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COMPOSITION OPERATORS ACTING ON WEIGHTED HILBERT SPACES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we considered composition operators on weighted Hilbert spaces of analytic functions and observed that a formula for the essential norm, gives a Hilbert-Schmidt characterization and characterizes the membership in Schatten-class for these operators. Also, closed range composition operators are investigated.

1. Introduction

Let $\mathbb{D}$ denotes the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ and φ be an analytic self map of $\mathbb{D}$. The composition operator C_φ induced by φ is defined $C_\varphi f = f \circ \varphi$, for any $f \in H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$. This operator can be generalized to the weighted composition operator uC_φ,

$$uC_\varphi f(z) = u(z) f(\varphi(z)), \quad u \in H(\mathbb{D}).$$

We considered a *weight* as a positive integrable function $\omega \in C^2[0,1)$ which is radial, $\omega(z) = \omega(|z|)$. The weighted Hilbert space of analytic functions $\mathcal{H}_\omega$ consists of all analytic functions on $\mathbb{D}$ such that

$$||f'||_\omega^2 = \int_{\mathbb{D}} |f'(z)|^2 \omega(z) \, dA(z) < \infty,$$

equipped with the norm

$$||f||_{\mathcal{H}_\omega}^2 = |f(0)|^2 + ||f'||_\omega^2.$$

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Here dA is the normalized area measure on $\mathbb{D}$. Also the weighted Bergman spaces defined by

$$\mathcal{A}^2_\omega = \left\{ f \in H(\mathbb{D}) : ||f||^2_\omega = \int_{\mathbb{D}} |f(z)|^2 \omega(z)\ dA(z) < \infty \right\}.$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $f \in \mathcal{H}_\omega$ if and only if

$$||f||^2_{\mathcal{H}_\omega} = \sum_{n=0}^{\infty} |a_n|^2 \omega_n < \infty,$$

where $\omega_0 = 1$ and for $n \geq 1$

$$\omega_n = 2n^2 \int_0^1 r^{2n-1} \omega(r) dr,$$

and $f \in \mathcal{A}_\omega$ if and only if

$$||f||^2_{\mathcal{A}_\omega} = \sum_{n=0}^{\infty} |a_n|^2 p_n < \infty,$$

where

$$p_n = 2 \int_0^1 r^{2n+1} \omega(r) dr, \quad n \geq 0.$$

Let $\omega_\alpha(r) = (1 - r^2)^\alpha$ (standard weight), $\alpha > -1$, $\mathcal{H}_{\omega_\alpha} = \mathcal{H}_\alpha$. If $0 \leq \alpha < 1$, then $\mathcal{H}_\alpha = \mathcal{D}_\alpha$, the weighted Dirichlet space, and $\mathcal{H}_1 = H^2$, the Hardy space.

There are several papers that studied composition operators on various spaces of analytic functions. The best monographs for these operators are [1, 7]. In [2], Kellay and Lefèvre studied composition operators on weighted Hilbert space of analytic functions by using generalized Nevanlinna counting function. They characterized boundedness and compactness of these operators. Pau and Pérez [6] studied boundedness, essential norm, Schatten-class and closed range properties of these operators acting on weighted Dirichlet spaces.

Our aim in this paper is to generalize the results of [6] to a large class of spaces. Throughout the remainder of this paper, c will denote a positive constant, the exact value of which will vary from one appearance to the next. Also for the positiove numbers a and b the notation $a \asymp b$ means that there exist positive constants c_1 and c_2 such that $c_1 a \leq b \leq c_2 a$.

2. Preliminaries

In this section we give some notations and lemmas which will be used in our work.

Definition 2.1 ([2]). We assume that ω is a weight function, with the following properties

(W_1): ω is non-increasing,
(W_2): $\omega(r)(1-r)^{-(1+\delta)}$ is non-decreasing for some $\delta > 0$,
(W_3): $\lim_{r\to 1^-}\omega(r) = 0$,
(W_4): One of the two properties of convexity is fulfilled

$$\begin{cases} (W_4^{(I)}): & \omega \text{ is convex and } \lim_{r\to 1}\omega'(r) = 0, \\ or & \\ (W_4^{(II)}): & \omega \text{ is concave.} \end{cases}$$

Such a weight ω is called admissible.

If ω satisfies conditions (W_1)-(W_3) and $(W_4^{(I)})$ (resp. $(W_4^{(II)})$), we shall say that ω is (I)-admissible (resp. (II)-admissible). Also we use weights satisfy (L1) condition (due to Lusky [5]):

$$(L1) \quad \inf_k \frac{\omega(1-2^{-k-1})}{\omega(1-2^{-k})} > 0.$$

This is equivalent to the following condition (see[3]):

there are $0 < r < 1$ and $0 < c < \infty$ with $\frac{\omega(z)}{\omega(a)} \leq c$ for every $a, z \in \Delta(a,r)$, where $\Delta(a,r) = \{z \in \mathbb{D} : |\sigma_a(z)| < r\}$ and $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is the Mobius transformation on $\mathbb{D}$.
All characterizations in this paper are needed to the generalized counting Nevanlinna function. Let φ be an analytic self map of $\mathbb{D}$ ($\varphi(\mathbb{D}) \subset \mathbb{D}$). The generalized counting Nevanlinna function associated to a weight ω is defined as follows

$$N_{\varphi,\omega}(z) = \sum_{a:\varphi(a)=z} \omega(a), \quad z \in \mathbb{D}\backslash\{\varphi(0)\}.$$

By using the change of variables formula we have: if f be a non-negative function on $\mathbb{D}$, then

$$\int_{\mathbb{D}} f(\varphi(z))|\varphi'(z)|^2\omega(z)dA(z) = \int_{\mathbb{D}} f(z)N_{\varphi,\omega}(z)dA(z). \tag{2.1}$$

Also the generalized counting Nevanlinna function has the sub-mean value property (Lemmas 2.2 and 2.3 [2]). Let ω be an admissible weight. Then for every $r > 0$ and $z \in \mathbb{D}$ such that $D(z,r) \subset \mathbb{D}\backslash D(0,1/2)$

$$N_{\varphi,\omega}(z) \leq \frac{2}{r^2}\int_{|\zeta-z|<r} N_{\varphi,\omega}(\zeta)dA(\zeta). \tag{2.2}$$

Lemma 2.2 ([2]). *If ω is a weight satisfying (W_1) and (W_2), then there exists $c > 0$ such that*

$$\frac{1}{c}\omega(z) \leq \omega(\sigma_{\varphi(0)}(z)) \leq c\omega(z), \quad z \in \mathbb{D}.$$

Lemma 2.3 ([2])**.** *Let ω be a weight satisfying (W_1) and (W_2). Let $a \in \mathbb{D}$ and*

$$f_a(z) = \frac{1}{\sqrt{\omega(a)}} \frac{(1-|a|^2)^{1+\delta}}{(1-\overline{a}z)^{1+\delta}}.$$

Then $||f_a||_{\mathcal{H}_\omega} \asymp 1$.

3. **Essential Norm**

Recall that the essential norm $||T||_e$ of a bounded operator T between Banach spaces X and Y is defined as the distance from T to $K(X,Y)$, the space of all compact operators between X and Y.

Theorem 3.1. *Let ω be an admissible weight and C_φ be a bounded operator on $\mathcal{H}_\omega$. Then*

$$||C_\varphi||_e \asymp \limsup_{|z|\to 1^-} \frac{N_{\varphi,\omega}(z)}{\omega(z)}.$$

Proof. Consider the test function defined in Lemma 2.3. Then $\{f_a\}_{a\in\mathbb{D}}$ is bounded in $\mathcal{H}_\omega$ and converges uniformly on compact subsets of $\mathbb{D}$ to 0 as $|a| \to 1^-$. Then for every compact operator K on $\mathcal{H}_\omega$, $\lim_{|a|\to 1^-} ||Kf_a||_{\mathcal{H}_\omega} = 0$. There exists a constant $c > 0$ such that

$$\begin{aligned} ||C_\varphi - K|| \geq & c \limsup_{|a|\to 1^-} ||C_\varphi f_a - K f_a||_{\mathcal{H}_\omega} \\ \geq & c \limsup_{|a|\to 1^-} ||C_\varphi f_a||_{\mathcal{H}_\omega}. \end{aligned}$$

Therefore

$$||C_\varphi||_e \geq c \limsup_{|a|\to 1^-} ||C_\varphi f_a||_{\mathcal{H}_\omega}.$$

On the other hand

$$\begin{aligned} ||C_\varphi f_a||^2_{\mathcal{H}_\omega} =& |f_a(\varphi(0))|^2 + \int_{\mathbb{D}} |f_a'(\varphi(z))|^2 |\varphi'(z)|^2 \omega(z) dA(z) \\ =& |f_a(\varphi(0))|^2 + \int_{\mathbb{D}} |f_a'(z)|^2 N_{\varphi,\omega}(z) dA(z) \\ \geq & c \frac{(1-|a|^2)^{2+2\delta}|a|^2}{\omega(a)} \int_{D(a,\frac{1-|a|}{2})} \frac{N_{\varphi,\omega}(z)}{|1-\overline{a}z|^{4+2\delta}} dA(z). \end{aligned}$$

If $|a|$ is close enough to 1, then $\varphi(0) \notin D(a, \frac{1-|a|}{2})$. So $|1-\overline{a}z| \asymp (1-|a|)$ for $z \in D(a, \frac{1-|a|}{2})$. We have

$$\limsup_{|a|\to 1^-} ||C_\varphi f_a||^2_{\mathcal{H}_\omega} \geq c \limsup_{|a|\to 1^-} \frac{|a|^2}{\omega(a)(1-|a|)^2} \int_{D(a,\frac{1-|a|}{2})} N_{\varphi,\omega}(z) dA(z).$$

By the sub-mean value property of $N_{\varphi,\omega}$, we get

$$\limsup_{|a|\to 1^-} ||C_\varphi f_a||^2_{\mathcal{H}_\omega} \geq c \limsup_{|a|\to 1^-} \frac{N_{\varphi,\omega}(a)}{\omega(a)}.$$

Now

$$||C_\varphi||^2_e \geq c \limsup_{|a|\to 1^-} \frac{N_{\varphi,\omega}(a)}{\omega(a)},$$

and the lower estimate is obtained. The upper estimate comes from p. 136 [1]. □

Since the space of compact operators is a closed subspace of space of bounded operators, then a bounded operator T is compact if and only if $||T||_e = 0$. According to this fact we have the following corollary which is Theorem 1.4 [2].

Corollary 3.2. *Let ω be an admissible weight. Then C_φ is compact on $\mathcal{H}_\omega$ if and only if*

$$\lim_{|z|\to 1^-} \frac{N_{\varphi,\omega}(z)}{\omega(z)} = 0.$$

4. Hilbert-Schmidt and Schatten-class

For studying Schatten-class we need the Toeplitz operator. For more information about relations between Toeplitz operator and Schatten-class see [9]. Let ψ be a positive function in $L^1(\mathbb{D}, dA)$ and ω be a weight. The Toeplitz operator associated to ψ is defined by

$$T_\psi f(z) = \frac{1}{\omega(z)} \int_{\mathbb{D}} \frac{f(t)\psi(t)\omega(t)}{(1-\overline{z}t)^2} dA(t).$$

$T_\psi \in S_p(\mathcal{A}^2_\omega)$ if and only if the function

$$\widehat{\psi}_r(z) = \frac{1}{(1-|z|^2)^2\omega(z)} \int_{\Delta(z,r)} \psi(t)\omega(t) dA(t),$$

is in $L^p(\mathbb{D}, d\lambda)$, [4], where $d\lambda = (1-|z|^2)^{-2} dA(z)$ is the hyperbolic measure on $\mathbb{D}$. According to the description of [6] pages 8 and 9, $C_\varphi \in S_p(\mathcal{H}_\omega)$ if and only if $\varphi' C_\varphi \in S_p(\mathcal{A}^2_\omega)$.

Theorem 4.1. *Let ω be an admissible weight satisfies (L1) condition. Then $C_\varphi \in S_p(\mathcal{H}_\omega)$ if and only if*

$$\psi(z) = \frac{N_{\varphi,\omega}(z)}{\omega(z)} \in L^{p/2}(\mathbb{D}, d\lambda).$$

Proof. For any $f, g \in \mathcal{H}_\omega$ we have

$$\begin{aligned}\langle(\varphi' C_\varphi)^*(\varphi' C_\varphi)f, g\rangle =& \langle(\varphi' C_\varphi)f, (\varphi' C_\varphi)g\rangle \\ =& \int_{\mathbb{D}} f(\varphi(z))\overline{g(\varphi(z))}|\varphi'(z)|^2\omega(z)dA(z) \\ =& \int_{\mathbb{D}} f(z)\overline{g(z)}N_{\varphi,\omega}(z)dA(z).\end{aligned}$$

On the other hand, since $\frac{1}{(1-\bar{z}t)^2}$ is the reproducing kernel in $\mathcal{A}^2$,

$$\begin{aligned}T_\psi f(z) &= \frac{1}{\omega(z)}\int_{\mathbb{D}} \frac{f(t)N_{\varphi,\omega}(t)}{(1-\bar{z}t)^2}dA(t) \\ &= \frac{N_{\varphi,\omega}(z)f(z)}{\omega(z)}.\end{aligned}$$

Therefore

$$\langle T_\psi f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}N_{\varphi,\omega}(z)dA(z).$$

Thus

$$T_\psi = (\varphi' C_\varphi)^*(\varphi' C_\varphi).$$

Theorem 1.4.6 [11] implies that $\varphi' C_\varphi \in S_p(\mathcal{A}^2_\omega)$ if and only if

$$(\varphi' C_\varphi)^*(\varphi' C_\varphi) \in S_{p/2}(\mathcal{A}^2_\omega).$$

We get $\varphi' C_\varphi \in S_p(\mathcal{A}^2_\omega)$ if and only if $T_\psi \in S_{p/2}(\mathcal{A}^2_\omega)$ if and only if $\widehat{\psi}_r(z) \in L^{p/2}(\mathbb{D}, d\lambda)$.

It is clear that $\Delta(z, r)$ contains an Euclidian disk centered at z of radius $\eta(1-|z|)$ with η depending only on r. By the sub-mean value property of $N_{\varphi,\omega}$ we have

$$\begin{aligned}\psi(z) = \frac{N_{\varphi,\omega}(z)}{\omega(z)} \leq& \frac{2}{r^2\omega(z)}\int_{\Delta(z,r)} N_{\varphi,\omega}(t)dA(t) \\ \leq& \frac{2}{r^2(1-|z|^2)^2\omega(z)}\int_{\Delta(z,r)} \psi(t)\omega(t)dA(t) \\ =& \frac{2}{r^2}\widehat{\psi}_r(z).\end{aligned}$$

So $\widehat{\psi}_r(z) \in L^{p/2}(\mathbb{D}, d\lambda)$ implies $\psi(z) \in L^{p/2}(\mathbb{D}, d\lambda)$. Now, suppose that $\psi(z) \in L^{p/2}(\mathbb{D}, d\lambda)$. From the arguments above, noting that

$$\begin{aligned}(1-|t|^2) &\asymp (1-|z|^2) \\ &\asymp |1-\bar{t}z|\end{aligned}$$

and $\frac{\omega(t)}{\omega(z)} \leq c$, for $t \in \Delta(z,r)$, we have

$$\begin{aligned}\widehat{\psi}_r(z)^{p/2} \leq& \frac{c}{\omega(z)^{p/2}} \sup\{\psi(t)^{p/2}\omega(t)^{p/2} : t \in \Delta(z,r)\}\\ \leq& \frac{c}{\omega(z)^{p/2}} \sup_{t\in\Delta(z,r)} \int_{\Delta(t,r)} \psi(s)^{p/2}\omega(s)^{p/2} dA(s)\\ \leq& \frac{c}{\omega(z)^{p/2}} \sup_{t\in\Delta(z,r)} \int_{\Delta(t,r)} \frac{(1-|z|^2)^2}{|1-\overline{s}z|^4}\psi(s)^{p/2}\omega(s)^{p/2} dA(s).\end{aligned}$$

Since $t \in \Delta(z,r)$, we can choose $\Delta(t,r)$ so that $\Delta(t,r) \subset \Delta(z,r)$. Then

$$\begin{aligned}\widehat{\psi}_r(z)^{p/2} \leq& \frac{c}{\omega(z)^{p/2}} \int_{\Delta(z,r)} \frac{(1-|z|^2)^2}{|1-\overline{s}z|^4}\psi(s)^{p/2}\omega(s)^{p/2} dA(s)\\ \leq& c\int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-\overline{s}z|^4}\psi(s)^{p/2} dA(s).\end{aligned}$$

By Fubini's Theorem and the well known Theorem 1.12 [10], we get

$$\int_{\mathbb{D}} \widehat{\psi}_r(z)^{p/2} d\lambda(z) \leq c\int_{\mathbb{D}} \psi(s)^{p/2} d\lambda(s).$$

□

If $p = 2$, then we have a characterization for Hilbert-Schmidt composition operators.

Corollary 4.2. *Let ω be an admissible weight satisfies (L1) condition. Then C_φ is Hilbert-Schmidt on $\mathcal{H}_\omega$ if and only if*

$$\int_{\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\omega(z)(1-|z|^2)^2} dA(z) = \int_{\mathbb{D}} \frac{N_{\varphi,\omega}(z)}{\omega(z)} d\lambda(z) < \infty.$$

5. Closed Range

It is well known that having the closed range for a bounded operator acting on a Hilbert space H is equivalent to existing a positive constant c such that for every $f \in H$, $||Tf||_H \geq c||f||_H$. Consider the function

$$\tau_{\varphi,\omega}(z) = \frac{N_{\varphi,\omega}(z)}{\omega(z)}.$$

Proposition 5.1. *Let ω be an admissible weight and C_φ be a bounded operator on $\mathcal{H}_\omega$. Then C_φ has closed range if and only if there exists a constant $c > 0$ such that for all $f \in \mathcal{H}_\omega$*

$$\int_{\mathbb{D}} |f'(z)|^2 \tau_{\varphi,\omega}(z)\omega(z)\ dA(z) \geq c\int_{\mathbb{D}} |f'(z)|^2 \omega(z)\ dA(z). \tag{5.1}$$

Proof. If $\varphi(0)=0$, then we can consider C_φ acting on $\dot{\mathcal{H}}_\omega$, the closed subspace of $\mathcal{H}_\omega$ consisting all functions with $f(0)=0$. Note that C_φ has closed range if and only if there exists a constant $c>0$ such that $||C_\varphi f||_{\mathcal{H}_\omega} \geq ||f||_{\mathcal{H}_\omega}$. But

$$\begin{aligned}||C_\varphi f||^2_{\mathcal{H}_\omega} &= \int_{\mathbb{D}} |f'(\varphi(z))|^2|\varphi'(z)|^2\omega(z)\ dA(z)\\ &= \int_{\mathbb{D}} |f'(z)|^2 N_{\varphi,\omega}\ dA(z)\\ &= \int_{\mathbb{D}} |f'(z)|^2\tau_{\varphi,\omega}(z)\omega(z)\ dA(z).\end{aligned}$$

Thus, in this case the proposition is proved. If $\varphi(0)=a\neq 0$, define the function $\psi=\sigma_a\circ\varphi$. Then $C_\varphi = C_\psi C_{\sigma_a}$ and C_{σ_a} is invertible on $\mathcal{H}_\omega$. Therefore C_φ has closed range if and only if C_ψ has closed range. Since $\psi(0)=0$, the argument above shows that C_ψ has closed range if and only if there exists a constant $c>0$ such that

$$\int_{\mathbb{D}} |f'(z)|^2\tau_{\psi,\omega}(z)\omega(z)\ dA(z) \geq c\int_{\mathbb{D}} |f'(z)|^2\omega(z)\ dA(z). \tag{5.2}$$

We just prove that (5.1) and (5.2) are equivalent. If (5.1) holds, then

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^2\tau_{\psi,\omega}(z)\omega(z)\ dA(z) &= \int_{\mathbb{D}} |f'(z)|^2 N_{\psi,\omega}\ dA(z)\\ &= \int_{\mathbb{D}} |(f\circ\psi)'(z)|^2\omega(z)\ dA(z)\\ &= \int_{\mathbb{D}} |(f\circ\sigma_a)'(\varphi(z))|^2|\varphi'(z)|^2\omega(z)\ dA(z)\\ &= \int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^2\tau_{\varphi,\omega}(z)\omega(z)\ dA(z)\\ &\geq c\int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^2\omega(z)\ dA(z)\\ &= c\int_{\mathbb{D}} |f'(z)|^2\omega(\sigma_a(z))\ dA(z)\\ &\asymp c\int_{\mathbb{D}} |f'(z)|^2\omega(z)\ dA(z).\end{aligned}$$

The last equation is due to Lemma 2.2. Hence (5.2) holds. Since $\varphi = \sigma_a\circ\psi$, the proof of converse part is similar. □

Fredholm composition operator is an example of composition operators with closed range property. Recall that a bonded operator T between two Banach spaces X, Y is called Fredholm if Kernel T and T^* are finite dimensional.

Example 5.2. Suppose that $C_\varphi : \mathcal{H}_\omega \to \mathcal{H}_\omega$ be a Fredholm operator. By theorem 3.29 [1], φ is an authomorphism of $\mathbb{D}$. Then $N_{\varphi,\omega}(z) = \omega(\varphi^{-1}(z))$. If $\varphi(0) = 0$, Schwarz lemma implies that $|\varphi^{-1}(z)| \leq |z|$. Since ω is non-increasing,

$$\omega(\varphi^{-1}(z)) = \omega(|\varphi^{-1}(z)|) \geq \omega(|z|) = \omega(z).$$

Now (5.1) holds. If $\varphi(0) \neq 0$, then the same argument can be applied.

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