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# COMPARISON OF THE ACCELERATION TECHNIQUES ON ANALYTICAL METHODS FOR SOLVING DIFFERENTIAL EQUATIONS OF INTEGER AND FRACTIONAL ORDER

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Abstract. The work addressed in this paper is a comparative study between convergence of the acceleration techniques, diagonal padé approximants and shanks transforms, on Homotopy analysis method and Adomian decomposition method for solving differential equations of integer and fractional orders.

## 1. Introduction

It is obvious that mathematical modeling of many physical systems leads to nonlinear differential equations. Also in recent years, it has been turned out that fractional differential equations can be used successfully to model many phenomena in various sciences. Therefore, there is an increasing interest to study of the fractional differential equations because of their various applications such as in viscoelasticity, anomalous diffusion, fluid mechanics, biology, chemistry, acoustics, control theory, etc. The exact solutions of the nonlinear differential equations can help us to know the described process. So, in the past decades, mathematicians have made many efforts in the study of exact solutions of nonlinear differential equations. But, for most differential equations, no exact solution is known and, in some cases, it is not even clear whether a unique solution exists. So, approximation methods, such as numerical and analytical methods, have been developed. Numerical methods give approximate solutions only at discrete points and they may also give rise to numerical instabilities such as oscillations, false equilibrium states, etc.

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In analytical methods [6] we obtain a continues solution for a differential equation. Adomian decomposition method [1, 4, 5] efficiently works with different types of linear and nonlinear equations and gives an analytic solution for all these types of equations without linearization or discretization. As an analytic tool, to solve nonlinear differential equations, homotopy analysis method(HAM) [2, 3, 8] provides a convenient way to guarantee the convergence of solution series. There also exist some techniques to accelerate the convergence of a series solution, such as padé and Shanks transform techniques[7]. In this paper comparisons are made between Adomian decomposition method, homotopy analysis method and accelerated solutions by padé technique and Shanks transformation.
In the remainder of this section we present some definitions.

**Definition 1.1.** We say that $f(t)$ is a function of class $\zeta$, if $f(t)$ is piecewise continuous on $(0, +\infty)$ and integrable on any finite subinterval of $(0, +\infty)$.

**Definition 1.2.** Let $f(t)$ be a function of class $\zeta$, then the Riemann-Liouville fractional integral of $f(t)$ of order $\beta$ is defined as

$$J_t^\beta f(t) = \int_0^t \frac{(t-\tau)^{\beta-1}}{\Gamma(\beta)} f(\tau) d\tau, \quad \beta > 0, \tag{1.1}$$

where $\Gamma(.)$ is Euler's gamma function.

The fractional integral satisfies the following equality

$$J_t^\nu t^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)} t^{\mu+\nu}, \quad \nu \geq 0,\ \mu > -1. \tag{1.2}$$

**Definition 1.3.** Let $f(t)$ be a function of class $\zeta$ and $\alpha$ be a positive real number satisfying

$$m - 1 < \alpha \leq m, \quad m \in \mathbf{N}^+,$$

then the Rimann-Liouville fractional derivative of $f(t)$ of order $\alpha$, when it exists, is defined as

$$D_t^\alpha f(t) = \frac{d^m}{dt^m}(J_t^{m-\alpha} f(t)), \quad t > 0. \tag{1.3}$$

**Definition 1.4.** Let $\alpha$ be a positive real number, such that $m - 1 < \alpha \leq m$, $m \in \mathbf{N}^+$, and $f^{(m)}(t)$ exists and be a function of class $\zeta$, then the Caputo fractional derivative of $f(t)$ of order $\alpha$ is defind as

$$D_t^\alpha f(t) = J_t^{m-\alpha} f^{(m)}(t), \quad t > 0. \tag{1.4}$$

## 2. Analytical Methods

2.1. **The Adomian decomposition method.** Let us consider the nonlinear differential equation

$$L(u) + R(u) + N(u) - g(x,t) = 0, \tag{2.1}$$

where $L$ is the highest order derivative which assumed to be invertible, $R$ is a linear differentiable operator of order less than $L$ and $N$ is a nonlinear operator. Define the solution $u(x,t)$ by the series

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \tag{2.2}$$

and the nonlinear terms can be decomposed into the infinite series of polynomials given by

$$N(u) = \sum_{k=0}^{\infty} A_k, \tag{2.3}$$

where $A_k$ are so called the Adomains polynomials which given by

$$A_k = \frac{1}{k!}\left[\frac{d^k}{d\lambda_k}\left[N\left(\sum_{k=0}^{\infty} \lambda^i u_i(x,t)\right)\right]\right]_{\lambda=0}. \tag{2.4}$$

So, the components $u_k(x,t)$ are determind by the following recursive relationship

$$u_0(x,t) = g(x,t), \qquad u_{k+1}(x,t) = -L^{-1}[R(u_k) + A_k], \qquad k \geq 0. \tag{2.5}$$

2.2. **Homotopy analysis method.** We consider the following differential equation

$$N[u(x,t)] = 0, \tag{2.6}$$

where $N$ represents a general nonlinear operator involving both linear and nonlinear terms, $u(x,t)$ is an unknown function and $x$ and $t$ denote spatial and temporal independent variables, respectively. By means of generalizing the traditional homotopy method, Liao [3] constructed the following zero-order deformation equation

$$(1-p)L[\phi(x,t;p) - u_0(x,t)] = p\hbar N[\phi(x,t;p)]. \tag{2.7}$$

Obviously, when $p = 0$ and $p = 1$, it holds

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t), \tag{2.8}$$

respetively. Thus, as $p$ increases from 0 to 1, the solution $\phi(x,t;p)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expending $\phi(x,t;p)$

in Taylor series with respect to $p$, one has

$$\phi(x,t;p) = u_0(x,t) + \sum_{m=1}^{+\infty} u_m(x,t)p^m, \tag{2.9}$$

where

$$u_m(x,t) = \frac{1}{m!}\frac{\partial^m \phi(x,t;p)}{\partial p^m}\bigg|_{p=0}. \tag{2.10}$$

Differentiating Eq.(2.7) $m$ times with respect to the embedding parameter $p$ and then setting $p = 0$ and finally dividing them by $m!$, we have the so-called $m$th-order deformation equation

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar \Re_m[u_{\overrightarrow{m-1}}(x,t)], \tag{2.11}$$

where

$$\Re_m(u_{\overrightarrow{m-1}}) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,t;p)]}{\partial p^{m-1}}\bigg|_p = 0, \tag{2.12}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m \geq 2. \end{cases} \tag{2.13}$$

In this way, it is easily to obtain $u_m(x,t)$ for $m \geq 1$, at $M$th order, we have

$$u(x,t) = \sum_{k=0}^{M} u_k(x,t).$$

When $M \to \infty$, we get an accurate approximation of the original equation (2.6).

2.3. **Convergence accelearation techniques.** Convergence acceleration techniques are used to accelerate convergence rate of a sequence or series and also for extending the region of convergence. In Shanks transformation starting from $S_n^0 = \phi_n(t)$, then we continue by the following

$$S_n^k = \frac{S_n^{(k-1)} S_{n+2}^{(k-1)} - (S_{n+1}^{(k-1)})^2}{S_n^{(k-1)} + S_{n+2}^{(k-1)} - 2S_{n+1}^{(k-1)}}, \qquad k \geq 1. \tag{2.14}$$

So, it yields one expression if $m = 3, 5, 7, \ldots$, while two expressions if $m = 4, 6, 8, \ldots$. Also, a Padé approximant is the ratio of two polynomials constructed from the coefficients of the Taylor series expansion of a function $u(x)$. The $[\frac{L}{M}]$ Padé approximant for a function $u(x)$ is given by

$$\left[\frac{L}{M}\right] = \frac{P_L(x)}{Q_M(x)}, \tag{2.15}$$

where $P_L(x)$ is polynomial of degree at most $L$ and $Q_M(x)$ is a polynomial of degree at most $M$. Using the formal power series

$$u(x) = \sum_{i=1}^{\infty} a_i x^i,$$

$$u(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}),$$

one can determine the coefficients of $P_L(x)$ and $Q_M(x)$ by comparing the coefficient of like powers.

## 3. Applications

**Example 3.1.** We first consider the following KdV equation

$$u_t - 6uu_x + u_{xxx} = 0, \qquad x \epsilon R, \tag{3.1}$$

with initial condition

$$u(x,0) = \frac{-k^2}{2}\text{sech}^2\left[\frac{k}{2}x\right], \tag{3.2}$$

where $k$ is a parameter. The exact solution is given by

$$u(x,t) = \frac{-k^2}{2}\sec^2\left[\frac{k}{2}(x - k^2 t)\right]. \tag{3.3}$$

Following the Adomian decomposition method we consider the equation as an operator equation

$$\begin{aligned} Lu + Ru + 6N(u) &= 0, \qquad x \epsilon R, \\ u(x,0) &= \frac{-k^2}{2}\text{sech}^2\left[\frac{k}{2}x\right], \end{aligned}$$

where

$$Lu = \frac{\partial u}{\partial t}, \qquad Ru = \frac{\partial^3 u}{\partial x^3}, \qquad N(u) = u\frac{\partial u}{\partial x}.$$

Considering the given initial condition, it is straightforward to choose

$$u(x,0) = \frac{-k^2}{2}\text{sech}^2\left[\frac{k}{2}x\right]$$

as an initial approximation. Now, using the recursive relation (2.5), we obtain the following relations

$$u_0(x,t) = \frac{-k^2}{2}\text{sech}^2\left[\frac{k}{2}x\right],$$

$$u_1(x,t) = -\frac{1}{4}k^5t(1+\cosh[kx])\text{sech}\left[\frac{kx}{2}\right]^4\tanh\left[\frac{kx}{2}\right],$$

$$u_2(x,t) = -\frac{1}{8}k^8t^2(-2+\cosh[kx])\text{sech}\left[\frac{kx}{2}\right]^4,$$

$$u_3(x,t) = -\frac{1}{24}k^{11}t^3(-5+\cosh[kx])\text{sech}\left[\frac{kx}{2}\right]^4\tanh\left[\frac{kx}{2}\right],$$

$$u_4(x,t) = \frac{1}{384}k^{14}t^4(-33+26\cosh[kx]-\cosh[2kx])\text{sech}\left[\frac{kx}{2}\right]^6,$$

$$u_5(x,t) = \frac{k^{17}t^5\text{sech}\left[\frac{kx}{2}\right]^6\left(57\text{sech}\left[\frac{kx}{2}\right]\sinh\left[\frac{3kx}{2}\right]\right)}{3840}, + \frac{-\text{sech}\left[\frac{kx}{2}\right]\sinh\left[\frac{5kx}{2}\right]-302\tanh\left[\frac{kx}{2}\right]\Big)}{3840}$$

$$\vdots$$

Next we apply two common convergence acceleration techniques on these results. In table 1, we make a comparison between the exact solution with the results of ADM, accelerated ADM solutions by iterated Shanks transforms and diagonal padé approximant respectively.

TABLE 1. Numerical comparison of methods for Example 3.1.

| $t$ | $\lvert u_{ADM} - u_{exact}\rvert$ | $\lvert u_{ADM-SHANKS} - u_{exact}\rvert$ | $\lvert u_{ADM-PADE} - u_{exact}\rvert$ |
|---|---|---|---|
| $-4$ | 16.9028 | 1.53533 | 0.00161449 |
| $-3$ | 1.34279 | 0.715353 | 0.000359940 |
| $-2$ | 0.0328700 | 0.00209987 | 0.0000234224 |
| $-1$ | 0.00004250260 | $0..0182239 \times 10^{-4}$ | $6.43979 \times 10^{-8}$ |
| 0 | 0 | 0 | 0 |

**Example 3.2.** We consider the quadratic Riccati differential equation

$$D^\mu y = 2y(t) - y^2(t) + 1, \quad 0 < \mu \leq 1, t > 0, \tag{3.4}$$

subject to initial condition $y(0) = 0$. Exact solution when $\mu = 1$ is

$$y(t) = 1 + \sqrt{2}\tanh(\sqrt{2}t) + \frac{1}{2}\ln\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right). \tag{3.5}$$

Following the HAM, the solution of the $m$th-order deformation equation for $m \geq 1$ becomes

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar J_t^{\mu}[\Re_m(\vec{y_{m-1}})], \quad m \geq 1, \tag{3.6}$$

where

$$\Re_m(\vec{y_{m-1}}) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t;q)]}{\partial q^{m-1}}\bigg|_{q=0} = D^{\mu} y_{m-1}(t) - (1-\chi_m) - 2y_{m-1}(t) + \Sigma_{k=0}^{m-1} y_k(t) y_{m-1-k}(t).$$

Starting with

$$y_0(t) = \frac{t^{\mu}}{\Gamma(\mu+1)}, \tag{3.7}$$

we obtain

$$y_1(t) = \frac{-2\hbar t^{2\mu}}{\Gamma(1+2\mu)} + \frac{4^{\mu}\hbar t^{3\mu}\Gamma\left(\frac{1}{2}+\mu\right)}{\sqrt{\pi}\mu\Gamma(\mu)\Gamma(1+3\mu)},$$

$$\begin{aligned} y_2(t) = &-\frac{2\hbar t^{2\mu}}{\Gamma(1+2\mu)} - \frac{2\hbar^2 t^{2\mu}}{\Gamma(1+2\mu)} + \frac{4\hbar^2 t^{3\mu}}{\Gamma(1+3\mu)} + \frac{2^{2\mu}\hbar t^{3\mu}\Gamma\left(\frac{1}{2}+\mu\right)}{\sqrt{\pi}\mu\Gamma(\mu)\Gamma(1+3\mu)} \\ &+ \frac{2^{2\mu}\hbar^2 t^{3\mu}\Gamma\left(\frac{1}{2}+\mu\right)}{\sqrt{\pi}\mu\Gamma(\mu)\Gamma(1+3\mu)} - \frac{2^{1+2\mu}\hbar^2 t^{4\mu}\Gamma\left(\frac{1}{2}+\mu\right)}{\sqrt{\pi}\mu\Gamma(\mu)\Gamma(1+4\mu)} \\ &- \frac{12\hbar^2 t^{4\mu}\Gamma(3\mu)}{\Gamma(\mu)\Gamma(1+2\mu)\Gamma(1+4\mu)} + \frac{2^{3+2\mu}\hbar^2 t^{5\mu}\Gamma(4\mu)\Gamma\left(\frac{1}{2}+\mu\right)}{\sqrt{\pi}\Gamma(\mu)\Gamma(1+\mu)\Gamma(1+3\mu)\Gamma(1+5\mu)} \\ &\vdots \end{aligned}$$

TABLE 2. Numerical comparison of methods for Example 3.2. when $\mu = 1$.

| $t$ | $y_{ADM} or y_{HAM} (\hbar = -1)$ | $y_{Shanks}$ | $y_{HP}[5,5]$ | $y_{Exact}$ |
|---|---|---|---|---|
| 0.5 | 0.756067 | 0.756056 | 0.756014 | 0.756014 |
| 0.6 | 0.953933 | 0.953488 | 0.953466 | 0.953466 |
| 0.7 | 1.1548 | 1.15198 | 1.15295 | 1.15295 |
| 0.8 | 1.35368 | 1.3432 | 1.34636 | 1.34636 |
| 0.9 | 1.55065 | 1.52407 | 1.52691 | 1.52691 |
| 1.0 | 1.75534 | 1.67655 | 1.6895 | 1.6895 |

Also by the Adomian decomposition method, we obtain the following components

$$
\begin{aligned}
y_0(t) &= \frac{t^\mu}{\Gamma(\mu+1)},\\
y_1(t) &= -\frac{2^{1-2\mu}t^{2\mu}\cos(\pi\mu)\Gamma\left(\frac{1}{2}-\mu\right)}{\sqrt{\pi}\mu\Gamma(\mu)} - \frac{t^{3\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+3\mu)},\\
y_2(t) &= \frac{2^{3-2\mu}t^{3\mu}\cos(\pi\mu)\Gamma\left(\frac{1}{2}-\mu\right)\Gamma(2\mu)}{\sqrt{\pi}\Gamma(\mu)\Gamma(1+3\mu)} - \frac{2t^{4\mu}\Gamma(1+2\mu)}{\Gamma(1+\mu)^2\Gamma(1+4\mu)}\\
&\quad + \frac{2t^{5\mu}\Gamma(1+2\mu)\Gamma(1+4\mu)}{\Gamma(1+\mu)^3\Gamma(1+3\mu)\Gamma(1+5\mu)} + \frac{12t^{4\mu}\Gamma(-2\mu)\Gamma(3\mu)\sin(2\pi\mu)}{\pi\Gamma(\mu)\Gamma(1+4\mu)},\\
&\ \ \vdots
\end{aligned}
$$

Next we apply two common convergence acceleration techniques on these results. In table 2 we compare the results of exact solution, accelerated solutions of homomtopy analysis method, Adomian decomposition method by pad'e technique and shanks transform at some points.

## 4. CONCLUSION

In this work, we made a comparative study between the results given by standard ADM, HAM and accelerated solutions by diagonal padé approximants and shanks transforms. It was shown that convergence region can be enlarged by making use of both Shanks transforms and Padé approximant. Also the rate of convergence increased by means of acceleration techniques.

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