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CONVERGENCE ANALYSIS OF PRODUCT INTEGRATION METHOD FOR NONLINEAR WEAKLY SINGULAR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS

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Abstract. In this paper, we studied the numerical solution of nonlinear weakly singular Volterra-Fredholm integral equations by using the product integration method. Also, we shall study the convergence behavior of a fully discrete version of a product integration method for numerical solution of the nonlinear Volterra-Fredholm integral equations. The reliability and efficiency of the proposed scheme are demonstrated by some numerical experiments.

1. Introduction

Integral equations are extensively used to solve the problems of economy, ecology, medicine as well as those of physics and engineering [12], [11]. In the last few years, much consideration has been given to numerical methods for these equations. For detailed studies of approximate methods for Volterra and Fredholm integral equations including weak singularity see, e.g., [1]-[15] and the references therein. Integral equations with weakly singular kernels arise in many modelling problems in mathematical physics, and chemical reactions, such as stereology, heat conduction, crystal growth, electrochemistry, superfluidity [12], [11], the radiation of heat from a semi infinite solid [8], and many other practical applications. We remark here that equations of this type have been the focus of many papers [1]-[15] in recent years.

The product integration method was originally introduced to handle approximations based on numerical integration of the integral equations.

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The resulting solution is found first at the set of quadrature node points, and then is extended to all points by means of a special, and generally quite accurate, interpolation formula. The numerical method is much simple to implement on a computer, but the error analysis is more sophisticated than some numerical methods.

In this paper, we shall study the convergence behavior of a fully discrete version of a product integration method for numerical solution of the nonlinear Volterra-Fredholm integral equations of the form

$$\begin{aligned} y(t) &= f(t) + \lambda_1 \int_0^t p_1(t,s)k_1(t,s,y(s))ds \\ &+ \lambda_2 \int_0^T p_2(t,s)k_2(t,s,y(s))ds, \quad t \in I = [0,T], \end{aligned} \tag{1.1}$$

where $k_1 \in C(D \times \mathbb{R})$, $k_2 \in C(I \times I \times \mathbb{R})$ and $D = \{(t,s) : \ 0 \le s \le t \le T\}$ and $p_r : \Delta \to \mathbb{R}$, $f : I \to \mathbb{R}$ are some continuous functions, and $\lambda_r, \ r = 1,2$ denotes (real or complex) parameters. Here

$$\Delta := \{(t,s) : \ 0 \le t \le T, \ 0 \le s \le T, \ t \ne s\}.$$

The kernels $p_r(t,s)$, $r = 1,2$ are weakly singular and $p_r(t,s)$ is given function. Typical forms of $p_r(t,s)$, $r = 1,2$ are

(i) $p_r(t,s) = |t-s|^{-\alpha}, \quad 0 < \alpha < 1,$

(ii) $p_r(t,s) = \log|t-s|.$

We assume that there exist some constants $c > 0$ and $0 < \nu < 1$ so that

$$| \ p_r(t,s) \ | \le c \ | \ t-s \ |^{-\alpha}, \quad (t,s) \in \Delta. \tag{1.2}$$

Moreover, we assume that problem (1.1) is uniquely solvable and its solution $y(t)$ is a continuous function.

The numerical treatment of (1.1) is not simple, because, as it is well known, the solutions of weakly singular integral equations usually have a weak singularity, even when the inhomogeneous term is regular. Specific methods have been proposed by several authors for equations with smooth solution and nonsmooth solution [1]-[15].

It is well known that for integral equations with bounded kernels, the smoothness of the kernel and the forcing function $f(x)$ determines the smoothness of the solution on the closed interval $[0,T]$, with $T > 0$. If we allow weakly singular kernels, then the resulting solutions are typically non-smooth at the initial point of the interval of integration, where their derivatives become unbounded. Some results concerning the behavior of the exact solution of equations of type (1.1) are given in [3]. The numerical solvability of weakly singular integral equations and other related equations have been pursued by several authors.

In the present paper, we further develop the works carried out in [2], [14] and [16]. Also, we provide a new strategy of product integration algorithm for numerical solution of weakly singular equation (1.1). The offered disceretization scheme uses Gaussian quadratures based on the new orthogonal polynomials which proposed in [4]. Using the roots of new orthogonal polynomials which will be introduced, we will show that a more exact quadrature can be obtained by which the integral part of (1.1) can be well-approximated even in the nonlinear case.

The layout of this paper is as follows. In sec.2, we are briefly introduced the new families of orthogonal polynomials and a Gauss-type numerical quadrature which has presented in [6]. Also, product integration methods for initial value problems which have singular points are introduced and applied to a variety of problems arising in engineering, and physics. In sec.3 the convergence analysis of the proposed method is investigated and finally in sec.4 some numerical experiments are reported to clarify the method and some comparisons are made with existing methods in the literature.

2. Preliminaries

For convenience of the reader, we will present a review of the Chelyshkov polynomials [4]. Then by the use of this polynomials, we developed the standard product integration method, for solving the equation (1.1).

Recently, Chelyshkov has introduced sequences of polynomials in [4], which are orthogonal over the interval $[0, 1]$ with the weight function $\omega(t) = 1$. These polynomials are explicitly defined by

(2.1)
$$P_{mk}(t) = \sum_{j=0}^{m-k} (-1)^j \binom{m-k}{j} \binom{m+k+1+j}{m-k} t^{k+j}, \quad k = 0, 1, \dots, m.$$

The polynomials have properties, which are analogous to the properties of the classical orthogonal polynomials. Since m is fixed, by orthogonality property in $[0, 1]$, the polynomials $P_{mk}(t)$ can be immediately connected to a fixed set of the Jacobi polynomials $P_m^{(\alpha,\beta)}(t)$ [4]. Precisely

$$P_{mk}(t) = (-1)^{m-k} t^k P_{n-k}^{(0,2k+1)}(2t - 1).$$

The orthogonalization procedure is only the starting point for examining polynomials. Realizing the procedure, one can suppose that the explicit definition of the polynomials $P_{mk}(t)$ is defined by (2.1). This yields the Rodrigues' type representation,

$$P_{mk}(t) = \frac{1}{(m-k)!} \frac{1}{t^{k+1}} \frac{d^{m-k}}{dt^{m-k}} \left(t^{m+k+1}(1-t)^{m-k} \right), \quad k = 0, 1, \dots, m.$$

Investigating more on (2.1), we deduced that in the family of orthogonal polynomials $\{P_{mk}(t)\}_{k=0}^{m}$ every member has degree m with $m-k$ simple roots. Hence, for every m the polynomial $P_{m0}(t)$ has exactly m simple roots in $(0,1)$. Following [4], it can be shown that the sequence of polynomials $\{P_{m0}(t)\}_{m=0}^{\infty}$ generate a family of orthogonal polynomials on $[0,1]$ which possesses all the properties of other classic orthogonal polynomials, e.g. Legendre or Chebyshev polynomials. Therefore, if the roots of $P_{m0}(t)$ are chosen as collocation points, then we can obtain an accurate numerical quadrature [4].

Now, consider the product rules

$$\begin{aligned}\int_0^1 p(t,s)f(s)ds &\approx \sum_{i=1}^{m} w_{m,i}(t)f(t_{m,i})\\ &= I_N(f;t),\end{aligned} \tag{2.2}$$

with $p(t,s) = \mid t-s \mid^{\nu},\ \nu > -1$, or $p(t,s) = \log \mid t-s \mid$, of interpolatory type, based on the zeros of some classes of generalized Jacobi orthogonal polynomials.

The error term of (2.2), is as follows

$$R_N(f;t) = \int_0^1 p(t,s)f(s)ds - I_N(f;t). \tag{2.3}$$

Following [13], we know that $R_N(f) = O(N^{-n})$ where $f \in C^n[0,1]$. Now, we suppose that the function f be a weakly singular function in end point, e.g. in $t = 1$, it means $f(t) = (1-t)^{\sigma},\ \sigma > -1$. It is clear that if f be a polynomial of degree N then $R_N(f;t) = 0$, so as defined in [5], we can write for all polynomial P_N of degree N

$$R_N(f;t) = \int_0^1 p(t,s)[f(s) - P_N(s)]ds - \int_0^1 p(t,s)L_N(f-P_N;s)ds,$$

where $L_N(f;t)$ is the Lagrange interpolation polynomial which interpolates $f(t)$ in the points $\{t_i\}_{i=0}^N$, and is given by

$$L_N(f;t) = \sum_{i=0}^{N} f(t_i)l_{N,j}(t),$$

and the fundamental polynomial $l_{N,j}(t)$ is in the form

$$l_{N,j}(t) = \prod_{\substack{i=0\\ i\neq j}}^{N} \frac{t-t_i}{t_j-t_i}, \quad j = 0,1,\ldots,N. \tag{2.4}$$

By a proper choice of the sequence of polynomial $\{P_N\}$, we will be able to derive an upper bounds for the two terms

$$R_{1,N}(f;t) = \int_0^1 \mid p(t,s) \mid\mid f(s) - P_N(s) \mid ds,$$

$$R_{2,N}(f;t) = \int_0^1 \mid p(t,s) \mid\mid L_N(f - P_N; s) \mid ds,$$

where $p(t,s)$ is the weakly singular kernel of type (i) or (ii).

Theorem 2.1 ([16])**.** *Let $f(t) = (1-t)^\sigma$, $\sigma > -1$ (not an integer) and $\nu > -1$, with $\sigma + \nu > -1$, then*

$$\int_0^1 \mid f(t) - L_N(f;t) \mid\mid t - s \mid^\nu dt \le C \begin{cases} N^{-2-2\sigma-2\nu} \log N, & \mid s \mid \le 1, \nu < 0, \\ N^{-2-2\sigma} \log N, & \mid s \mid \le 1, \nu \ge 0, \end{cases} \tag{2.5}$$

where C is constant and independent of s and N.

Corollary 2.2 ([16])**.** *Let $f(t) = (1-t)^\sigma$, $\sigma > -1$ and not integer, then we have*

$$\int_0^1 \mid f(t) - L_N(f;t) \mid\mid \log \mid t - s \mid\mid dt \le C \begin{cases} N^{-2-2\sigma} \log^2 N, & \mid s \mid \le 1, \\ N^{-2-2\sigma} \log N, & 0 \le s < 1, \end{cases} \tag{2.6}$$

where C is constant and independent of s and N.

3. Product integration method

To achieve our goal, we consider the nonlinear Volterra-Fredholm integral equations of the form

$$\begin{aligned} y(t) = g(t) + \lambda_1 &\int_0^t p_1(t,s) k_1(t,s,y(s)) ds \\ &+ \lambda_2 \int_0^T p_2(t,s) k_2(t,s,y(s)) ds, \quad t \in I = [0,T], \end{aligned} \tag{3.1}$$

where $k_1 \in C(D \times \mathbb{R})$, $k_2 \in C(I \times I \times \mathbb{R})$ and $D = \{(t,s) : 0 \le s \le t \le T\}$ and $p_r : \Delta \to \mathbb{R}$, $f : I \to \mathbb{R}$ are some continuous functions, and λ_r, $r = 1,2$ denotes (real or complex) parameters. Here

$$\Delta := \{(t,s) : 0 \le t \le T, 0 \le s \le T, t \ne s\}.$$

If we use the $(N+1)$-point quadrature rule and collocate (3.1) at the nodes $\{t_i\}_{i=1}^N \bigcup \{t_0 = 0\}$, then we have

$$\begin{aligned} y(t_i) = g(t_i) + \lambda_1 &\int_0^{t_i} p_1(t_i,s) k_1(t_i,s,y(s)) ds \\ &+ \lambda_2 \int_0^T p_2(t_i,s) k_2(t_i,s,y(s)) ds. \end{aligned} \tag{3.2}$$

For approximating the integral terms, where $p_r(t,s)$ is a weakly singular kernel, we use the Lagrange interpolating polynomial to approximate

$k_r(t_i, s, y(s))$ as

$$L_N(k_r; s) := \sum_{j=0}^{N} l_{N,j}(s) k_r(t_i, t_j, y_N(t_j)), \quad i = 0, 1, \dots, N, \tag{3.3}$$

with

$$l_{N,j}(t) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{t - t_j}{t_i - t_j}, \quad j = 0, 1, \dots, N.$$

Now, by substituting (3.3) in the integral terms of equation (3.2), we have

$$\int_0^{t_i} p_1(t_i, s) k_1(t_i, s, y(s)) ds = \sum_{j=0}^{N} \int_0^{t_i} p_1(t_i, s) l_{Nj}(s) k_1(t_i, t_j, y_N(t_j)) ds,$$

and

$$\int_0^{T} p_1(t_i, s) k_2(t_i, s, y(s)) ds = \sum_{j=0}^{N} \int_0^{T} p_2(t_i, s) l_{Nj}(s) k_2(t_i, t_j, y_N(t_j)) ds.$$

Defining

$$W_{i,j} := \int_0^{t_i} p_1(t_i, s) l_{Nj}(s) ds, \tag{3.4}$$

and

$$V_{i,j} := \int_0^{T} p_2(t_i, s) l_{Nj}(s) ds, \tag{3.5}$$

the product integration method on nods $\{t_i\}_{i=1}^N \bigcup \{t_0 = 0\}$ gives

$$\begin{aligned} y_N(t_i) = g(t_i) + \sum_{j=0}^{N} [&\lambda_1 W_{i,j} k_1(t_i, t_j, y_N(t_j)) \\ &+ \lambda_2 V_{i,j} k_2(t_i, t_j, y_N(t_j))], \quad i = 0, 1, \dots, N, \end{aligned} \tag{3.6}$$

where $\{t_i\}_{i=1}^N$ are the roots of Nst-degree orthogonal polynomial $P_{N0}(t)$ and $W_{i,j}$, $V_{i,j}$ are the weight coefficients which can be obtained from (3.4) and (3.5).

Using these notations, we can summarize the procedure in the following algorithm:

Note that, the equation (3.6) is a $(N+1)\times(N+1)$ nonlinear system of equations which has a unique solution [6]. Solving this nonlinear system, determines the values of $y(t_i), i = 0, 1, \dots, N$ which are the solutions of (3.1) in the points $\{t_i\}_{i=1}^N \bigcup \{t_0 = 0\}$.

Algorithm 1.

Input:
Number of mesh points N;
begin
$t_0 = 0$;
For i=1,2,...,N:
Compute t_i simple roots of $P_{m,0}(t)$ from (2);
For j=0,1,...,N:
Compute W_{ij} from (3.4);
Compute V_{ij} from (3.5);
Compute $y_N(t_i)$ from (3.6) and solve this nonlinear system;
end.

In order to obtain the numerical solution in any arbitrary point $\eta \in I_h \subseteq [0,1]$, we are concerned with a rule which depends on β and $y(\eta)$ as follows

$$\int_0^1 y(s)ds \approx \sum_{j=0}^{N} \beta_j y(t_j) + \beta y(\eta). \tag{3.7}$$

Clearly, this quadrature is exact for polynomials of degree $\leq 2N$ (see [9] and [10]). However, in the case of quadrature rules which include among their nodes, in addition the point η as a collocation points, we will obtain $(N+1) \times (N+1)$ nonlinear system of equations whose solutions give the value of any of our grid points especially at the point $t_{N+1} = \eta$, [16].

4. Convergence Analysis

Without loss of generality, in our convergence analysis we examine the linear test equation

$$y(t) = g(t) + \int_0^t p_1(t,s)y(s)ds + \int_0^1 p_2(t,s)y(s)ds. \tag{4.1}$$

where $p_r : \Delta \to \mathbb{R}$, $g : I \to \mathbb{R}$ are some continuous functions, and $\lambda_r, r = 1,2$ denotes (real or complex) parameters. Here

$$\Delta := \{(t,s) : \ 0 \leq t \leq T, 0 \leq s \leq T, t \neq s\}.$$

If, for a given mesh $\{t_i\}_{i=1}^N \bigcup \{t_0 = 0\}$, we apply the method (3.6) to the test equation (4.1), we obtain $y_N(t)$ as an approximate solution as

$$y_N(t) = g(t) + \sum_{j=0}^{N} [W_j(p_1;t) + V_j(p_2;t)] y_N(t_j),$$

where

$$W_j(p_1;t) = \int_0^t p_1(t,s) l_{Nj}(s) ds, \qquad V_j(p_2;t) = \int_0^1 p_2(t,s) l_{Nj}(s) ds.$$

In order to examine the uniform convergence of the approximate solution $y_N(t)$ to the exact solution $y(t)$ of (4.1), notice that

$$y(t) - y_N(t) = \sum_{j=0}^{N} W_j^*(p_1,p_2;t)[y(t_j) + y_N(t_j)] + t_N(p_1,p_2,y;t),$$

where

$$\begin{aligned} t_N(p_1,p_2,y;t) &= t_{1N}(p_1,y;t) + t_{2N}(p_2,y;t), \\ W_j^*(p_1,p_2;t) &= W_j(p_1,t) + V_j(p_2,t), \end{aligned} \tag{4.2}$$

and $t_{1N}(p_1,y;t)$ and $t_{2N}(p_2,y;t)$ are the local truncation errors defined by

$$\begin{aligned} t_{1N}(p_1,y;t) &= \int_0^t p_1(t,s) y(s) ds - \sum_{j=0}^{N} W_j(p_1,t) y(t_j), \\ t_{2N}(p_2,y;t) &= \int_0^1 p_2(t,s) y(s) ds - \sum_{j=0}^{N} V_j(p_2,t) y(t_j). \end{aligned} \tag{4.3}$$

Hence, we obtain

$$\| y(t) - y_N(t) \|_\infty \leq \|(I - T_N)^{-1}\|_\infty \|t_N\|_\infty, \tag{4.4}$$

where T_N is the linear operator from $C([0,1])$ into $C([0,1])$, defined by

$$T_N y(t) = \sum_{j=0}^{N} W_j^*(p_1,p_2;t) y(t), \quad y \in C([0,1]), \quad t \in [0,1].$$

Our final goal is to determine an upper bound for $\| y(t) - y_N(t) \|_\infty$. For this purpose, firstly we recall the following auxiliary lemmas regarding kernels of type (i) and (ii) from [14].

Theorem 4.1. *(From* [14] *) Let* $\{t_j\}_{j=0}^N$ *be the zeros of* $(N+1)$ *st-degree member of a set of polynomials that are orthogonal on* $[-1,1]$ *with respect to the wight function*

$$\omega(s) = u(s)(1-s)^{\bar{\alpha}}(1+s)^{\bar{\beta}}, \quad -1 < \bar{\alpha} \leq \frac{3}{2}, \quad \bar{\beta} \geq -\frac{1}{2}.$$

Here, $u(t)$ *is positive and continuous in* $[-1,1]$ *and the modulus of continuity* φ *of* u *satisfies*

$$\int_0^1 \varphi(u,\delta) \frac{d\delta}{\delta} < \infty.$$

Let $L_N(y;s)$ denote the interpolating polynomial of degree $\leq N$ that coincides with the function y at the nodes $\{t_j\}_{j=0}^N$. Moreover, suppose $p_r(t,s),\ r=1,2$ are kernels of type (i) or (ii). Then, for every function y containing only endpoint singularity and in particular for every function $y \in C([-1,1])$, there holds

$$\lim_{N\to\infty} \| \int_{-1}^{t} p_r(t,s)\{y(s) - L_N(y;s)\}ds\| = 0.$$

In particular, for $0 < \alpha < 1$, we have the bounds

$$\|t_{1N}(|t-s|^{-\alpha}, y; t)\|_\infty = O\{(N+1)^{-2-2\sigma+2\alpha}\log(N+1)\},$$
$$\|t_{1N}(\log|t-s|, y; t\|_\infty = O\{(N+1)^{-2-2\sigma}\log^2(N+1)\}.$$

By using the Equation (4.3), Theorem 2.1 and Corollary 2.2, we have

$$\begin{aligned} \mid t_{1N}(p_1,y;t) \mid &\leq \int_0^t \mid p_1(t,s) \mid\mid y(s) - L_N(y;s) \mid ds \\ &\leq \int_0^1 \mid p_1(t,s) \mid\mid y(s) - L_N(y;s) \mid ds, \\ \mid t_{2N}(p_2,y;t) \mid &\leq \int_0^1 \mid p_2(t,s) \mid\mid y(s) - L_N(y;s) \mid ds. \end{aligned} \tag{4.5}$$

Hence, from (1.2) and (4.5) we have

$$\begin{aligned} \mid t_N(p_1,p_2,y;t) \mid &\leq \mid t_{1N}(p_1,y;t) \mid + \mid t_{2N}(p_2,y;t) \mid \\ &\leq \int_0^1 \{\mid p_1(t,s) \mid + \mid p_2(t,s) \mid\} \mid y(s) - L_N(y;s) \mid ds \\ &\leq C\int_0^1 \mid t-s \mid^{-\alpha}\mid y(s) - L_N(y;s) \mid ds. \end{aligned} \tag{4.6}$$

Now, form [14], Theorem 4.1 and Equation (4.6), we have

$$\|t_N(p_1(t,s), p_2(t,s), y; t)\|_\infty = O\{h\}, \tag{4.7}$$

where

$$h = \min\{(N+1)^{-2-2\sigma+2\alpha}\log(N+1), (N+1)^{-2-2\sigma}\log^2(N+1)\}.$$

It should be noted that our main concern is estimation of (4.4). The behavior of tern $\| (I - T_m)^{-1} \|_\infty$ in (4.4) has been investigated under some assumptions in [14]. Based on this idea, we can establish the boundedness of the $\| (I - T_m)^{-1} \|_\infty$.

Finally, we can state the following results.

Theorem 4.2. *Let $t_0 = 0$ and the nodes $\{t_i\}_{i=1}^N$ be the zeroes of the polynomial $P_{N0}(t)$ and $p_r(t,s)$ be kernel functions of the form (i) or (ii). Then the approximate solution $y_N(t)$ converges uniformly to the* *exact solution $y(t)$. Moreover the rate of convergence coincides with the product integration quadrature choosed to approximate the integral term (4.1).*

Proof. The proof follows immediately from the estimate (4.4) together with Theorem 4.1. The bound (4.7) supplies an estimate of the rate of convergence. □

5. Presentation of results

To illustrate the Algorithm 1, we considered some test problems. For computational purposes, the following test problems of different forms of kernels were considered. All the computations were carried out with software Maple.

Example 5.1. The linear Volterra-Fredholm integral equation in

$$y(t) = -\frac{4}{3}t^{\frac{3}{2}} - \frac{1}{2} + \int_0^t |t-s|^{-\frac{1}{2}}\, y(s)ds + \int_0^1 (t + y(s))ds, \quad t \in [0,1]$$

has the analytical solution $y(t) = t$.

Example 5.2. The linear Volterra-Fredholm integral equation in

$$y\,(t) = g\,(t) + \int_0^t \ln\left(|t-s|\right) y\,(s)\, ds + \int_0^1 \ln\left(|t-s|\right) y\,(s)\, ds, \quad t \in [0,1]$$

where

$$g(t) = \frac{3t}{2} - \frac{t^2}{4}\left(2\,\ln(t) - 3\right) + \frac{1}{4} - \frac{t^2}{2}\left(\ln(t) + \ln(1-t)\right) - \frac{1}{2}\,\ln(1-t)\,,$$

has the analytical solution $y(t) = t$.

Example 5.3. The nonlinear Volterra-Fredholm integral equation in $[0,1]$

$$y(t) = g(t) + \int_0^t \ln(|t-s|)(ts^2 - y^2(s))ds + \int_0^1 \frac{y^2(s)}{\sqrt{|t-s|}}ds,$$

where

$$g(t) = \sqrt{t} - \frac{t^2}{36}((12t^2 - 18)\ln(t) + 27 - 22t^2) - \frac{4t^{\frac{3}{2}}}{3} - \frac{4t\sqrt{1-t}}{3} - \frac{2\sqrt{1-t}}{3},$$

has the following analytical solution $y(t) = \sqrt{t}$.

Example 5.4. The nonlinear Volterra-Fredholm integral equation in $[0,1]$

$$y(t) = g(t) - \int_0^t \ln(|t-s|) y^2(s) ds + \int_0^1 \frac{y^2(s)}{\sqrt{|t-s|}} ds,$$

where

$$g(t) = \sqrt{t} - \frac{t^2(2\ln(t)-3)}{4} - \frac{4t^{\frac{3}{2}}}{3} - \frac{2\sqrt{1-t}}{3}$$

has the following analytical solution $y(t) = \sqrt{t}$.

Example 5.5. Consider the nonlinear weakly singular Volterra-Fredholm integral equation with algebraic nonlinearity

$$y(t) = g(t) + \int_0^t \frac{y^2(s)}{\sqrt{|t-s|}} ds + \int_0^1 \frac{1+y^2(s)}{\sqrt{|t-s|}} ds,$$

with exact solution $y(t) = t$ and

$$g(t) = t - \frac{32t^{\frac{5}{2}}}{15} - 2\sqrt{t} - \frac{12\sqrt{1-t}}{5} - \frac{8t\sqrt{1-t}}{15} - \frac{16t^2\sqrt{1-t}}{15},$$

and therefore, provides an example to verify the accuracy of this method.

We refrain from going into details. Using the same notations and methods as implemented in the previous examples, we give the maximum errors found by presented method compared with the exact solution in Table 1.

Table 1. Maximum errors of numerical solution and exact solution in Chelyshkov points in addition to the end points $t = 0, 1$.

Example	$\frac{N=10}{\|e\|_\infty}$	$\frac{N=20}{\|e\|_\infty}$	$\frac{N=30}{\|e\|_\infty}$
5.1	10^{-9}	10^{-12}	10^{-16}
5.2	10^{-8}	10^{-10}	10^{-15}
5.3	10^{-8}	10^{-17}	10^{-19}
5.4	10^{-9}	10^{-14}	10^{-17}
5.5	10^{-8}	10^{-12}	10^{-16}

6. Conclusion

By introducing the product integration method, the weakly singular nonlinear Volterra-Fredholm integral equations in applied sciences and physics, can be transformed into nonlinear algebraic system, which may be solved using classical methods. Both the applicability and the accuracy of this method for the solution of nonlinear equations, have been examined by means of several problems. Another considerable advantage of the method is that unknown coefficients of the solution are found very easily by using the computer programs.

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