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SOME RELATIONSHIPS BETWEEN G-FRAMES AND FRAMES

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Abstract. In this paper we proved that every g-Riesz basis for Hilbert space H with respect to K by adding a condition is a Riesz basis for Hilbert $B(K)$-module $B(H,K)$. This is an extension of [A. Askarizadeh, M. A. Dehghan, *G-frames as special frames*, Turk. J. Math., 35, (2011) 1-11]. Also, we derived similar results for g-orthonormal and orthogonal bases. Some relationships between dual frame, dual g-frame and exact frame and exact g-frame are presented too.

1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [3] for study of nonharmonic Fourier series. They are reintroduced and developed in 1986 by Daubechies, Grossmann and Meyer [2], and popularized from then on. If H is a Hilbert space, and I a set which is finite or countable. A system $\{f_i\}_{i\in I} \subseteq H$ is called a frame for H if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} | \langle f, f_i\rangle |^2 \leq B\|f\|^2, \tag{1.1}$$

for all $f \in H$. The constants A and B are called frame bounds. If $A = B$ we call this frame a tight frame and if $A = B = 1$ we call it Parseval frame.

In [6], Sun introduced a generalization of frames and showed that this includes more other cases of generalizations of frame concept and proved that many basic properties can be derived within this more general context. Another generalization of frames in Hilbert spaces is frame

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in Hilbert C*-module [4]. In [1], it is proved that every g-frame in Hilbert space H with respect to Hilbert space K is a frame for Hilbert $B(K)$-module $B(H,K)$ and vice versa. Also, it is shown that every g-Riesz(g-orthogonal basis) in Hilbert space H with respect to Hilbert space K is a Riesz(orthogonal basis) for Hilbert $B(K)$-module $B(H,K)$ but the inverse is not valid. In this paper, we proved that by adding some conditions the both side of them are valid. Also, we investigated the relationships between dual frames.

2. Preliminaries

Definition 2.1. Let U and V be two Hilbert spaces and $\{V_i : i \in I\}$ is a sequence of subspaces of V, where I is a subset of $\mathbf{Z}$. $B(U, V_i)$ is the collection of all bounded linear operators from U into V_i . We call sequence $\{\Lambda_i \in B(U, V_i) : i \in I\}$ a generalized frame, or simply a g-frame, for U with respect to $\{V_i : i \in I\}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leq \sum_{i\in I} \|\Lambda_i f\|^2 \leq B\|f\|^2, \tag{2.1}$$

for all $f \in U$. The constants A and B are called g-frame bounds. If $A = B$ we call this g-frame a tight g-frame and if A = B =1 we call it Parseval g-frame.

Definition 2.2. Let $\{\Lambda_i : i \in I\}$ be a sequence in $B(H,K)$,

(i) $\{\Lambda_i : i \in I\}$ is called a g-orthonormal basis for H with respect to K if $\langle \Lambda_i^* f, \Lambda_i^* g\rangle = \delta_{i,j}\langle f, g\rangle$ for any $i, j \in I$ and $f, g \in K$ and also
$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2,$$
for any $f \in H$.

(ii) If $\{f : \Lambda_i f = 0,\ i \in I\} = \{0\}$, then $\{\Lambda_i : i \in I\}$ is called g-complete.

(iii) $\{\Lambda_i : i \in I\}$ is called a g-Riesz basis for H with respect to K if it is g-complete and there are positive constants A and B such that
$$A\sum_{i\in I_1} \|g_i\|^2 \leq \left\|\sum_{i\in I_1} \Lambda_i^* g_i\right\|^2 \leq B\sum_{i\in I_1} \|g_i\|^2,$$
for any finite subset I_1 of I and $\{g_i\}_{i\in I_1} \subseteq K$.

In [7] Zhu proved the following characterization for g-Riesz bases.

Theorem 2.3 ([7]). *Let $\{\Lambda_i : i \in I\}$ is a sequence in $B(H,K)$. Then the following two statements are equivalent:*

(i) *The sequence* $\{\Lambda_i : i \in I\}$ *ia a g-Riesz basis for* H *with respect to* K *with bounds* A *and* B*.*
(ii) *The sequence* $\{\Lambda_i : i \in I\}$ *ia a g-frame for* H *with respect to* K *with bounds* A *and* B*, and* $\{\Lambda_i : i \in I\}$ *is an* $\ell^2(K)$*-linearly independent family, i.e., if* $\sum_{i\in I} \Lambda_i^* g_i = 0$ *for* $\{g_i\}_{i\in I} \in \ell^2(K)$*, then* $g_i = 0$ *for all* $i \in I$*.*

Definition 2.4. Let A be a C*-algebra with the involution *. An inner product A-module (or pre Hilbert A-module) is a complex linear space H which is a left A-module with map $\langle \cdot, \cdot \rangle : H \times H \to A$ which satisfies the following properties:

1) $\langle \alpha f + \beta g, h \rangle = \alpha < f, h > +\beta < g, h >$ for all $f, g, h \in H$ and $\alpha, \beta \in \mathbf{C}$,
2) $\langle af, g \rangle = a\langle f, g \rangle$ for all $f, g \in H$ and $a \in A$,
3) $\langle f, g \rangle = \langle g, f \rangle^*$ for all $f, g \in H$,
4) $\langle f, f \rangle \geq 0$ for all $f \in H$ and $\langle f, f \rangle = 0$ iff $f = 0$.

For $f \in H$, we define a norm on H by $\|f\|_H = \|\langle f, f \rangle\|_A^{1/2}$. If H is complete with this norm, it is called a Hilbert C*-module over A or a Hilbert A-module.

An element a of a C*-algebra A is positive if $a^* = a$ and its spectrum is a subset of positive real number. We write $a \geq 0$ to mean that a is positive. It is easy to see that $\langle f, f \rangle \geq 0$ for every $f \in H$, hence we define $|f| = \langle f, f \rangle^{1/2}$.

Frank and Larson [4] defined the frames, orthogonal bases and Riesz bases in Hilbert C*-modules.

Definition 2.5. Let A be a unital C*-algebra. A sequence $\{x_i\}_{i\in I}$ of elements in Hilbert A-module H is called a frame for H if there exist two constants $A, B > 0$, such that

$$A\langle x, x \rangle \leq \sum_{i\in I} \langle x, x_i \rangle \langle x_i, x \rangle \leq B\langle x, x \rangle,$$

for every $x \in H$.

Definition 2.6. A sequence $\{x_i\}_{i\in I}$ in a Hilbert A-module H is called an orthogonal basis for H if it is a generating set (i.e., the A-linear hull of $\{x_i\}_{i\in I}$ is weak-dense in H) such that

(i) $\langle x_i, x_j \rangle = 0$ for each $i \neq j$,
(ii) $\|x_i\| = 1$ for each $i \in I$,
(iii) the A-linear combinations $\sum_{i\in S} a_i x_i$ with coefficients $\{a_i : i \in S\} \subseteq A$ and $S \subseteq I$ are equal to zero if and only if every summand $a_i x_i$ is equal to zero, $i \in S$.

Definition 2.7. A sequence $\{x_i\}_{i\in I}$ in a Hilbert A-module H is called a Riesz basis for H if it is a generating set with the additional property that A-linear combinations $\sum_{i\in S} a_i x_i$ with coefficients $\{a_i : i \in S\} \subseteq A$ and $S \subseteq I$ are equal to zero if and only if every summand $a_i x_i$ is equal to zero, $i \in S$.

For Hilbert spaces H and K, the Banach space $B(H,K)$ of all bounded linear operators from H into K is a Hilbert $B(K)$-module.

Askarizadeh and Dehghan in [1] proved that a sequence of operators in $B(H,H)$ is a g-frame for H with respect to K if and only if it is a frame for $B(H,K)$ considered as a Hilbert C*-module.

Theorem 2.8 ([1])**.** *Let $\{\Lambda_i \in B(H,K) : i \in I\}$ be a sequence in $B(H,K)$. Then it is a frame for $B(H,K)$ considered as a Hilbert C*-module if and only if it is a g-frame for H with respect to K.*

3. Main results

In [1], it is proved that every g-orthogonal basis in Hilbert space H with respect to Hilbert space K is an orthogonal basis for Hilbert $B(K)$-module $B(H,K)$ but the inverse is not valid. In the next theorem we proved that by adding some conditions the inverse is valid. Also see Corollary 2.13 in [5].

Theorem 3.1. *Let $\{\Lambda_i \in B(H,K) : i \in I\}$ is an orthogonal basis for $B(H,K)$ as a Hilbert C*-module. If $\Lambda_i\Lambda_i^* = I_K$ for any $i \in I$, then $\{\Lambda_i : i \in I\}$ is a g-orthonormal basis for H with respect to K.*

Proof. We must show that

$$\langle \Lambda_i^* g, \Lambda_j^* h\rangle = \delta_{i,j}\langle g, h\rangle,$$

and

$$\sum_{i\in I} \|\Lambda_i f\|^2 = \|f\|^2,$$

for any $g, h \in K$ and $f \in H$.
By Definition 2.6 $\langle \Lambda_i f, \Lambda_j f\rangle = 0$ for all $f \in H$ and $i \neq j$ and therefore $\Lambda_i\Lambda_j^* = 0$. Hence we obtain

$$\begin{aligned}\langle \Lambda_i^* g, \Lambda_j^* h\rangle &= \langle g, \Lambda_i\Lambda_j^* h\rangle \\ &= \langle g, 0\rangle = 0, \quad \forall g, h \in K.\end{aligned}$$

Now, by assumptions

$$\begin{aligned}\langle \Lambda_i^* g, \Lambda_i^* h\rangle &= \langle g, \Lambda_i\Lambda_i^* h\rangle \\ &= \langle g, h\rangle, \quad \forall g, h \in K.\end{aligned}$$

We prove $\sum_{i\in I}\|\Lambda_i f\|^2=\|f\|^2$ for any $f\in H$. By Definition 2.6 for any $T\in B(H,K)$ there exists $\{U_i\in B(K):i\in I\}$ such that $T=\sum_{i\in I}U_i\Lambda_i$. Fix $i_0\in I$, then we have

$$\begin{aligned}\langle T,\Lambda_{i_0}\rangle &= \left\langle \sum_{i\in I}U_i\Lambda_i,\Lambda_{i_0}\right\rangle\\ &=\sum_{i\in I}U_i\ \langle\Lambda_i,\Lambda_{i_0}\rangle\\ &=U_{i_0}\langle\Lambda_{i_0},\Lambda_{i_0}\rangle=U_{i_0}.\end{aligned}$$

Hence we obtain

$$\begin{aligned}T&=\sum_{i\in I}\langle T,\Lambda_i\rangle\Lambda_i\\ &=\sum_{i\in I}T\Lambda_i^*\Lambda_i\\ &=T\sum_{i\in I}\Lambda_i^*\Lambda_i.\end{aligned}$$

It follows that for any $f\in H$

$$\begin{aligned}\|f\|^2&=\langle f,f\rangle\\ &=\left\langle\sum_{i\in I}\Lambda_i^*\Lambda_i f,f\right\rangle\\ &=\sum_{i\in I}\langle\Lambda_i^*\Lambda_i f,f\rangle\\ &=\sum_{i\in I}\langle\Lambda_i f,\Lambda_i f\rangle\\ &=\sum_{i\in I}\|\Lambda_i f\|^2.\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let $\{\Lambda_i\in B(H,K):i\in I\}$ be a Riesz basis for $B(H,K)$ considered as a Hilbert C^*-module and $\langle\Lambda_i^*f,\Lambda_j^*g\rangle=\delta_{i,j}\langle f,g\rangle$ for any $f,g\in K$ and $i,j\in I$. Then $\{\Lambda_i:i\in I\}$ is a g-Riesz basis for H with respect to K.*

Proof. With regarding to Theorem 2.3 and 2.8 it is enough to prove if

$$\sum_{i\in I}\Lambda_i^*g_i=0$$

for any $\{g_i\}_{i\in I} \in \ell^2(K)$, then $g_i = 0$ for all $i \in I$.
Fix $i_0 \in I$, then we have

$$\begin{aligned} 0 &= \left\langle \sum_{i\in I} \Lambda_i^* g_i, \Lambda_{i_0}^* g_{i_0} \right\rangle \\ &= \sum_{i\in I} \langle \Lambda_i^* g_i, \Lambda_{i_0}^* g_{i_0} \rangle \\ &= \langle g_{i_0}, g_{i_0} \rangle \\ &= \|g_{i_0}\|^2. \end{aligned}$$

Hence $g_{i_0} = 0$. It follows that $g_i = 0$ for all $i \in I$. □

Remark 3.3 ([1])**.** Let $\{\Lambda_i \in B(H,K) : i \in I\}$ be a g-frame for H. The g-frame operator of $\{\Lambda_i : i \in I\}$ is defined by

$$S_g : H \to H, \quad f \mapsto \sum_{i\in I} \Lambda_i^* \Lambda_i f.$$

Also, the frame operator of the frame $\{\Lambda_i : i \in I\}$ is defined by

$$\begin{aligned} ST &= \sum_{i\in I} \langle T, \Lambda_i \rangle \Lambda_i \\ &= \sum_{i\in I} T \Lambda_i^* \Lambda_i. \end{aligned}$$

Therefore, $ST = TS_g$, and from this equation, for any $T \in B(H,K)$ a reconstruction formula is derived by $T = S^{-1} T S_g$.

Next proposition says that canonical dual for both cases are equal.

Proposition 3.4. *Let $\{\Lambda_i \in B(H,K) : i \in I\}$ be a frame for $B(H,K)$. Suppose $\{\Gamma_i \in B(H,K) : i \in I\}$ and $\{\Theta_i \in B(H,K) : i \in I\}$ be the canonical dual frame for $B(H,K)$ and canonical dual g-frame for H with respect to K. Then $\Gamma_i = \Theta_i$ for all $i \in I$.*

Proof. Let S and S_g are respectively frame operator and g-frame operator of $\{\Lambda_i : i \in I\}$. Then we have $\Gamma_i = S^{-1}\Lambda_i$ and $\Theta_i = \Lambda_i S_g^{-1}$.
Now, by Remark 3.3, $ST = TS_g$ or equivalently $TS_g^{-1} = S^{-1}T$ for any $T \in B(H,K)$. It implies that $\Gamma_i = \Theta_i$ for all $i \in I$. □

Corollary 3.5. *Let $\{\Lambda_i \in B(H,K) : i \in I\}$ be an operator sequence in $B(H,K)$. Then it is an exact frame for $B(H,K)$ considered as a Hilbert C^*-module if and only if it is an exact g-frame for H with respect to K.*

Proof. By the Theorem 2.8. □

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