COMPOSITION OPERATORS ACTING ON WEIGHTED HILBERT SPACES OF ANALYTIC FUNCTIONS

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\textbf{Abstract.} In this paper, we considered composition operators on weighted Hilbert spaces of analytic functions and observed that a formula for the essential norm, gives a Hilbert-Schmidt characterization and characterizes the membership in Schatten-class for these operators. Also, closed range composition operators are investigated.

1. Introduction

Let $\mathbb{D}$ denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and $\varphi$ be an analytic self map of $\mathbb{D}$. The composition operator $C_\varphi$ induced by $\varphi$ is defined $C_\varphi f = f \circ \varphi$, for any $f \in H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$. This operator can be generalized to the weighted composition operator $uC_\varphi$,

$$uC_\varphi f(z) = u(z)f(\varphi(z)), \quad u \in H(\mathbb{D}).$$

We considered a \textit{weight} as a positive integrable function $\omega \in C^2[0,1)$ which is radial, $\omega(z) = \omega(|z|)$. The weighted Hilbert space of analytic functions $H_\omega$ consists of all analytic functions on $\mathbb{D}$ such that

$$\|f\|_{H_\omega}^2 = \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \, dA(z) < \infty,$$

equipped with the norm

$$\|f\|_{H_\omega}^2 = |f(0)|^2 + \|f'|_{\omega}^2.$$
Here $dA$ is the normalized area measure on $\mathbb{D}$. Also the weighted Bergman spaces defined by

$$A^2_\omega = \left\{ f \in H(\mathbb{D}) : ||f||^2_\omega = \int_{\mathbb{D}} |f(z)|^2 \omega(z) \, dA(z) < \infty \right\}.$$  

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f \in \mathcal{H}_\omega$ if and only if

$$||f||^2_{\mathcal{H}_\omega} = \sum_{n=0}^{\infty} |a_n|^2 \omega_n < \infty,$$

where $\omega_0 = 1$ and for $n \geq 1$

$$\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) \, dr,$$

and $f \in \mathcal{A}_\omega$ if and only if

$$||f||^2_{\mathcal{A}_\omega} = \sum_{n=0}^{\infty} |a_n|^2 p_n < \infty,$$

where

$$p_n = 2 \int_0^1 r^{2n+1} \omega(r) \, dr, \quad n \geq 0.$$

Let $\omega_\alpha(r) = (1 - r^2)^\alpha$ (standard weight), $\alpha > -1$, $\mathcal{H}_\omega = \mathcal{H}_\alpha$. If $0 \leq \alpha < 1$, then $\mathcal{H}_\alpha = \mathcal{D}_\alpha$, the weighted Dirichlet space, and $\mathcal{H}_1 = H^2$, the Hardy space.

There are several papers that studied composition operators on various spaces of analytic functions. The best monographs for these operators are [1, 7]. In [2], Kellay and Lefevre studied composition operators on weighted Hilbert space of analytic functions by using generalized Nevanlinna counting function. They characterized boundedness and compactness of these operators. Pau and Pérez [6] studied boundedness, essential norm, Schatten-class and closed range properties of these operators acting on weighted Dirichlet spaces.

Our aim in this paper is to generalize the results of [6] to a large class of spaces. Throughout the remainder of this paper, $c$ will denote a positive constant, the exact value of which will vary from one appearance to the next. Also for the positive numbers $a$ and $b$ the notation $a \asymp b$ means that there exist positive constants $c_1$ and $c_2$ such that $c_1 a \leq b \leq c_2 a$.

2. Preliminaries

In this section we give some notations and lemmas which will be used in our work.
Definition 2.1 (\[2\]). We assume that \( \omega \) is a weight function, with the following properties

(W\(_1\)): \( \omega \) is non-increasing,
(W\(_2\)): \( \omega(r)(1-r)^{-(1+\delta)} \) is non-decreasing for some \( \delta > 0 \),
(W\(_3\)): \( \lim_{r \to 1^-} \omega(r) = 0 \),
(W\(_4\)): One of the two properties of convexity is fulfilled

\[
\begin{align*}
&\text{or} &\quad (W_{4}(I)) &\quad \omega \text{ is convex and } \lim_{r \to 1^-} \omega'(r) = 0, \\
&\quad (W_{4}(II)) &\quad \omega \text{ is concave.}
\end{align*}
\]

Such a weight \( \omega \) is called admissible.

If \( \omega \) satisfies conditions (W\(_1\))-(W\(_3\)) and (W\(_4(I)\)), we shall say that \( \omega \) is (I)-admissible. Also we use weights satisfy (L1) condition (due to Lusky [5]):

\[
(L1) \quad \inf_k \frac{\omega(1 - 2^{-k-1})}{\omega(1 - 2^{-k})} > 0.
\]

This is equivalent to the following condition (see [3]):

there are \( 0 < r < 1 \) and \( 0 < c < \infty \) with \( \frac{\omega(z)}{\omega(a)} \leq c \) for every \( a, z \in \Delta(a, r) \), where \( \Delta(a, r) = \{ z \in \mathbb{D} : |\sigma_a(z)| < r \} \) and \( \sigma_a(z) = \frac{a-z}{1-\bar{a}z} \) is the Mobius transformation on \( \mathbb{D} \).

All characterizations in this paper are needed to the generalized counting Nevanlinna function. Let \( \varphi \) be an analytic self map of \( \mathbb{D} \) (\( \varphi(\mathbb{D}) \subset \mathbb{D} \)). The generalized counting Nevanlinna function associated to a weight \( \omega \) is defined as follows

\[
N_{\varphi, \omega}(z) = \sum_{a: \varphi(a) = z} \omega(a), \quad z \in \mathbb{D}\setminus\{\varphi(0)\}.
\]

By using the change of variables formula we have: if \( f \) be a non-negative function on \( \mathbb{D} \), then

\[
(2.1) \quad \int_{\mathbb{D}} f(\varphi(z))|\varphi'(z)|^2 \omega(z) dA(z) = \int_{\mathbb{D}} f(z) N_{\varphi, \omega}(z) dA(z).
\]

Also the generalized counting Nevanlinna function has the sub-mean value property (Lemmas 2.2 and 2.3 [2]). Let \( \omega \) be an admissible weight. Then for every \( r > 0 \) and \( z \in \mathbb{D} \) such that \( D(z, r) \subset \mathbb{D}\setminus D(0, 1/2) \)

\[
(2.2) \quad N_{\varphi, \omega}(z) \leq \frac{2}{r^2} \int_{|\zeta - z| < r} N_{\varphi, \omega}(\zeta) dA(\zeta).
\]

Lemma 2.2 ([2]). If \( \omega \) is a weight satisfying (W\(_1\)) and (W\(_2\)), then there exists \( c > 0 \) such that

\[
\frac{1}{c} \omega(z) \leq \omega(\sigma_{\varphi(0)}(z)) \leq c \omega(z), \quad z \in \mathbb{D}.
\]
Lemma 2.3. Let \( \omega \) be a weight satisfying \((W_1)\) and \((W_2)\). Let \( a \in \mathbb{D} \) and

\[
f_a(z) = \frac{1}{\sqrt{\omega(a)}} \frac{(1 - |a|^2)^{1+\delta}}{(1 - \overline{a}z)^{1+\delta}}.
\]

Then \( \|f_a\|_{\mathcal{H}_\omega} \asymp 1 \).

3. Essential Norm

Recall that the essential norm \( \|T\|_e \) of a bounded operator \( T \) between Banach spaces \( X \) and \( Y \) is defined as the distance from \( T \) to \( K(X,Y) \), the space of all compact operators between \( X \) and \( Y \).

Theorem 3.1. Let \( \omega \) be an admissible weight and \( C_\varphi \) be a bounded operator on \( \mathcal{H}_\omega \). Then

\[
\|C_\varphi\|_e \asymp \limsup_{|z| \to 1^-} \frac{N_{\varphi,\omega}(z)}{\omega(z)}.
\]

Proof. Consider the test function defined in Lemma 2.3. Then \( \{f_a\}_{a \in \mathbb{D}} \) is bounded in \( \mathcal{H}_\omega \) and converges uniformly on compact subsets of \( \mathbb{D} \) to 0 as \( |a| \to 1^- \). Then for every compact operator \( K \) on \( \mathcal{H}_\omega \), \( \lim_{|a| \to 1^-} \|Kf_a\|_{\mathcal{H}_\omega} = 0 \). There exists a constant \( c > 0 \) such that

\[
\|C_\varphi - K\| \geq c \limsup_{|a| \to 1^-} \|C_\varphi f_a - Kf_a\|_{\mathcal{H}_\omega}
\]

\[
\geq c \limsup_{|a| \to 1^-} \|C_\varphi f_a\|_{\mathcal{H}_\omega}.
\]

Therefore

\[
\|C_\varphi\|_e \geq c \limsup_{|a| \to 1^-} \|C_\varphi f_a\|_{\mathcal{H}_\omega}.
\]

On the other hand

\[
\|C_\varphi f_a\|_{\mathcal{H}_\omega}^2 = \|f_a(\varphi(0))\|^2 + \int_{\mathbb{D}} |f'_a(\varphi(z))|^2 \varphi'(z)^2 \omega(z) dA(z)
\]

\[
\geq c \frac{(1 - |a|^2)^{2+2\delta} |a|^2}{\omega(a)} \int_{D(a,1-|a|)} \frac{N_{\varphi,\omega}(z)}{|1 - \overline{a}z|^{4+2\delta}} dA(z).
\]

If \( |a| \) is close enough to 1, then \( \varphi(0) \not\in D(a, \frac{1-|a|}{2}) \). So \( |1 - \overline{a}z| \asymp (1 - |a|) \) for \( z \in D(a, \frac{1-|a|}{2}) \). We have

\[
\limsup_{|a| \to 1^-} \|C_\varphi f_a\|_{\mathcal{H}_\omega}^2 \geq c \limsup_{|a| \to 1^-} \frac{|a|^2}{\omega(a)(1 - |a|)^2} \int_{D(a,1-|a|)} N_{\varphi,\omega}(z) dA(z).
\]
By the sub-mean value property of $N_{\varphi, \omega}$, we get

$$\limsup_{|a| \to 1^-} \|C_{\varphi}f_a\|_{H_\omega}^2 \geq c \limsup_{|a| \to 1^-} \frac{N_{\varphi, \omega}(a)}{\omega(a)},$$

Now

$$\|C_{\varphi}\|^2 \geq c \limsup_{|a| \to 1^-} \frac{N_{\varphi, \omega}(a)}{\omega(a)},$$

and the lower estimate is obtained. The upper estimate comes from p. 136 [1]. □

Since the space of compact operators is a closed subspace of space of bounded operators, then a bounded operator $T$ is compact if and only if $\|T\|_e = 0$. According to this fact we have the following corollary which is Theorem 1.4 [2].

**Corollary 3.2.** Let $\omega$ be an admissible weight. Then $C_{\varphi}$ is compact on $H_\omega$ if and only if

$$\lim_{|z| \to 1^-} \frac{N_{\varphi, \omega}(z)}{\omega(z)} = 0.$$

### 4. Hilbert-Schmidt and Schatten-class

For studying Schatten-class we need the Toeplitz operator. For more information about relations between Toeplitz operator and Schatten-class see [3]. Let $\psi$ be a positive function in $L^1(\mathbb{D}, dA)$ and $\omega$ be a weight. The Toeplitz operator associated to $\psi$ is defined by

$$T_\psi f(z) = \frac{1}{\omega(z)} \int_{\mathbb{D}} \frac{f(t)\psi(t)\omega(t)}{(1-\overline{z}t)^2}dA(t).$$

$T_\psi \in S_p(A_\omega^2)$ if and only if the function

$$\widehat{\psi}_r(z) = \frac{1}{(1-|z|^2)^2 \omega(z)} \int_{\Delta(z,r)} \psi(t)\omega(t)dA(t),$$

is in $L^p(\mathbb{D}, d\lambda)$, [4], where $d\lambda = (1-|z|^2)^{-2}dA(z)$ is the hyperbolic measure on $\mathbb{D}$. According to the description of [5] pages 8 and 9, $C_{\varphi} \in S_p(H_\omega)$ if and only if $\varphi^*C_{\varphi} \in S_p(A_\omega^2)$.

**Theorem 4.1.** Let $\omega$ be an admissible weight satisfies (L1) condition. Then $C_{\varphi} \in S_p(H_\omega)$ if and only if

$$\psi(z) = \frac{N_{\varphi, \omega}(z)}{\omega(z)} \in L^{p/2}(\mathbb{D}, d\lambda).$$
Proof. For any \( f, g \in H_\omega \) we have
\[
\langle (\varphi' C_\varphi)^*(\varphi' C_\varphi) f, g \rangle = \langle (\varphi' C_\varphi) f, (\varphi' C_\varphi) g \rangle = \int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}|\varphi'(z)|^2 \omega(z) dA(z)
\]
\[
= \int_{\mathbb{D}} f(z)\overline{g(z)} N_{\varphi,\omega}(z) dA(z).
\]
On the other hand, since \( \frac{1}{(1-\overline{z}t)^2} \) is the reproducing kernel in \( \mathbb{A}^2 \),
\[
T_\psi f(z) = \frac{1}{\omega(z)} \int_{\mathbb{D}} \frac{f(t)N_{\varphi,\omega}(t)}{(1-\overline{z}t)^2} dA(t)
\]
\[
= N_{\varphi,\omega}(z) f(z) \frac{\omega(z)}{\omega(z)}
\]
Therefore
\[
\langle T_\psi f, g \rangle = \int_{\mathbb{D}} f(z)\overline{g(z)} N_{\varphi,\omega}(z) dA(z).
\]
Thus
\[
T_\psi = (\varphi' C_\varphi)^*(\varphi' C_\varphi).
\]
Theorem 1.4.6 \[11\] implies that \( \varphi' C_\varphi \in S_p(\mathbb{A}^2_\omega) \) if and only if
\[
(\varphi' C_\varphi)^*(\varphi' C_\varphi) \in S_p(\mathbb{A}^2_\omega).
\]
We get \( \varphi' C_\varphi \in S_p(\mathbb{A}^2_\omega) \) if and only if \( T_\psi \in S_p(\mathbb{A}^2_\omega) \) if and only if \( \widehat{\psi}_r(z) \in L^{p/2}(\mathbb{D}, d\lambda) \).

It is clear that \( \Delta(z, r) \) contains an Euclidian disk centered at \( z \) of radius \( \eta(1-|z|) \) with \( \eta \) depending only on \( r \). By the sub-mean value property of \( N_{\varphi,\omega} \) we have
\[
\psi(z) = \frac{N_{\varphi,\omega}(z)}{\omega(z)} \leq \frac{2}{\nu^2 \omega(z)} \int_{\Delta(z, r)} N_{\varphi,\omega}(t) dA(t)
\]
\[
\leq \frac{2}{\nu^2(1-|z|^2)^2 \omega(z)} \int_{\Delta(z, r)} \psi(t) \omega(t) dA(t)
\]
\[
= \frac{2}{\nu^2} \widehat{\psi}_r(z).
\]
So \( \widehat{\psi}_r(z) \in L^{p/2}(\mathbb{D}, d\lambda) \) implies \( \psi(z) \in L^{p/2}(\mathbb{D}, d\lambda) \). Now, suppose that \( \psi(z) \in L^{p/2}(\mathbb{D}, d\lambda) \). From the arguments above, noting that
\[
(1-|t|^2) \asymp (1-|z|^2)
\]
\[
\asymp |1-\overline{t}z|
\]
and \( \frac{\omega(t)}{\omega(z)} \leq c \), for \( t \in \Delta(z,r) \), we have

\[
\psi_r(z)^{p/2} \leq \frac{c}{\omega(z)^{p/2}} \sup\{ \psi(t)^{p/2} \omega(t)^{p/2} : t \in \Delta(z,r) \}
\]

\[
\leq \frac{c}{\omega(z)^{p/2}} \sup_{t \in \Delta(z,r)} \int_{\Delta(t,r)} \psi(s)^{p/2} \omega(s)^{p/2} dA(s)
\]

\[
\leq \frac{c}{\omega(z)^{p/2}} \sup_{t \in \Delta(z,r)} \int_{\Delta(t,r)} \frac{(1 - |z|^2)^2}{|1 - \overline{s}z|^4} \psi(s)^{p/2} \omega(s)^{p/2} dA(s).
\]

Since \( t \in \Delta(z,r) \), we can choose \( \Delta(t, r) \) so that \( \Delta(t, r) \subset \Delta(z, r) \). Then

\[
\psi_r(z)^{p/2} \leq \frac{c}{\omega(z)^{p/2}} \int_{\Delta(z,r)} \frac{(1 - |z|^2)^2}{|1 - \overline{s}z|^4} \psi(s)^{p/2} \omega(s)^{p/2} dA(s)
\]

\[
\leq c \int_{\Delta} \frac{(1 - |z|^2)^2}{|1 - \overline{s}z|^4} \psi(s)^{p/2} dA(s).
\]

By Fubini’s Theorem and the well known Theorem 1.12 [10], we get

\[
\int_{\Delta} \psi_r(z)^{p/2} d\lambda(z) \leq c \int_{\Delta} \psi(s)^{p/2} d\lambda(s).
\]

\[\square\]

If \( p = 2 \), then we have a characterization for Hilbert-Schmidt composition operators.

**Corollary 4.2.** Let \( \omega \) be an admissible weight satisfies (L1) condition. Then \( C_\phi \) is Hilbert-Schmidt on \( H_\omega \) if and only if

\[
\int_{\Delta} \frac{N_{\phi,\omega}(z)}{\omega(z)(1 - |z|^2)^2} dA(z) = \int_{\Delta} \frac{N_{\phi,\omega}(z)}{\omega(z)} d\lambda(z) < \infty.
\]

5. **Closed Range**

It is well known that having the closed range for a bounded operator acting on a Hilbert space \( H \) is equivalent to existing a positive constant \( c \) such that for every \( f \in H \), \( ||Tf||_H \geq c ||f||_H \). Consider the function

\[
\tau_{\phi,\omega}(z) = \frac{N_{\phi,\omega}(z)}{\omega(z)}.
\]

**Proposition 5.1.** Let \( \omega \) be an admissible weight and \( C_\phi \) be a bounded operator on \( H_\omega \). Then \( C_\phi \) has closed range if and only if there exists a constant \( c > 0 \) such that for all \( f \in H_\omega \)

\[
\int_{\Delta} |f'(z)|^2 \tau_{\phi,\omega}(z) \omega(z) dA(z) \geq c \int_{\Delta} |f'(z)|^2 \omega(z) dA(z).
\]
Proof. If \( \varphi(0) = 0 \), then we can consider \( C_\varphi \) acting on \( \mathcal{H}_\omega \), the closed subspace of \( \mathcal{H}_\omega \) consisting all functions with \( f(0) = 0 \). Note that \( C_\varphi \) has closed range if and only if there exists a constant \( c > 0 \) such that \( \| C_\varphi f \|_{\mathcal{H}_\omega} \geq \| f \|_{\mathcal{H}_\omega} \). But

\[
\| C_\varphi f \|_{\mathcal{H}_\omega}^2 = \int_D |f'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) \, dA(z)
= \int_D |f'(z)|^2 N_{\varphi_0} \, dA(z)
= \int_D |f'(z)|^2 r_{\varphi_0} \omega(z) \, dA(z).
\]

Thus, in this case the proposition is proved. If \( \varphi(0) = a \neq 0 \), define the function \( \psi = \sigma_a \circ \varphi \). Then \( C_\varphi = C_\psi C_{\sigma_a} \) and \( C_{\sigma_a} \) is invertible on \( \mathcal{H}_\omega \). Therefore \( C_\varphi \) has closed range if and only if \( C_\psi \) has closed range. Since \( \psi(0) = 0 \), the argument above shows that \( C_\psi \) has closed range if and only if there exists a constant \( c > 0 \) such that

\[
\int_D |f'(z)|^2 \tau_{\psi_0} \omega(z) \, dA(z) \geq c \int_D |f'(z)|^2 \omega(z) \, dA(z).
\]

We just prove that (5.1) and (5.2) are equivalent. If (5.1) holds, then

\[
\int_D |f'(z)|^2 \tau_{\psi_0} \omega(z) \, dA(z) = \int_D |f'(z)|^2 N_{\psi_0} \, dA(z)
= \int_D |(f \circ \psi)'(z)|^2 \omega(z) \, dA(z)
= \int_D |(f \circ \sigma_a)'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) \, dA(z)
= \int_D |(f \circ \sigma_a)'(z)|^2 \tau_{\varphi_0} \omega(z) \, dA(z)
\geq c \int_D |(f \circ \sigma_a)'(z)|^2 \omega(z) \, dA(z)
= c \int_D |f'(z)|^2 \omega(\sigma_a(z)) \, dA(z)
\asymp c \int_D |f'(z)|^2 \omega(z) \, dA(z).
\]

The last equation is due to Lemma 2.2. Hence (5.2) holds. Since \( \varphi = \sigma_a \circ \psi \), the proof of converse part is similar. \( \square \)

Fredholm composition operator is an example of composition operators with closed range property. Recall that a bounded operator \( T \) between two Banach spaces \( X, Y \) is called Fredholm if Kernel \( T \) and \( T^* \) are finite dimensional.
Example 5.2. Suppose that $C_\varphi : \mathcal{H}_\omega \to \mathcal{H}_\omega$ be a Fredholm operator. By theorem 3.29 [11], $\varphi$ is an authomorphism of $\mathbb{D}$. Then $N_{\varphi,\omega}(z) = \omega(\varphi^{-1}(z))$. If $\varphi(0) = 0$, Schwarz lemma implies that $|\varphi^{-1}(z)| \leq |z|$. Since $\omega$ is non-increasing,

$$\omega(\varphi^{-1}(z)) = \omega(|\varphi^{-1}(z)|) \geq \omega(|z|) = \omega(z).$$

Now (5.1) holds. If $\varphi(0) \neq 0$, then the same argument can be applied.

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References


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