = 5, \\ 0, & n \neq 5, \end{cases} d\tau \\ &+ 1 + (1-25\pi^2)t^\alpha E_{\alpha,\alpha+1}\left((1-25\pi^2)t^\alpha\right).\end{aligned} \tag{5.26}$$

Now, if we take the Laplace transform from both side of (5.26) we obtain

$$L[(B_0)_n(t)] = 0, \quad n \neq 5$$

and

$$\begin{aligned}L[(B_0)_5(t)] &= L\left[\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left((1-25\pi^2)\tau^\alpha\right)H(t-\tau)d\tau\right] \\ &+ L\left[1 + (1-25\pi^2)t^\alpha E_{\alpha,\alpha+1}\left((1-25\pi^2)t^\alpha\right)\right] \\ &= \frac{1}{s^\alpha - 1 + 25\pi^2} \times L\left[H(t)\right] + \frac{1}{s} \\ &+ (1-25\pi^2)L\left[\sum_{k=0}^{\infty} t^\alpha \frac{(1-25\pi^2)^k t^{\alpha k}}{\Gamma(\alpha k + \alpha + 1)}\right] \\ &= \frac{1}{s^\alpha - 1 + 25\pi^2} \times \left(\frac{\Gamma(\gamma+1)}{s^{\gamma+1-\alpha} + (25\pi^2 - 1)(\frac{1}{s^{\gamma+1}} + \frac{1}{s})}\right) \\ &+ \frac{1}{s} + \frac{1-25\pi^2}{s(s^\alpha - (1-25\pi^2))} \\ &= \frac{1}{s} + \frac{\Gamma(\gamma+1)}{s^{\gamma+1}}.\end{aligned} \tag{5.27}$$

Hence from (5.26) and (5.27), we obtain

$$(B_0)_5(t) = \begin{cases} t^\gamma + 1, & n = 5, \\ 0, & n \neq 5. \end{cases}$$

Thus, the solution for (5.18) with above Neumann boundary conditions takes the form as

$$\tilde{W}_0(x,t) = (t^\gamma + 1)\cos(5\pi x).$$

Hence like as Example 5.1, and by some computational algebra we derive

$$(\tilde{W})_i(x,t) \equiv 0, \quad i = 1, 2, \ldots.$$

Then the analytical solution for the fractional Fisher's equation with given conditions is as follows:

$$u(x,t) = (t^\gamma + 1)\cos(5\pi x) + t^4x^2 + t^3x.$$

Example 5.3. One more time we consider the fractional Fisher's equation as follows

$$D_t^\alpha u(x,t) = u_{xx}(x,t) + u(x,t)\left(1 - u(x,t)\right) + f(x,t), \tag{5.28}$$

with the initial and Robin boundary conditions as

$$\begin{array}{ll} \mathrm{u(x,0) = 0}, & 0 \leq x \leq 1, \\ u(0,t) - \frac{1}{\pi}u_x(0,t) = t^3 - \frac{1}{\pi}t^5, & t \geq 0, \\ u(1,t) - \frac{1}{\pi}u_x(1,t) = (1 - \frac{1}{\pi})t^5 + t^3, & t \geq 0, \end{array} \tag{5.29}$$

and

$$\begin{aligned} f(x,t) = {} & (\cos(\pi x) + \sin(\pi x))\left[\frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha} + \pi^2t^4 - t^4 + 2t^9x + 2t^7\right] \\ & + \frac{\Gamma(6)}{\Gamma(6-\alpha)}t^{5-\alpha}x + \frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} - t^5x - t^3 \\ & + t^8\left(\cos(\pi x) + \sin\pi x\right)^2 + t^{10}x^2 + t^6 + 2t^8x. \end{aligned} \tag{5.30}$$

Next, by assuming

$$\begin{aligned} u(x,t) &= \bar{W}(x,t) + \bar{V}(x,t) \\ &= \bar{W}(x,t) + t^5x + t^3, \end{aligned}$$

we get

$$\begin{cases} D_t^\alpha \bar{W}(x,t) = \frac{\partial^2 \bar{W}(x,t)}{\partial x^2} + \bar{W}(x,t) - (\tilde{W}(x,t) + t^5x + t^3)^2 + \tilde{f}(x,t), \\ \bar{W}(x,0) = 0, \qquad 0 \leq x \leq 1, \\ \bar{W}(0,t) - \frac{1}{\pi}\bar{W}_x(0,t) = 0, \qquad t \geq 0, \\ \bar{W}(1,t) - \frac{1}{\pi}\bar{W}_x(1,t) = 0, \qquad t \geq 0, \end{cases} \tag{5.31}$$

in which

$$\begin{aligned} \tilde{f}(x,t) = {} & (\cos(\pi x) + \sin(\pi x))\left[\frac{\Gamma(5)}{\Gamma(5-\alpha)}t^{4-\alpha} + \pi^2t^4 - t^4 + 2t^9x + 2t^7\right] \\ & + t^8\left(\cos(\pi x) + \sin\pi x\right)^2 + t^{10}x^2 + t^6 + 2t^8x. \end{aligned}$$

To solve (5.31) we use MHPM as

$$D_t^{\alpha}\bar{W}(x,t) = \frac{\partial^2 \bar{W}(x,t)}{\partial x^2} + \bar{W}(x,t) + ph(\bar{W}+\bar{V}) + \tilde{f}_1(x,t) + p\tilde{f}_2(x,t). \tag{5.32}$$

Therefore if we assume $\tilde{W}(x,t) = \sum_{i=0}^{\infty} \tilde{W}_i p^i$, and substitute it in (5.32), we obtain

$$p^0 : \begin{cases} D_t^{\alpha}\bar{W}_0(x,t) = \frac{\partial^2 \bar{W}_0(x,t)}{\partial x^2} + \bar{W}_0(x,t) + \tilde{f}_1(x,t), \\ \bar{W}_0(x,0) = 0, & 0 \le x \le L, \\ \bar{W}_0(0,t) - \frac{1}{\pi}(\bar{W}_0)_x(0,t) = 0, & t \ge 0, \\ \bar{W}_0(1,t) - \frac{1}{\pi}(\bar{W}_0)_x(1,t) = 0, & t \ge 0. \end{cases} \tag{5.33}$$

$$p^1 : \begin{cases} D_t^{\alpha}\bar{W}_1(x,t) = \frac{\partial^2 \bar{W}_1(x,t)}{\partial x^2} + \bar{W}_1(x,t) + \tilde{f}_2(x,t) + A_0, \\ \bar{W}_1(x,0) = 0, & 0 \le x \le L, \\ \bar{W}_1(0,t) - \frac{1}{\pi}(\bar{W}_1)_x(0,t) = 0, & t \ge 0, \\ \bar{W}_1(1,t) - \frac{1}{\pi}(\bar{W}_1)_x(1,t) = 0, & t \ge 0. \end{cases} \tag{5.34}$$

$$\vdots$$

$$p^k : \begin{cases} D_t^{\alpha}\bar{W}_k(x,t) = \frac{\partial^2 \bar{W}_k(x,t)}{\partial x^2} + \bar{W}_k(x,t) + A_{k-1}, \\ \bar{W}_k(x,0) = 0, & 0 \le x \le L, \\ \bar{W}_k(0,t) - \frac{1}{\pi}(\bar{W}_k)_x(0,t) = 0, & t \ge 0, \\ \bar{W}_k(1,t) - \frac{1}{\pi}(\bar{W}_k)_x(1,t) = 0, & t \ge 0, \end{cases} \tag{5.35}$$

in which we have $\tilde{f}_1(x,t) + \tilde{f}_2(x,t) = \tilde{f}(x,t)$ and $\tilde{f}_1(x,t)$ must be satisfied in initial and boundary conditions (5.33) [8]. Here,

$$\tilde{f}_1(x,t) = (\cos(\pi x) + \sin(\pi x)) \left[\frac{\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} + \pi^2 t^4 - t^4 \right].$$

With a similar manner as Examples 5.1 and 5.2, we use separation method for the corresponding homogeneous equation in (5.33). We obtain the eigenvalue and eigenfunction of the Sturm-Liouville problem as follows

$$\lambda_n = n^2\pi^2, \qquad (X_0)_n(x) = \cos(n\pi x) + \sin(n\pi x), \quad n = 1, 2, \ldots. \tag{5.36}$$

By assuming that

$$\bar{W}_0(x,t) = \sum_{n=1}^{\infty} (B_0)_n(t) \left(\cos(n\pi x) + \sin(n\pi x) \right), \tag{5.37}$$

and substituting in (5.33) we derive

$$D_t^{\alpha}(B_0)_n(t) + \left[(n\pi)^2 - 1 \right] (B_0)_n(t) = (\tilde{f}_1)_n(t), \tag{5.38}$$

so, like as Examples 5.1 and 5.2, we have

$$H(t) = \frac{\Gamma(5)}{\Gamma(5-\alpha)} t^{4-\alpha} + (\pi^2 - 1)t^4.$$

Since $\bar{W}_0(x,t)$ satisfies the initial conditions in (5.33), we have

$$\sum_{n=1}^{\infty}(B_0)_n(0)(\cos(n\pi x)+\sin(n\pi x)) = \cos(\pi x)+\sin(n\pi x), \tag{5.39}$$

which gives

$$\begin{aligned}(B_0)_n(0) &= \frac{2}{1}\int_0^1 (\cos(\pi x)+\sin(\pi x))(\cos(n\pi x)+\sin(n\pi x))\,dx \\ &= \begin{cases} 1, & n=1, \\ 0, & n\neq 1, \end{cases}\end{aligned} \tag{5.40}$$

where

$$\begin{aligned}(\tilde{f}_1)_n(t) &= \frac{2}{1}\int_0^1 \tilde{f}_1(x,t)(\cos(n\pi x)+\sin(n\pi x))\,dx \\ &= \begin{cases} H(t), & n=1, \\ 0, & n=1. \end{cases}\end{aligned}$$

Furthermore Lemma 2.8 implies that

$$\begin{aligned}(B_0)_n(t) &= \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left((1-\pi^2)\tau^\alpha\right)(\tilde{f}_1)_n(t-\tau)d\tau \\ &= \int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}\left((1-\pi^2)\tau^\alpha\right)\begin{cases} H(t-\tau), & n=1, \\ 0, & n=1, \end{cases} d\tau.\end{aligned} \tag{5.41}$$

Now, if we take the Laplace transform from both side of (5.41) we obtain

$$L[(B_0)_n(t)] = 0, \quad n\neq 1$$

and

$$\begin{aligned}L[(B_0)_1(t)] &= \frac{24}{s^\alpha-1+\pi^2}\times\frac{s^\alpha+\pi^2-1}{s^5} \\ &= \frac{24}{s^5}.\end{aligned} \tag{5.42}$$

Thus from (5.41) and (5.42), we obtain

$$(B_0)_1(t) = \begin{cases} t^4, & n=1, \\ 0, & n\neq 1. \end{cases}$$

Hence, the solution for (5.33) with above Robin boundary conditions takes the form as

$$\bar{W}_0(x,t) = t^4\left(\cos(\pi x)+\sin(\pi x)\right).$$

Therefore, we derive

$$(\bar{W}_2)_i(x,t)\equiv 0, \quad i=1,2,\ldots.$$

Then the analytical solution for the fractional Fisher's equation with given conditions is as follows:

$$u(x,t) = t^4(\cos(\pi x)+\sin(\pi x))+t^5x+t^3.$$

6. Conclusion

In this article, we obtained analytical solutions for the time-fractional Fisher's nonlinear differential equation. We showed that by choosing proper functions $\tilde{f}_1$ and $\tilde{f}_2$, the solution can be obtained only in one iteration of MHPM. Finally, we illustrated the effectiveness of this method by some examples.

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