

GENERALIZED MULTIVALUED F -WEAK CONTRACTIONS ON COMPLETE METRIC SPACES

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ABSTRACT. In this paper, we introduce the notion of generalized multivalued F - weak contraction and we prove some fixed point theorems related to introduced contraction for multivalued mapping in complete metric spaces. Our results extend and improve the results announced by many others with less hypothesis. Also, we give some illustrative examples.

1. INTRODUCTION

Throughout the paper, \mathbb{N} and \mathbb{N}_0 denote the set of positive integers and the set of nonnegative integers. Respectively similarly, let \mathbb{R} , \mathbb{R}^+ and \mathbb{R}_0^+ represent the set of real numbers, positive real numbers and the set of nonnegative real numbers, respectively.

Recently many results of the fixed point problems for maps on metric spaces have been proved; see, for instance [3, 6, 7, 8, 9, 10, 11, 12, 16]. Wardowski [18] has introduced the concept of an F -contraction as follows:

Definition 1.1 ([18]). Let \mathcal{F} be the family of all functions $F: \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

- (F1) F is strictly increasing, i.e. for all $x, y \in \mathbb{R}^+$ such that $x < y$, $F(x) < F(y)$;
- (F2) For each sequence $\{\alpha_n\}_{n=1}^{\infty}$ of positive numbers, $\lim_{n \rightarrow \infty} \alpha_n = 0$ if and only if $\lim_{n \rightarrow \infty} F(\alpha_n) = -\infty$;
- (F3) There exist $k \in (0, 1)$ such that $\lim_{\alpha \rightarrow 0^+} \alpha^k F(\alpha) = 0$.

2010 *Mathematics Subject Classification.* 47H10, 54H25, 54E99, 54D99.

Key words and phrases. Multivalued F - weak contraction, Fixed point, Multivalued mappings.

Received: 09 April 2015, Accepted: 25 May 2015.

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Definition 1.2 ([18]). Let (X, d) be a metric space. A mapping $T : X \rightarrow X$ is said to be an F -contraction on (X, d) if there exist $F \in \mathcal{F}$ and $\tau > 0$ such that

$$(1.1) \quad \forall x, y \in X, \quad d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Theorem 1.3 ([18]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an F -contraction. Then T has a unique fixed point $x^* \in X$.

Remark 1.4. From (F1) and (1.1) it is easy to conclude that every F -contraction is necessarily continuous.

Very recently, Piri and Kumam [15] extended the result of Wardowski [18] by replacing the condition (F3) in Definition 1.1 with the following one:

$$(F3') \quad F \text{ is continuous on } (0, \infty).$$

Let \mathfrak{F} denote the family of all functions $F : \mathbb{R}_+ \rightarrow \mathbb{R}$ which satisfy conditions (F1), (F2) and (F3'). Under this new set-up, they proved a fixed point result that generalized the result of Wardowski [18].

Remark 1.5 ([15]). Note that, the conditions (F3) and (F3') are independent of each other. Indeed, for $p \geq 1$, $F(\alpha) = \frac{-1}{\alpha^p}$ satisfies the conditions (F1) and (F2) but it does not satisfy (F3), while it satisfies the condition (F3'). Therefore, $\mathfrak{F} \not\subseteq \mathcal{F}$. On the other hand, for $a > 1$, $t \in (0, 1/a)$, $F(\alpha) = \frac{-1}{(\alpha + [\alpha])^t}$, where $[\alpha]$ denotes the integer part of α , satisfies the conditions (F1) and (F2) but it does not satisfy (F3'), while it satisfies the condition (F3) for any $k \in (1/a, 1)$. Therefore, $\mathcal{F} \not\subseteq \mathfrak{F}$. Also, if we take $F(\alpha) = \ln \alpha$, then $F \in \mathcal{F}$ and $F \in \mathfrak{F}$. Therefore, $\mathcal{F} \cap \mathfrak{F} \neq \emptyset$.

Definition 1.6 ([15]). Let (X, d) be a metric space and let $F \in \mathfrak{F}$. A mapping $T : X \rightarrow X$ is said to be a F -Suzuki-contraction if there exists $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}d(x, Tx) < d(x, y) \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y)).$$

Theorem 1.7 ([15]). Let (X, d) be a complete metric space and $T : X \rightarrow X$ be an F -Suzuki-contraction. Then

- (1) T has a unique fixed point $x^* \in X$.
- (2) For all $x \in X$, the sequence $\{T^n x\}$ is convergent to x^* .

Following Wardowski, Minak et. al. [13] introduced the concept of Ćirić type generalized F -contraction by combining the idea of Wardowski [18] and Ćirić [9] and gave fixed point result for this type mapping on complete spaces as follows:

Definition 1.8 ([13]). Let (X, d) be a metric space and $T : X \rightarrow X$ be a mapping. Then T is said to be Ćirić type generalized F - contraction on (X, d) if $F \in \mathcal{F}$ and there exists $\tau > 0$ such that

$$\forall x, y \in X, \quad d(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(d(Tx, Ty)) \leq F(M(x, y)),$$

where

$$M(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2} \right\}.$$

Theorem 1.9 ([13]). *Let (X, d) be a complete metric space and $T : X \rightarrow X$ be a Ćirić type generalized F - contraction. If T or F is continuous, then T has a unique fixed point $x^* \in X$.*

Minak et al. was unaware that Definition 1.8 was given previously by Wardowski and Van Dung [17] as F -weak contraction and also Theorem 1.9 was proved by them.

On the other hand, using the concept of the Hausdorff metric, Nadler [14], introduced the notion of multivalued contraction mappings and proved a multivalued version of the well known Banach contraction principle. Let (X, d) be a metric space. Let $P(X)$, $B(X)$, $CB(X)$ and $K(X)$ represent the family of all nonempty subsets of X , the family of all nonempty and bounded subsets of X , the family of all nonempty, closed and bounded subsets of X and the family of all nonempty compact subsets of X , respectively. For $A, B \in B(X)$, we define

$$\delta(A, B) = \sup \{d(a, b) : a \in A, b \in B\}.$$

It is well known that, $H : CB(X) \times CB(X) \rightarrow \mathbb{R}$ is defined by,

$$H(A, B) = \max \left\{ \sup_{a \in A} D(a, B), \sup_{b \in B} D(b, A) \right\},$$

for every $A, B \in CB(X)$, is a metric on $CB(X)$, which is called Hausdorff metric induced by d , where $D(a, B) = \inf \{d(a, b) : b \in B\}$. Let $T : X \rightarrow CB(X)$ be a map, then T is called multivalued contraction if for all $x, y \in X$ there exists $L \in [0, 1)$ such that

$$H(Tx, Ty) \leq Ld(x, y).$$

Then Nadler [14] proved that every multivalued contraction mapping on a complete metric space has a fixed point.

Considering the Pompeiu-Hausdorff metric H , both Theorems 1.3 and 1.9 were extended to multivalued cases in [4] and [2] respectively (see also [1]).

It is our main aim in this work to establish the main result of Altun et al., [4] without the assumptions $F2$ and $F3$ used in their proof and also, we give the main result of Acar et. al. [2] for a mapping $T : X \rightarrow CB(X)$

instead of a mapping $T : X \rightarrow K(X)$ and we give their result without the continuity of T and assumptions $F2$ and $F3$ used in their proof.

2. MAIN RESULTS

Let \mathfrak{F}_G denotes the family of all functions $F : \mathbb{R}^+ \rightarrow \mathbb{R}$ which satisfy conditions $(F1)$ and $(F3')$.

Definition 2.1. Let (X, d) be a metric space. A mapping $T : X \rightarrow CB(X)$ is said to be a generalized multivalued F -weak contraction of type (A) on (X, d) , if there exist $F \in \mathfrak{F}_G$ and $\tau > 0$ such that

$$\forall x, y \in X, \quad H(Tx, Ty) > 0 \quad \Rightarrow \quad \tau + F(H(Tx, Ty)) \leq F(M_T(x, y)),$$

where

$$M_T(x, y) = \max \left\{ d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2} \right\}.$$

Theorem 2.2. Let (X, d) be a complete metric space and $T : X \rightarrow CB(X)$ be a generalized multivalued F -weak contraction of type (A) on (X, d) . Then T has a fixed point in X .

Proof. Let $x_0 \in X$ be an arbitrary point. Since for every $x \in X$, $Tx \neq \emptyset$, so we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, for all $n \in \mathbb{N}_0$. If there exists $n_0 \in \mathbb{N}$ for which $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of T and so the proof is completed. So, we assume that $x_n \neq x_{n+1}$, for all $n \in \mathbb{N}$. So

$$(2.1) \quad H(Tx_{n-1}, Tx_n) \geq d(x_n, x_{n+1}) > 0, \quad \forall n \in \mathbb{N}.$$

Since T is a multivalued F -weak contraction of type (A), from the inequality (2.1) and $(F1)$, we have

$$\begin{aligned} \tau + F(d(x_n, x_{n+1})) &\leq \tau + F(H(Tx_{n-1}, Tx_n)) \\ &\leq F(M_T(x_{n-1}, x_n)) \\ &= F \left(\max \left\{ \begin{array}{l} d(x_{n-1}, x_n), D(x_{n-1}, Tx_{n-1}), \\ D(x_n, Tx_n), \frac{D(x_{n-1}, Tx_n) + D(x_n, Tx_{n-1})}{2} \end{array} \right\} \right) \end{aligned}$$

$$\leq F \left(\max \left\{ \begin{array}{l} d(x_{n-1}, x_n), \\ d(x_{n-1}, x_n) + D(x_n, Tx_{n-1}), \\ d(x_n, x_{n+1}) + D(x_{n+1}, Tx_n), \\ \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{2} \\ + \frac{D(x_{n+1}, Tx_n) + D(x_n, Tx_{n-1})}{2} \end{array} \right\} \right)$$

$$= F \left(\max \left\{ \begin{array}{l} d(x_{n-1}, x_n), d(x_{n-1}, x_n) + 0, \\ d(x_n, x_{n+1}) + 0, \\ \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 0 + 0}{2} \end{array} \right\} \right)$$

$$(2.2) \quad \leq F(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}).$$

If there exists $n \in \mathbb{N}$ such that

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_n, x_{n+1}),$$

then (2.2) becomes

$$\tau + F(d(x_n, x_{n+1})) \leq F(d(x_n, x_{n+1})),$$

which is a contradiction. Thus, we conclude that

$$\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} = d(x_{n-1}, x_n),$$

for all $n \in \mathbb{N}$. Hence, the inequality (2.2) turns into

$$(2.3) \quad F(d(x_n, x_{n+1})) \leq F(d(x_{n-1}, x_n)) - \tau, \quad \text{for all } n \in \mathbb{N}.$$

Since $\tau > 0$, so we get

$$(2.4) \quad F(d(x_n, x_{n+1})) < F(d(x_{n-1}, x_n)), \quad \text{for all } n \in \mathbb{N}.$$

It follows from (2.4) and (F1) that

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N}.$$

Therefore $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}}$ is a nonnegative nonincreasing sequence, and hence

$$\lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = \gamma \geq 0.$$

Now, we claim that $\gamma = 0$. Arguing by contradiction, we assume that $\gamma > 0$. Since $\{d(x_n, x_{n+1})\}_{n \in \mathbb{N}}$ is a nonnegative nonincreasing sequence,

so there exists $n_1 \in \mathbb{N}$ such that

$$(2.5) \quad d(x_n, x_{n+1}) > \gamma, \quad \forall n > n_1.$$

So from (2.3), (2.5) and (F1), we get

$$(2.6) \quad \begin{aligned} F(\gamma) &< F(d(x_n, x_{n+1})) \\ &\leq F(d(x_{n-1}, x_n)) - \tau \\ &\leq F(d(x_{n-2}, x_{n-1})) - 2\tau \\ &\vdots \\ &\leq F(d(x_{n_1}, x_{n_1+1})) - (n - n_1)\tau, \quad \forall n > n_1. \end{aligned}$$

Since $F(\gamma) \in \mathbb{R}$ and

$$\lim_{n \rightarrow \infty} (F(d(x_{n_1}, x_{n_1+1})) - (n - n_1)\tau) = -\infty,$$

so there exists $n_2 \in \mathbb{N}$ such that

$$(2.7) \quad F(d(x_{n_1}, x_{n_1+1})) - (n - n_1)\tau < F(\gamma), \quad \forall n > n_2.$$

Setting $n_3 = \max\{n_1, n_2\}$, it follows (2.6) and (2.7), that

$$F(\gamma) < F(d(x_{n_1}, x_{n_1+1})) - (n - n_1)\tau < F(\gamma), \quad \forall n > n_3,$$

which is a contradiction. Therefore, we have

$$(2.8) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+1}) = 0.$$

Now, we claim that

$$(2.9) \quad \lim_{n, m \rightarrow \infty} d(x_n, x_m) = 0.$$

Arguing by contradiction, we assume that there exist $\epsilon > 0$ and subsequences $\{p(n)\}_{n=1}^{\infty}$ and $\{q(n)\}_{n=1}^{\infty}$ of natural numbers such that for all $n \in \mathbb{N}$,

$$(2.10) \quad p(n) > q(n) > n, \quad d(x_{p(n)}, x_{q(n)}) \geq \epsilon, \quad d(x_{p(n)-1}, x_{q(n)}) < \epsilon.$$

Thus, for all $n \in \mathbb{N}$,

$$(2.11) \quad \begin{aligned} \epsilon &\leq d(x_{p(n)}, x_{q(n)}) \\ &\leq d(x_{p(n)}, x_{p(n)-1}) + d(x_{p(n)-1}, x_{q(n)}) \\ &\leq d(x_{p(n)}, x_{p(n)-1}) + \epsilon. \end{aligned}$$

It follows from (2.8), (2.11) and the Sandwich theorem that

$$(2.12) \quad \lim_{n \rightarrow \infty} d(x_{p(n)}, x_{q(n)}) = \epsilon.$$

From (2.12), there exists $n_4 \in \mathbb{N}$ such that

$$0 < \frac{\epsilon}{2} < d(x_{p(n)}, x_{q(n)}) \leq H(Tx_{p(n)}, Tx_{q(n)}), \quad \forall n > n_4.$$

Since $x_{p(n)+1} \in Tx_{p(n)}$, $x_{q(n)+1} \in Tx_{q(n)}$ and T is a multivalued F -weak contraction of type (A), then we have

$$\begin{aligned} & \tau + F(d(x_{p(n)}, x_{q(n)})) \\ & \leq \tau + F(H(Tx_{p(n)}, Tx_{q(n)})) \\ & \leq F(M_T(x_{p(n)}, x_{q(n)})) \\ & = F\left(\max\left\{\begin{array}{l} d(x_{p(n)}, x_{q(n)}), D(x_{p(n)}, Tx_{p(n)}), \\ D(x_{q(n)}, Tx_{q(n)}), \\ \frac{D(x_{p(n)}, Tx_{q(n)}) + D(x_{q(n)}, Tx_{p(n)})}{2} \end{array}\right\}\right) \\ & \leq F\left(\max\left\{\begin{array}{l} d(x_{p(n)}, x_{q(n)}), \\ d(x_{p(n)}, x_{p(n)+1}) + D(x_{p(n)+1}, Tx_{p(n)}), \\ d(x_{q(n)}, x_{q(n)+1}) + D(x_{q(n)+1}, Tx_{q(n)}), \\ \frac{1}{2}[d(x_{p(n)}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}) \\ + D(x_{q(n)+1}, Tx_{q(n)}) + d(x_{q(n)}, x_{p(n)}) \\ + d(x_{p(n)}, x_{p(n)+1}) + D(x_{p(n)+1}, Tx_{p(n)})] \end{array}\right\}\right) \\ & = F\left(\max\left\{\begin{array}{l} d(x_{p(n)}, x_{q(n)}), d(x_{p(n)}, x_{p(n)+1}) + 0, \\ d(x_{q(n)}, x_{q(n)+1}) + 0, \\ \frac{d(x_{p(n)}, x_{q(n)}) + d(x_{q(n)}, x_{q(n)+1}) + 0}{2} \\ + \frac{d(x_{q(n)}, x_{p(n)}) + d(x_{p(n)}, x_{p(n)+1}) + 0}{2} \end{array}\right\}\right). \end{aligned}$$

Letting $n \rightarrow \infty$ in the above inequality and by taking (2.8), (2.12) and $(F3')$ into account, we get

$$\tau + F(\epsilon) \leq F(\epsilon).$$

Since $\tau > 0$, this is a contradiction. Hence

$$\lim_{n,m \rightarrow \infty} d(x_n, x_m) = 0.$$

Therefore, we conclude that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X . Since (X, d) is a complete metric space, so there exists $x^* \in X$ such that

$$(2.13) \quad \lim_{n \rightarrow \infty} d(x_n, x^*) = 0.$$

In this case we claim $x^* \in Tx^*$. Suppose contrary that $x^* \notin Tx^*$. In this case there exists $n_0 \in \mathbb{N}_0$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $D(x_{n_k+1}, Tx^*) > 0$ for all $n_k \geq n_0$ (otherwise, there exists $n_1 \in \mathbb{N}_0$ such that $x_n \in Tx^*$ for all $n \geq n_1$, which implies that $x^* \in Tx^*$. This is a contradiction, since $x^* \notin Tx^*$). Since $D(x_{n_k+1}, Tx^*) > 0$ for all $n_k \geq n_0$, and T is multivalued F -weak contraction of type (A), we obtain

$$\begin{aligned} & \tau + F(D(x_{n_k+1}, Tx^*)) \\ & \leq \tau + F(H(Tx_{n_k}, Tx^*)) \\ & \leq F(M(x_{n_k}, x^*)) \\ & = F\left(\max\left\{\begin{array}{l} d(x_{n_k}, x^*), D(x_{n_k}, Tx_{n_k}), D(x^*, Tx^*), \\ \frac{D(x_{n_k}, Tx^*) + D(x^*, Tx_{n_k})}{2} \end{array}\right\}\right) \\ & \leq F\left(\max\left\{\begin{array}{l} d(x_{n_k}, x^*), D(x^*, Tx^*), \\ d(x_{n_k}, x_{n_k+1}) + D(x_{n_k+1}, Tx_{n_k}), \\ \frac{D(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1}) + D(x_{n_k+1}, Tx_{n_k})}{2} \end{array}\right\}\right) \\ & = F\left(\max\left\{\begin{array}{l} d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}) + 0, D(x^*, Tx^*), \\ \frac{D(x_{n_k}, Tx^*) + d(x^*, x_{n_k+1}) + 0}{2} \end{array}\right\}\right). \end{aligned}$$

Letting $n_k \rightarrow \infty$ in the above inequality and by taking (2.8), (2.13) and $(F3')$ into account, we get that

$$\tau + F(D(x^*, Tx^*)) \leq F(D(x^*, Tx^*)),$$

which is a contradiction. Therefore, we have $x^* \in Tx^*$. Hence x^* is a fixed point of T . \square

Definition 2.3. Let (X, d) be a metric space. A mapping $T : X \rightarrow CB(X)$ is said to be a generalized multivalued F -weak contraction of type (B) on (X, d) , if there exist $F \in \mathfrak{F}$ and $\tau > 0$ such that for all $x, y \in X$

$$\min \{ \delta(Tx, Ty), d(x, y) \} > 0 \Rightarrow \tau + F(H(Tx, Ty)) \leq F(M_T(x, y)),$$

where

$$M_T(x, y) = \max \left\{ d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2} \right\}.$$

By the careful analysis of the proof of Theorem 2.2, we have the following theorem.

Theorem 2.4. Let (X, d) be a complete metric space. If $T : X \rightarrow CB(X)$ be a continuous generalized multivalued F -weak contraction of type (B), then T has a fixed point in X .

Example 2.5. Let $X = \{0\} \cup [\frac{5}{2}, 3]$ and $d(x, y) = |x - y|$. Obviously (X, d) is a complete metric space. Suppose that $F(\alpha) = \frac{-1}{\alpha} + \alpha \in \mathfrak{F}$ and $\tau \in (0, \frac{1}{8})$. We define the self-mapping $T : X \rightarrow CB(X)$ as follows:

$$Tx = \begin{cases} \{\frac{5}{2}\}, & x \in \{0\} \cup [\frac{5}{2}, 3), \\ \{0\}, & x = 3. \end{cases}$$

Since T is not continuous, T is not a F -contraction by Remark 1.4. Now, we show that T is a multivalued F -weak contraction of type (A).

Let

$$M_T(x, y) = \max \left\{ d(x, y), D(x, Tx), D(y, Ty), \frac{D(x, Ty) + D(y, Tx)}{2} \right\}.$$

Now, we will consider the inequality

$$(2.14) \quad H(Tx, Ty) > 0,$$

when $x \neq y$, and the inequality

$$(2.15) \quad \tau + F(H(Tx, Ty)) \leq F(M_T(x, y)),$$

for those $x, y \in X$ (with $x \neq y$) which satisfy (2.14).

Case 1: For $x = 0$ and $y = 3$, we have

$$H(T0, T3) = H \left(\left\{ \frac{5}{2} \right\}, \{0\} \right) = \frac{5}{2} > 0,$$

and hence (2.14) is satisfied. Since

$$H(T0, T3) = \frac{5}{2} < 3 = d(0, 3) \leq M_T(0, 3),$$

so, we have

$$-\frac{1}{H(T0, T3)} \leq -\frac{1}{M(0, 3)}.$$

Thus, we get

$$\begin{aligned} \tau + F(H(T0, T3)) &= \tau - \frac{1}{H(T0, T3)} + H(T0, T3) \\ &\leq \tau - \frac{1}{M(0, 3)} + \frac{5}{2} \\ &\leq \frac{1}{8} - \frac{1}{M(0, 3)} + \frac{5}{2} \\ &\leq \frac{1}{2} - \frac{1}{M(0, 3)} + \frac{5}{2} \\ &= -\frac{1}{M(0, 3)} + 3 \\ &= -\frac{1}{M(0, 3)} + d(0, 3) \\ &\leq -\frac{1}{M(0, 3)} + M_T(0, 3) \\ &= F(M_T(0, 3)), \end{aligned}$$

so (2.15) holds.

Case 2: For $x \in [\frac{5}{2}, 3)$ and $y = 0$, we have

$$H(Tx, T0) = H\left(\left\{\frac{5}{2}\right\}, \left\{\frac{5}{2}\right\}\right) = 0,$$

Thus (2.14) is not true. Hence we do not need to check (2.15).

Case 3: For $x = 3$ and $y \in [\frac{5}{2}, 3)$, we have

$$H(T3, Ty) = H\left(\{0\}, \left\{\frac{5}{2}\right\}\right) = \frac{5}{2} > 0.$$

and hence (2.14) is satisfied. Since

$$H(T3, Ty) = \frac{5}{2} < 3 = D(3, T3) \leq M_T(3, y),$$

so, we have $-\frac{1}{H(T3, Ty)} \leq -\frac{1}{M_T(3, y)}$. By using some calculations as used in Case 1, we conclude that

$$\tau + F(H(T3, Ty)) \leq F(M_T(T3, Ty)),$$

so (2.15) holds. Hence T is a multivalued F -weak contraction.

Acknowledgment. We would like to thank the reviewers and editors for their very very careful reading and valuable comments on our manuscript, which provided insights that helped to improve the paper.

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