SOME NEW PROPERTIES OF FUZZY STRONGLY $g^*$-CLOSED SETS AND $\delta g^*$-CLOSED SETS IN FUZZY TOPOLOGICAL SPACES

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Abstract. In this paper, a new class of fuzzy sets called fuzzy strongly $g^*$-closed sets is introduced and its properties are investigated. Moreover, we study some more properties of this type of closed spaces.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh in his classical paper [21]. Subsequently, many researchers have been worked in this area and other related areas which have applications in different fields based on this concept. Topology is Analysis in a general setting. As a generalization of topological spaces Chang [1] introduced the concept of fuzzy topological space in 1968. In 1986, Maki [10] continued the works of Levine [8] and Dunham [7] on generalized closed sets and closure operators by introducing the notion of a generalized $\Lambda$-set in a topological space $(X, \tau)$ and by defining an associated closure operator, i.e. the $\Lambda$-closure operator. He studied the relationships between the given topology $\tau$ and the topology $\tau^\Lambda$ generated by the family of generalized $\Lambda$-sets. $g^*$-closed sets were introduced and studied by Veerakumar [19] for general topology. Recently, Parimalazhagan and Pillai introduced strongly $g^*$-closed sets in topological space [14].

In the present paper, we introduce fuzzy strongly $g^*$-closed sets in fuzzy topological space (FTS) and investigate certain basic properties of these fuzzy sets.

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2. PRELIMINARIES

A family $\tau$ of fuzzy sets of $X$ is called a fuzzy topology on $X$ if 0 and 1 belong to $\tau$ and $\tau$ is closed with respect to arbitrary union and finite intersection. The members of $\tau$ are called fuzzy open sets and their complements are fuzzy closed sets. Throughout this paper $(X, \tau)$ represent non empty topological spaces on which no separation axioms are assumed unless otherwise mentioned. For a subset $A$ of a space $(X, \tau)$, $cl(A)$ and $int(A)$ denote the closure and the interior of $A$ respectively. $(X, \tau)$ will be replaced by $X$ if there is no chance of confusion. We define the closure of a fuzzy set $A$ by

$$cl(A) = \bigwedge \{\mu : \mu \geq A, \ 1 - \mu \in \tau\}$$

and interior

$$int(A) = \bigvee \{\mu : \mu \leq A, \ \mu \in \tau\}.$$  

The semi closure [17] (respectively pre-closure [3], $\alpha$-closure [11] and semi pre-closure [11]) of a fuzzy set $A$ of $(X, \tau)$ is the intersection of all $fs$-closed (respectively $fp$-closed, $fa$-closed and $fsp$-closed) sets that contain $A$ and is denoted by $scl(A)$ (respectively $pcl(A)$, $acl(A)$ and $spcl(A)$).

We recall some basic definitions and notations of most essential concepts needed in the following.

**Definition 2.1.** A fuzzy set $A$ of $(X, \tau)$ is called,

(i) semi open (in short, $fs$-open) if $A \leq cl(int(A))$ and a fuzzy semi closed ($fs$-closed) if $int(cl(A)) \leq A$.

(ii) fuzzy pre-open ($fp$-open) if $A \leq int(cl(A))$ and a fuzzy pre-closed ($fp$-closed) if $cl(int(A)) \leq A$.

(iii) fuzzy $\alpha$-open ($fa$-open) if $A \leq int[cl(int(A))]$ and a fuzzy $\alpha$-closed ($fa$-closed) if $cl(int[cl(A)]) \leq A$.

(iv) fuzzy semi pre-open ($fsp$-open) if $A \leq cl(int[cl(A)])$ and a fuzzy semi pre-closed ($fsp$-closed) if $int[cl(int(A))] \leq A$.

(v) fuzzy $\theta$-open ($f\theta$-open) if $A = int_\theta(A)$ and a fuzzy $\theta$-closed ($f\theta$-closed) if $A = cl_\theta(A)$ where $cl_\theta(A) = \bigwedge \{cl(\mu) : A \leq \mu, \ \mu \in \tau\}$.

(vi) fuzzy generalized closed ($fg$-closed) if $cl(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy open set in $X$.

(vii) fuzzy generalized semi closed ($gf$s-closed) if $scl(A) \leq H$, whenever $A \leq H$ and $H$ is $fs$-open set in $X$. This set is also called generalized fuzzy weakly semi closed set.

(viii) fuzzy generalized semi closed ($fg$s-closed) if $scl(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy open set in $X$.

(ix) fuzzy generalized pre-closed ($fgp$-closed) if $pcl(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy open set in $X$. 

(x) fuzzy α-generalized closed (fαg-closed) if αcl (A) \leq H, whenever A \leq H and H is fuzzy open set in X.

(xi) fuzzy generalized α-closed (fαg-closed) if αcl (A) \leq H, whenever H is fuzzy open set in X.

(xii) fuzzy generalized semi pre-closed (fsp-closed) if spcl (A) \leq H, whenever A \leq H and H is fuzzy open set in X.

(xiii) fuzzy semi pre-generalized closed (fspg-closed) if spcl (A) \leq H, whenever A \leq H and H is fuzzy open set in X.

(xiv) fuzzy g*-generalized closed (fg*-closed) if cl (A) \leq H, whenever A \leq H and H is fuzzy open set in X.

(xv) fuzzy g*-closed (fg*-closed) if cl (A) \leq H, whenever A \leq H and H is fg*-open in X.

**Definition 2.2.** A fuzzy set in X is called a fuzzy point if and only if it takes the value 0 for all y \in X except one, say x \in X. If its value at x is p (0 < p \leq 1), we denote this fuzzy point by x_p, where the point x is called its support.

**Definition 2.3.** A fuzzy point x_p \in A is said to be quasi-coincident with the fuzzy set A denoted by x_pA if and only if p + A (x) > 1. A fuzzy set A is quasi-coincident with a fuzzy set B denoted by AqB if and only if there exists x \in X such that A (x) + B (x) > 1. If A and B are not quasi-coincident, then we write A\overline{q} B. Note that A \leq B if and only if A\overline{q} (1 - B).

3. **δg*-closed Sets**

We need the following definitions in the sequel.

**Definition 3.1.** A fuzzy point x_\alpha is said to be a fuzzy δ-cluster point of a fuzzy set A in a fts X if every fuzzy regular open quasi neighborhood of H of x_\alpha is quasi coincident with A.

**Definition 3.2.** The union of all fuzzy δ-cluster points of A is called the fuzzy δ-closure of A, is denoted by δcl (A).

**Definition 3.3.** A subset A of a topological space (X, \tau) is called a fuzzy g-closed set if fcl (A) \leq H, whenever A \leq H and H is open in (X, \tau).

**Definition 3.4.** The complement of a fg-closed set is called a fg-open set.

**Definition 3.5.** A subset A of a space X is called fδg-closed if fcl_\delta (A) \leq H, whenever A \leq H and H is a fuzzy open set.

**Definition 3.6.** A fuzzy set A in fts (X, \tau) is called fuzzy δg*-closed if and only if fcl_\delta (A) \leq B, whenever A \leq B and B is fuzzy g-open in X.
In the following, we investigate the on $\delta g^*$-closed sets and study some of its fundamental properties. Several characterizations of $\delta g^*$-closed sets are given.

**Theorem 3.7.** Every fuzzy $\delta$-closed set is a fuzzy $\delta g^*$-closed set in $(X, \tau)$.

*Proof.* Let $A$ be a fuzzy $\delta$-closed set in a fts $X$ and $B$ be a fuzzy $g$-open set in $X$ such that $A \subseteq B$. Since $A$ is a fuzzy $\delta$-closed, $fcl_\delta(A) = A$. Therefore $fcl_\delta(A) = A \subseteq B$. Hence $A$ is a fuzzy $\delta g^*$-closed set. □

**Theorem 3.8.** If $A$ is fuzzy $\delta$-open and fuzzy $\delta g^*$-closed in $(X, \tau)$, then $A$ is fuzzy $\delta$-closed in $(X, \tau)$.

*Proof.* Let $A$ be fuzzy $\delta$-open and fuzzy $\delta g^*$-closed in $X$. Suppose $A \subseteq A$, then $fcl_\delta(A) \subseteq A$. But $A \subseteq fcl_\delta(A)$, which implies that $fcl_\delta(A) = A$. Hence $A$ is fuzzy $\delta$-closed. □

**Theorem 3.9.** Let $(X, \tau)$ be a fts and $A$ be a fuzzy set of $X$. Then $A$ is fuzzy $\delta g^*$-closed if and only if $Aq B$ implies $fcl_\delta(A) q B$ for every fuzzy $\delta g^*$-closed set $B$ of $(X, \tau)$.

*Proof.* Suppose $A$ is a fuzzy $\delta g^*$-closed set of $X$. Let $B$ be a fuzzy $g$-closed set in $X$ such that $Aq B$. Then $A \subseteq 1 - B$ and $1 - B$ is a fuzzy $g$-open set of $X$. Therefore $fcl_\delta(A) \subseteq 1 - B$, as $A$ is fuzzy $\delta g^*$-closed. Hence $fcl_\delta(A) q B$. Conversely, let $D$ be a fuzzy $g$-open set in $X$ such that $A \subseteq D$. Then $Aq (1 - D)$ and $1 - D$ is a fuzzy $g$-closed set in $X$. By hypothesis, $fcl_\delta(A) q (1 - D)$ which implies, $cl_\delta(A) \subseteq D$. Hence $A$ is fuzzy $\delta g^*$-closed. □

**Theorem 3.10.** Let $A$ be a fuzzy $\delta g^*$-closed set in $(X, \tau)$ and $x_p$ be a fuzzy point of $(X, \tau)$ such that $x_pq fcl_\delta(A)$, then $fcl_\delta(x_p)q A$.

*Proof.* Let $A$ be a fuzzy $\delta g^*$-closed set in $(X, \tau)$ and $x_p$ be a fuzzy point of $(X, \tau)$ such that $x_pq fcl_\delta(A)$. Suppose $fcl_\delta(x_p)q A$, then $fcl_\delta(x_p) \subseteq 1 - A$ and hence $A \subseteq 1 - fcl_\delta(x_p)$. Now, $1 - fcl_\delta(x_p)$ is fuzzy $\delta$-open and hence fuzzy $\delta g$-open. Moreover, since $A$ is fuzzy $\delta g^*$-closed, $fcl_\delta(A) \subseteq 1 - fcl_\delta(x_p) \subseteq 1 - x_p$. Hence $x_pq fcl_\delta(A)$, which is a contradiction. □

**Theorem 3.11.** If $A$ is a fuzzy $\delta g^*$-closed set in $(X, \tau)$ and $A \subseteq B \subseteq fcl_\delta(A)$, then $B$ is a fuzzy $\delta g^*$-closed set in $(X, \tau)$.

*Proof.* Let $A$ be a fuzzy $\delta g^*$-closed set in $(X, \tau)$. Given $A \subseteq B \subseteq fcl_\delta(A)$. Suppose $B \subseteq H$ where $H$ is fuzzy $g$-open set. Since $A \subseteq B \subseteq H$ and $A$ is a fuzzy $\delta g^*$-closed set, we get $fcl_\delta(A) \subseteq H$. As $B \subseteq fcl_\delta(A)$,
$\text{fcl}_\delta(B) \leq \text{fcl}_\delta(\text{fcl}_\delta(A)) = \text{fcl}_\delta(A)$ we get $\text{fcl}_\delta(B) \leq H$. Hence $B$ is a fuzzy $\delta g^*$-closed set in $(X, \tau)$. \qed

**Theorem 3.12.** If $A$ is a fuzzy $\delta g^*$-open set in $(X, \tau)$ and $f\text{int}_\delta(A) \leq B \leq A$, then $B$ is a fuzzy $\delta g^*$-open set in $(X, \tau)$.

**Proof.** Let $A$ be fuzzy $\delta g^*$-open set and $B$ be any fuzzy set in $X$ such that $f\text{int}_\delta(A) \leq B \leq A$. Then $1 - A$ is a fuzzy $\delta g^*$-closed set and $1 - A \leq 1 - B \leq f\text{cl}_\delta(1 - A)$, as $1 - f\text{int}_\delta(A) = f\text{cl}_\delta(1 - A)$. Therefore $1 - B$ is a fuzzy $\delta g^*$-closed. Hence $B$ is fuzzy $\delta g^*$-open. \qed

4. **Fuzzy strongly $g^*$-closed sets in fuzzy topological spaces**

**Definition 4.1.** Let $(X, \tau)$ be a fuzzy topological space. A fuzzy set $A$ of $(X, \tau)$ is called fuzzy strongly $g^*$-closed if $\text{cl}(\text{int}(A)) \leq H$, whenever $A \leq H$ and $H$ is $fg$-open in $X$.

**Theorem 4.2.** Every fuzzy closed set is a fuzzy strongly $g^*$-closed set in the fuzzy topological space $(X, \tau)$.

**Proof.** Let $A$ be fuzzy closed set in $X$ and $H$ be a $fg$-open set in $X$ such that $A \leq H$. Since $A$ is fuzzy closed, $\text{cl}(A) = A$. Therefore $\text{cl}(A) \leq H$. Now, $\text{cl}(\text{int}(A)) \leq \text{cl}(A) \leq H$. Hence $A$ is fuzzy strongly $g^*$-closed set in $X$. \qed

**Remark 4.3.** The converse of the Theorem need not be true in general. For example, let $X = \{a, b, c\}$. Fuzzy sets $A$ and $B$ are defined by $A(a) = 0.7$, $A(b) = 0.3$, $A(c) = 0.5$, $B(a) = 0.2$, $B(b) = 0.1$ and $B(c) = 0.3$. Let $\tau = \{0, A, 1\}$. Then $B$ is a fuzzy strongly $g^*$-closed set but it is not a fuzzy closed set in $(X, \tau)$.

**Remark 4.4.** Every fuzzy $g^*$-closed set is a fuzzy strongly $g^*$-closed set in $X$. But the converse of this state need not be true. For example, let $X = \{a, b\}$, $\tau = \{0, B, 1\}$ and the fuzzy sets $A$ and $B$ be defined as follows: $A(a) = 0.3$, $A(b) = 0.3$, $B(a) = 0.5$ and $B(b) = 0.4$. Then $A$ is a fuzzy strongly $g^*$-closed while it is not $fg$-closed.

**Theorem 4.5.** Every fuzzy $g^*$-closed set is a fuzzy strongly $g^*$-closed set in $(X, \tau)$.

**Proof.** Suppose that $A$ is $fg^*$-closed in $X$. Let $H$ be a $fg$-open set in $X$ such that $A \leq H$. Then $\text{cl}(A) \leq H$, since $A$ is $fg^*$-closed. Now, $\text{cl}(\text{int}(A)) \leq \text{cl}(A) \leq H$, hence $A$ is fuzzy strongly $g^*$-closed set in $X$. \qed

However the converse of the Theorem need not be true as seen from the following example:
Example 4.6. Let $X = \{a, b\}$, $\tau = \{0, A, B, D, 1\}$ and fuzzy sets $A$, $B$, $D$ and $H$ are defined as follows: $A(a) = 0.2$, $A(b) = 0.4$, $B(a) = 0.6$, $B(b) = 0.7$, $D(a) = 0.4$, $D(b) = 0.6$, $H(a) = 0.4$ and $H(b) = 0.5$. Then $H$ is fuzzy strongly $g^*$-closed set but it is not $fg^*$-closed in $(X, \tau)$.

Theorem 4.7. A fuzzy set $A$ of fuzzy topological space $(X, \tau)$ is fuzzy strongly $g^*$-closed if and only if $A\bar{q}B$, then $cl(\text{int}(A))\bar{q}B$ for every $fg^*$-closed set $B$ of $(X, \tau)$.

Proof. Suppose that $A$ is a fuzzy strongly $g^*$-closed set of $X$ such that $A\bar{q}B$. Then $A \leq 1 - B$ and $1 - B$ is a fuzzy $g$-open set in $X$, which implies that $cl(\text{int}(A)) \leq 1 - B$, since $A$ is fuzzy strongly $g^*$-closed. Hence $cl(\text{int}(A))\bar{q}B$. Now, let $E$ be a fuzzy $g$-open set in $X$ such that $A \leq E$. Then $A\bar{q}1 - E$ and $1 - E$ is $fg^*$-closed set in $X$. By hypothesis, $cl(\text{int}(A))\bar{q}(1 - E)$ which implies that $cl(\text{int}(A)) \leq E$. Hence $A$ is fuzzy strongly $g^*$-closed in $X$. $\square$

Theorem 4.8. Let $A$ be a fuzzy strongly $g^*$-closed set in $(X, \tau)$ and $x_p$ be a fuzzy point of $(X, \tau)$ such that $x_p q cl(\text{int}(A))$ then $cl(\text{int}(x_p)) q A$.

Proof. Let $A$ be a fuzzy strongly $g^*$-closed set in $(X, \tau)$ and $x_p$ be a fuzzy point of $(X, \tau)$ such that $x_p q cl(\text{int}(A))$. Suppose $cl(\text{int}(x_p)) q A$, then $cl(\text{int}(x_p)) q 1 - A$ and hence $A \leq 1 - cl(\text{int}(x_p))$. Now, $1 - cl(\text{int}(x_p))$ is fuzzy open. Moreover, since $A$ is fuzzy strongly $g^*$-closed, $cl(\text{int}(A)) \leq 1 - cl(\text{int}(x_p)) \leq 1 - x_p$. Hence $x_p \bar{q} cl(\text{int}(A))$, which is a contradiction. $\square$

Theorem 4.9. If $A$ is a fuzzy strongly $g^*$-closed set in $(X, \tau)$ and $A \leq B \leq cl(\text{int}(A))$, then $B$ is fuzzy strongly $g^*$-closed in $(X, \tau)$.

Proof. Let $A$ be a fuzzy strongly $g^*$-closed set in $(X, \tau)$ and $B \leq H$ where $H$ is a fuzzy $g$-open set in $X$. Then $A \leq H$. Since $A$ is a fuzzy strongly $g^*$-closed set, it follows that $cl(\text{int}(A)) \leq H$. Now, $B \leq cl(\text{int}(A))$ implies $cl(\text{int}(B)) \leq cl(\text{int}(cl(\text{int}(A)))) = cl(\text{int}(A))$. We get, $cl(\text{int}(B)) \leq H$. Hence, $B$ is a fuzzy strongly $g^*$-closed set in $(X, \tau)$. $\square$

Definition 4.10. A fuzzy set $A$ of $(X, \tau)$ is called fuzzy strongly $g^*$-open set in $X$ if and only if $1 - A$ is fuzzy strongly $g^*$-closed in $X$.

In other words, $A$ is fuzzy strongly $g^*$-open if and only if $H \leq cl(\text{int}(A))$, whenever $H \leq A$ of $H$ is $fg^*$-closed in $X$.

Theorem 4.11. Let $(Y, \tau_Y)$ be a subspace of a fuzzy topological space $(X, \tau)$ and $A$ be a fuzzy set of $Y$. If $A$ is fuzzy strongly $g^*$-closed in $X$, then $A$ is a fuzzy strongly $g^*$-closed in $Y$. 
Let $X$ be a fuzzy set in $(Y; \tau)$. If $A \leq H$, we have $H = G \cap Y$ where $G$ is $f^g$-open in $X$. Hence $A \leq H = G \cap Y$ implies $A \leq G$ and $A$ is fuzzy strongly $g^*$-open in $X$. We get $cl (\text{int} (A)) \leq G$. Therefore $cl (\text{int} (A)) \cap Y \leq G \cap Y = H$. Thus $cl (\text{int} (A)) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $g^*$-open in $Y$. Hence $A$ is fuzzy strongly $g^*$-open in $Y$.

Theorem 4.12. If a fuzzy set $A$ of a fuzzy topological space $X$ is both fuzzy open and fuzzy strongly $g^*$-closed, then it is fuzzy closed.

Proof. Suppose that a fuzzy set $A$ of $X$ is both fuzzy open and fuzzy strongly $g^*$-closed. Now, $A \geq cl (\text{int} (A)) \geq cl (A)$. That is $A \geq cl (A)$, since $A \leq cl (A)$. So we get $A = cl (A)$. Hence $A$ is fuzzy closed in $X$. □

Theorem 4.13. If a fuzzy set $A$ of a fuzzy topological space $X$ is both fuzzy strongly $g^*$-closed and fuzzy semi open, then it is $f^g$-closed.

Proof. Suppose a fuzzy set $A$ of $X$ is both fuzzy strongly $g^*$-closed and fuzzy semi open in $X$. Let $H$ be a $f^g$-open set such that $A \leq H$. Since $A$ is fuzzy strongly $g^*$-closed, therefore $cl (\text{int} (A)) \leq H$. Also since $A$ is $f^s$-open, $A \leq cl (\text{int} (A))$. We have $cl (A) \leq cl (\text{int} (A)) \leq H$. Hence $A$ is $f^g$-closed in $X$. □

Remark 4.14. Every $f^\theta$-closed set is a fuzzy strongly $g^*$-closed set, but the converse is not true in general. For example, Let $X = \{a, b\}$, $\tau = \{0, A, 1\}$ and fuzzy sets $A$ and $B$ are defined as follows: $A(a) = 0.3$, $A(b) = 0.7$, $B(a) = 0.6$ and $B(b) = 0.5$. Then $B$ is strongly $g^*$-closed but it is not $f^\theta$-closed.

Every $f^p$-closed, $fsp$-closed, $gfs$-closed, $fg^o$-closed and $fspg$-closed sets are fuzzy strongly $g^*$-closed, but the converse may not be true in general.

We are now ready to construct our main examples.

Example 4.15. Let $X = \{a, b\}$, $\tau = \{0, A, 1\}$ and fuzzy sets $A$ and $B$ are in defined by $A(a) = 0.8$, $A(b) = 0.2$, $B(a) = 0.9$ and $B(b) = 0.6$. Then $B$ is a fuzzy strongly $g^*$-closed in $(X, \tau)$, while it is not a $f^p$-closed set in $(X, \tau)$.

Example 4.16. Let $X = \{a, b\}$ and $\tau = \{0, A, 1\}$. Let fuzzy sets $A$ and $B$ are defined in $X$ by $A(a) = 0.2$, $A(b) = 0.6$, $B(a) = 0.5$ and $B(b) = 0.7$. Then $B$ is a fuzzy strongly $g^*$-closed set in $(X, \tau)$, while it is not a $fsp$-closed set in $(X, \tau)$.

Example 4.17. Let $X = \{a, b, c\}$, $\tau = \{0, A, 1\}$ and fuzzy sets $A$ and $B$ are defined by $A(a) = 0.8$, $A(b) = 0.3$, $A(c) = 0.1$, $B(a) = 0.8$, $B(b) = 0.6$ and $B(c) = 0.7$. Then $B$ is a fuzzy strongly $g^*$-closed set in $(X, \tau)$, while it is not a $fspg$-closed set in $(X, \tau)$.
$B(b) = 0.1$ and $B(c) = 0.1$. Then $B$ is a strongly $g^*$-closed set in $(X, \tau)$, while it is neither $fgs$-closed.

**Example 4.18.** Let $X = \{a, b\}$ and $\tau = \{0, A, 1\}$. Define fuzzy set $A$, $B$ and $D$ in $X$ by $A(a) = 0.3$, $A(b) = 0.6$, $B(a) = 0.6$, $B(b) = 0.6$, $D(a) = 0.9$ and $D(b) = 0.8$. Then $B$ is fuzzy strongly $g^*$-closed, while it is not $fspg$-closed.

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