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WEIGHTED COMPOSITION OPERATORS BETWEEN GROWTH SPACES ON CIRCULAR AND STRICTLY CONVEX DOMAINS

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Abstract. Let Ω_X be a bounded, circular and strictly convex domain of a Banach space X and $\mathcal{H}(\Omega_X)$ denote the space of all holomorphic functions defined on Ω_X. The growth space $\mathcal{A}^\omega(\Omega_X)$ is the space of all $f \in \mathcal{H}(\Omega_X)$ for which

$$|f(x)| \leqslant C\omega(r_{\Omega_X}(x)), \quad x \in \Omega_X,$$

for some constant $C > 0$, whenever r_{Ω_X} is the Minkowski functional on Ω_X and $\omega : [0, 1) \to (0, \infty)$ is a nondecreasing, continuous and unbounded function. Boundedness and compactness of weighted composition operators between growth spaces on circular and strictly convex domains were investigated.

1. Introduction

Let Ω_X, be a bounded, circular and strictly convex domain of a Banach space X. A domain $\Omega_X \subset X$ is said to be circular if $e^{i\theta}x \in \Omega_X$ for every point $x \in \Omega_X$ and any real number θ. An example of a circular domain is the annulus $\{x \in X : r_1 < \|x\| < r_2\}$, where $0 < r_1 < r_2$. We recall that a domain Ω_X is strictly convex if it is convex and contains the open line segment (x_1, x_2) for each pair of boundary points $x_1, x_2 \in \partial\Omega_X$. The open unit ball $\mathbb{B}_X$, is a good model of a circular and strictly convex domain of a Banach space X.

We recall that, $P_m : \Omega_X \to \mathbb{C}$ is said to be an m-homogeneous polynomial if there exists an m-linear mapping $A : \Omega_X^m \to \mathbb{C}$ such that

$$P_m(x) = A(x, \ldots, x).$$

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A mapping $f : \Omega_X \to \mathbb{C}$ is said to be holomorphic or analytic if for each $a \in \Omega_X$, there exist a ball $B(a, r) \subset \Omega_X$ and a sequence of continuous m-homogeneous polynomials $P_m : \Omega_X \to \mathbb{C}$ such that

$$f(x) = \sum_{m=0}^{\infty} P_m(x - a),$$

uniformly for $x \in B(a, r)$. For a general background about m-homogeneous polynomials and holomorphic mappings we refer to [10].

Let $\mathcal{H}(\Omega_X)$ be the class of all holomorphic functions on Ω_X and r_{Ω_X} denote the Minkowski functional on Ω_X, that is

$$r_{\Omega_X}(x) = \inf\{\lambda > 0 : \lambda^{-1} x \in \Omega_X\}.$$

Clearly r_{Ω_X} is a seminorm on Ω_X and $r_{\Omega_X}(x) < 1$ for all $x \in \Omega_X$ [4, Proposition 1.14]. Also r_{Ω_X} is continuous and $a\|x\| \leq r_{\Omega_X}(x) \leq b\|x\|$ for all $x \in X$ and fixed $a, b > 0$ [12, Proposition 8].

A non-negative continuous function τ on Ω_X is an exhaustion of Ω_X with radious Δ if and only if

$$0 \leq \sqrt{\tau} < \Delta = \sup \sqrt{\tau} \leq \infty,$$

and $\{x \in \Omega_X : \tau(x) \leq r^2\}$ is compact in Ω_X for all $r \in [0, \Delta]$. Let $n \in \mathbb{N}$, we assume that $\Omega_{\mathbb{C}^n}$ is a bounded, circular and strictly convex domain with boundary of class $\mathcal{C}^2$. Then $r^2_{\Omega_{\mathbb{C}^n}}$ is an exhaustion of $\Omega_{\mathbb{C}^n}$ with maximal radious $\Delta = 1$ [11, Example 2].

Let $\omega : [0, 1) \to (0, \infty)$ be a weight, that is, ω is a non-decreasing, continuous and unbounded function. The growth space $\mathcal{A}^{\omega}(\Omega_X)$ is the space of all $f \in \mathcal{H}(\Omega_X)$ satisfying

$$|f(x)| \leqslant C\omega(r_{\Omega_X}(x)), \quad x \in \Omega_X,$$

for some constant $C > 0$. Using [10, Proposition 9.13], it is easy to show that $\mathcal{A}^{\omega}(\Omega_X)$ is a Banach space with respect to the following norm:

$$\|f\|_{\mathcal{A}^{\omega}(\Omega_X)} := \sup_{x \in \Omega_X} \frac{|f(x)|}{\omega(r_{\Omega_X}(x))}.$$

Now we recall some particular cases. If $X = \mathbb{C}^n$ and $\omega(t) = \frac{1}{(1-t)^{\alpha}}$, $\alpha > 0$, we get the growth spaces $\mathcal{A}^{-\alpha}(\Omega_{\mathbb{C}^n})$. For $X = \mathbb{C}^n$ and $\omega(t) = \log \frac{e}{1-t}$, we have the logarithmic growth space $\mathcal{A}^{-\log}(\Omega_{\mathbb{C}^n})$. For more details on the growth spaces and logarithmic growth space we refer to [5, 6, 7]. When $\Omega_{\mathbb{C}^n} = \mathbb{B}_n$, the open unit ball of $\mathbb{C}^n$, we have $\mathcal{A}^{-\alpha}(\mathbb{B}_n) = H^{\infty}_{\alpha}$, the standard weighted Banach space of holomorphic functions [2, 3, 8, 9].

Let $\varphi : \Omega_Y \to \Omega_X$ be a holomorphic mapping and $\psi \in \mathcal{H}(\Omega_Y)$. Then the weighted composition operator

$$\psi C_{\varphi} : \mathcal{H}(\Omega_X) \to \mathcal{H}(\Omega_Y)$$

is defined by $\psi C_\varphi f = \psi(f \circ \varphi)$. When $\psi(x) \equiv 1$, the weighted composition operator $1C_\varphi = C_\varphi$ is called the composition operator.

Note that, the norm topology on growth space is stronger than the pointwise convergence topology. Therefore, if ψC_φ between growth spaces is well defined, then ψC_φ is bounded by closed graph theorem. As a consequence, to find out whether the weighted composition operator is bounded, it is enouph to find out whether it is well defined. We refer to [5, 6, 7] and particularly to the recent survey [1] for information about the weighted composition operators between growth spaces in the special cases. Throughout this paper, constants are denoted by C, they are positive and not necessarily the same in all accurrences.

2. **Results**

In this section, we characterize the boundedness and compactness of the operator $\psi C_\varphi : \mathcal{A}^\nu(\Omega_{\mathbb{C}^n}) \to \mathcal{A}^\omega(\Omega_X)$ by reverse estimates in $\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$.

A weight ν is called doubling, if there exists a constant $C > 1$ such that

$$\nu\left(1 - \frac{t}{2}\right) < C\nu(1-t), \quad 0 < t \leq 1.$$

To prove the boundedness of weighted composition operator, we need the following lemma.

Lemma 2.1. [1] *Let $\Omega_{\mathbb{C}^n} \subset \mathbb{C}^n$ be a bounded, circular and strictly convex domain with $\mathcal{C}^2$-boundary and ν be a doubling weight. There exist functions $f_m \in \mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$, $0 \leqslant m \leqslant M = M(\Omega_{\mathbb{C}^n})$ such that*

$$\sum_{m=0}^{M} |f_m(z)| \geqslant \nu(r_{\Omega_{\mathbb{C}^n}}(z)), \quad z \in \Omega_{\mathbb{C}^n}.$$

Theorem 2.2. *Let $\psi \in \mathcal{H}(\Omega_X)$, $\varphi : \Omega_X \to \Omega_{\mathbb{C}^n}$ be a holomorphic mapping and ν be a doubling weight. Then ψC_φ maps $\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$ into $\mathcal{A}^\omega(\Omega_X)$ if and only if*

$$L := \sup_{x \in \Omega_X} \frac{|\psi(x)|\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))}{\omega(r_{\Omega_X}(x))} < \infty. \tag{2.1}$$

Proof. Setting $z = \varphi(x)$ in Lemma 2.1 we get

$$\sum_{m=0}^{M} |f_m(\varphi(x))| \geq \nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x))), \quad x \in \Omega_X.$$

Then

$$\frac{|\psi(x)|\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))}{\omega(r_{\Omega_X}(x))} \leq \sum_{m=0}^{M} \frac{|\psi(x)||f_m(\varphi(x))|}{\omega(r_{\Omega_X}(x))}$$
$$= \sum_{m=0}^{M} \frac{|(\psi C_\varphi f_m)(x)|}{\omega(r_{\Omega_X}(x))}, \quad x \in \Omega_X.$$

Since $f_m \in \mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$ and

$$\psi C_\varphi(\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})) \subseteq \mathcal{A}^\omega(\Omega_X),$$

then $\psi C_\varphi f_m$ are in $\mathcal{A}^\omega(\Omega_X)$ for $0 \leq m \leq M$. This implies that (2.1) is satisfied.

Conversely for $f \in \mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$, we have

$$\begin{aligned}\|\psi C_\varphi f\|_{\mathcal{A}^\omega(\Omega_X)} &= \sup_{x\in\Omega_X} \frac{|\psi(x) f(\varphi(x))|}{\omega(r_{\Omega_X}(x))} \\ &= \sup_{x\in\Omega_X} \frac{|\psi(x)|\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))}{\omega(r_{\Omega_X}(x))} \times \frac{|f(\varphi(x))|}{\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))} \\ &\leq L\|f\|_{\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})},\end{aligned}$$

which implies that $\psi C_\varphi(\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})) \subseteq \mathcal{A}^\omega(\Omega_X)$. □

Moreover, the above theorem shows that $\psi C_\varphi : \mathcal{A}^\nu(\Omega_{\mathbb{C}^n}) \to \mathcal{A}^\omega(\Omega_X)$ is, infact, bounded if and only if (2.1) satisfies.

Lemma 2.3 ([1]). *Let Y be a linear metric space with translation metric. Then the operator $\psi C_\varphi : \mathcal{A}^\nu(\Omega_{\mathbb{C}^n}) \to Y$ is compact if and only if it is bounded and $\psi C_\varphi f_k$ converges to zero in the metric of Y for any bounded sequence f_k in $\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$ such that $f_k \to 0$ uniformly on compact subsets of $\Omega_{\mathbb{C}^n}$.*

Lemma 2.4 ([1]). *Let ν be a doubling weight. Then there exist test functions $f_\lambda \in \mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$, $\lambda \in \Omega_{\mathbb{C}^n}$, such that*

$$\begin{aligned}\|f_\lambda\|_{\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})} &\leq C, \\ C|f_\lambda(\lambda)| &\geq \nu(r_{\Omega_{\mathbb{C}^n}}(\lambda)),\end{aligned}$$

and $f_\lambda \to 0$ uniformly on compact subsets of $\Omega_{\mathbb{C}^n}$ as $r_{\Omega_{\mathbb{C}^n}}(\lambda) \to 1$, where the constant $C > 0$ does not depend on $\lambda \in \Omega_{\mathbb{C}^n}$.

Theorem 2.5. *Let $\psi \in \mathcal{H}(\Omega_X)$, $\varphi : \Omega_X \to \Omega_{\mathbb{C}^n}$ be a holomorphic mapping and ν be a doubling weight. Then $\psi C_\varphi : \mathcal{A}^\nu(\Omega_{\mathbb{C}^n}) \to \mathcal{A}^\omega(\Omega_X)$ is compact if and only if*

$$\lim_{r_{\Omega_{\mathbb{C}^n}}(\varphi(x))\to 1} \frac{|\psi(x)|\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))}{\omega(r_{\Omega_X}(x))} = 0. \tag{2.2}$$

Proof. Let (2.2) hold and $(f_k)_{k\in N}$ be a sequence in the closed unit ball of $\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$ such that converges uniformly to zero on compact subsets of $\Omega_{\mathbb{C}^n}$. Using Lemma 2.3 it is sufficient to show that

$$\lim_{k\to\infty} \|\psi C_\varphi f_k\|_{\mathcal{A}^\omega(\Omega_X)} = 0.$$

By (2.2), give $\varepsilon > 0$, there is $\delta \in (0,1)$, such that

$$\frac{|\psi(x)||f_k(\varphi(x))|}{\omega(r_{\Omega_X}(x))} \leq \|f_k\|_{\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})} \frac{|\psi(x)|\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))}{\omega(r_{\Omega_X}(x))} < \varepsilon, \tag{2.3}$$

if $r_{\Omega_{\mathbb{C}^n}}(\varphi(x)) > \delta$, since $\|f_k\|_{\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})} \leq 1$. Next, since $r^2_{\Omega_{\mathbb{C}^n}}$ is an exhaustion of $\Omega_{\mathbb{C}^n}$ with maximal radious $\Delta = 1$, the subset $\Omega^\delta_{\mathbb{C}^n} = \{z \in \Omega_{\mathbb{C}^n} : r_{\Omega_{\mathbb{C}^n}}(z) \leq \delta\}$ of $\Omega_{\mathbb{C}^n}$ is compact. Therefore

$$\frac{|f_k(z)|}{\nu(r_{\Omega_{\mathbb{C}^n}}(z))} \leq \frac{|f_k(z)|}{\nu(0)} \to 0,$$

uniformly on $\Omega^\delta_{\mathbb{C}^n}$ as $k \to \infty$. Hence, if $r_{\Omega_{\mathbb{C}^n}}(\varphi(x)) \leq \delta$ then

$$\frac{|f_k(\varphi(x))|}{\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))} < \varepsilon, \tag{2.4}$$

for k sufficiently large. Since

$$\psi C_\varphi(\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})) \subseteq \mathcal{A}^\omega(\Omega_X),$$

using Theorem 2.2, we have

$$\frac{|\psi(x)||f_k(\varphi(x))|}{\omega(r_{\Omega_X}(x))} \leq L \frac{|f_k(\varphi(x))|}{\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x)))}.$$

Applying this inequality along with (2.3) and (2.4), we have

$$\frac{|\psi(x)||f_k(\varphi(x))|}{\omega(r_{\Omega_X}(x))} < C\varepsilon, \quad x \in \Omega_X,$$

for k sufficiently large. It follows that

$$\|\psi C_\varphi f_k\|_{\mathcal{A}^\omega(\Omega_X)} \to 0 \text{ as } k \to \infty.$$

Hence $\psi C_\varphi : \mathcal{A}^\nu(\Omega_{\mathbb{C}^n}) \to \mathcal{A}^\omega(\Omega_X)$ is compact.

Conversely, let $(x_k)_{k\in\mathbb{N}}$ be a sequence in Ω_X such that

$$r_{\Omega_{\mathbb{C}^n}}(\varphi(x_k)) \to 1 \text{ as } k \to \infty.$$

By Lemma 2.4, there exist functions $f_k := f_{\varphi(x_k)} \in \mathcal{A}^\nu(\Omega_{\mathbb{C}^n})$, such that $\|f_k\|_{\mathcal{A}^\nu(\Omega_{\mathbb{C}^n})} \leq C$,

$$C|f_k(\varphi(x_k))| \geq \nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x_k))),$$

and $f_k \to 0$ uniformly on compact subsets of $\Omega_{\mathbb{C}^n}$ as $r_{\Omega_{\mathbb{C}^n}}(\varphi(x_k)) \to 1$. Since $\psi C_\varphi : \mathcal{A}^\nu(\Omega_{\mathbb{C}^n}) \to \mathcal{A}^\omega(\Omega_X)$ is compact, using Lemma 2.3 we have

$$\|\psi C_\varphi f_k\|_{\mathcal{A}^\omega(\Omega_X)} \to 0, \tag{2.5}$$

as $k \to \infty$. On the other hand

$$\begin{aligned}\|\psi C_\varphi f_k\|_{\mathcal{A}^\omega(\Omega_X)} &= \sup_{x\in\Omega_X} \frac{|\psi(x)||f_{\varphi(x_k)}(\varphi(x))|}{\omega(r_{\Omega_X}(x))} \\ &\geq \frac{|\psi(x_k)||f_{\varphi(x_k)}(\varphi(x_k))|}{\omega(r_{\Omega_X}(x_k))} \\ &\geq C\frac{|\psi(x_k)|\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x_k)))}{\omega(r_{\Omega_X}(x_k))}.\end{aligned}$$

Comparing this inequality with (2.5), we obtain

$$\lim_{k\to\infty} \frac{|\psi(x_k)|\nu(r_{\Omega_{\mathbb{C}^n}}(\varphi(x_k)))}{\omega(r_{\Omega_X}(x_k))} = 0,$$

which implies (2.2). □

References

1. E. Abakumov and E. Doubtsov, *Reverse estimates in growth spaces, Mathematische Zeitschrift*, 271 (2012) 392-413.
2. K. D. Bierstedt, J. Bonet and J. Taskinen, *Associated weights and spaces of holomorphic functions*, Studia Math. 127 (2) (1998) 137-168.
3. J. Bonet, P. Domanski, M. Lindstrom and J. Taskinen, *Composition operators between weighted Banach spaces of analytic functions*, J. Aust. Math. Soc. (Ser. A), 64 (1998) 101-118.
4. J. B. Conway, *A Course in Functional Analysis*, Spinger-Verlag New York, 1985.
5. E. Doubstov, *Growth spaces on circular domains: Composition operators and Carleson measure*, Comptes Rendus Mathematigue, 347 (2009), 609-611.
6. E. Doubtsov, *Carleson-Sobolev measure for weighted Bloch spaces*, Func. Anal. 258 (2010) 2801-2816.
7. E. S. Dubtsov, *Weighted composition operators on growth spaces*, Siberian Mathematical Journal, 50 (6) (2009) 998-1006.
8. W. Lusky, *On weighted spaces of harmonic and holomorphic functions*, J. London Math. Soc., 51 (1995) 309-320.
9. A. Montes-Rodrguez, *Weighted composition operators on weighted Banach spaces of analytic functions*, J. London Math. Soc., 61 (2) (2000) 872-884.
10. J. Mujica, *Complex Analysis in Banach Spaces*, Elsevier Science Publishing Company, 1985.
11. G. Patrizio, *Parabolic exhaustions for strictly convex domains*, Manuscripta Math. 47 (1984) 271-309.
12. E. Zeidler, *Applied Functional Analysis Applications to Mathematical Physics*, Applied Mathematical Sciences, Springer-Verlag New Yorc Inc. V. 108.

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