

SUPERSTABILITY OF m -ADDITIVE MAPS ON COMPLETE NON-ARCHIMEDEAN SPACES

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ABSTRACT. The stability problem of the functional equation was conjectured by Ulam and was solved by Hyers in the case of additive mapping. Baker et al. investigated the superstability of the functional equation from a vector space to real numbers. In this paper, we exhibit the superstability of m -additive maps on complete non-Archimedean spaces via a fixed point method raised by Diaz and Margolis.

1. INTRODUCTION

Let \mathbb{K} be a field. A non-Archimedean absolute value on \mathbb{K} is a function $|\cdot| : \mathbb{K} \rightarrow \mathbb{R}$ such that for any $a, b \in \mathbb{K}$ we have

- (i) $|a| \geq 0$ and equality holds if and only if $a = 0$,
- (ii) $|ab| = |a||b|$,
- (iii) $|a + b| \leq \max\{|a|, |b|\}$.

The condition (iii) is called the strict triangle inequality. By (ii), we have $|1| = |-1| = 1$. Thus, by induction, it follows from (iii) that $|n| \leq 1$ for each integer n . In addition, we always assume that $|\cdot|$ is non-trivial, i.e., there is an $a_0 \in \mathbb{K}$ such that $|a_0| \notin \{0, 1\}$.

Let X be a linear space over a scalar field \mathbb{K} with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \rightarrow \mathbb{R}$ is a non-Archimedean norm (valuation) if it satisfies the following conditions:

- (i) $\|x\| = 0$ if and only if $x = 0$,
- (ii) $\|rx\| = |r|\|x\|$ for all $r \in \mathbb{K}$ and $x \in X$,

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(iii) the strong triangle inequality (ultrametric), i.e.,

$$\|x + y\| \leq \max\{\|x\|, \|y\|\} \quad (x, y \in X).$$

Then, $(X, \|\cdot\|)$ is called a non-Archimedean space.

It follows from (iii) that

$$\|x_n - x_m\| \leq \max\{\|x_{j+1} - x_j\| : m \leq j \leq n - 1\} \quad (n > m),$$

therefore a sequence $\{x_m\}$ is Cauchy in X if and only if $\{x_{m+1} - x_m\}$ converges to zero in a non-Archimedean space. By a complete non-Archimedean space we mean one in which every Cauchy sequence is convergent.

The issue of stability of a functional equation can be expressed in the following way (see [10]-[13], [17]): *When is it true that a map which approximately satisfies a functional equation η must be near to an exact solution of η ?*

If the problem accepts a solution, we say that the equation η is stable. There are cases in which each approximate solution is actually a true solution. In such cases, we call the equation η superstable. In 1979, Baker, Lawrence, and Zorzitto [3] investigated the superstability of the functional equation from a vector space to real numbers.

It has been motivated by a question raised in 1940 by Ulam concerning approximate homomorphisms of groups (see [17]). The first partial answer to Ulam's question (in the case of Cauchy's functional equation in Banach spaces) was given by Hyers in [10]. Perhaps Aoki was the first author who generalized the theorem of Hyers (see [1]). Th.M. Rassias [14] provided a generalization of Hyers' Theorem which allows the Cauchy difference to be unbounded. Moreover, Bourgin [4] and Găvruta [8] considered the stability problem with unbounded Cauchy differences (see also [2]). On the other hand, J.M. Rassias [15] considered the Cauchy difference controlled by a product of different powers of norm. However, there was a singular case; for this singularity a counterexample was given by P. Găvruta [9].

Let X be a set. A function $d : X \times X \rightarrow [0, \infty]$ is called a generalized metric on X if d satisfies

- (i) $d(x, y) = 0$ if and only if $x = y$,
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the distinction between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity. Let (X, d) be a generalized metric space. A map $J : X \rightarrow X$ satisfies a Lipschitz condition with Lipschitz constant L if there exists a constant $L > 0$ such that $d(Jx, Jy) \leq Ld(x, y)$ for all $x, y \in X$. If the Lipschitz

constant L is less than 1, then the map J is called strictly contractive. We recall the following theorem by Diaz and Margolis (cf. [6, 16]).

Theorem 1.1. *Let (X, d) be a complete generalized metric space and $J : X \rightarrow X$ a strictly contractive map with Lipschitz constant $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that*

- (i) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$,
- (ii) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ,
- (iii) y^* is the unique fixed point of J in the set

$$Y = \{y \in X : d(J^{n_0} x, y) < \infty\},$$

- (iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

Throughout this paper we suppose that A and B are two complete non-Archimedean spaces. For convenience, we use the following abbreviation for a given map $f : A \rightarrow B$,

$$(1.1) \quad \Delta_m f(x, y) = mf\left(\frac{x+y}{m}\right) - f(x) - f(y)$$

for all $x, y \in A$, where m is a positive integer. We say that f is m -additive if $\Delta_m f(x, y) = 0$.

In this paper, we investigate the superstability of m -additive maps on complete non-Archimedean spaces via a fixed point method.

2. Superstability of m -additive maps

In this section, we demonstrate the superstability of m -additive maps on complete non-Archimedean spaces.

Lemma 2.1. *Let $f : A \rightarrow B$ be a map defined on vector spaces A and B .*

- (i) f is additive if and only if $\Delta_m f(x, y) = 0$ for all $x, y \in A$ and each positive integers $m \neq 2$,
- (ii) f is additive if and only if $f(0) = 0$ and $\Delta_2 f(x, y) = 0$ for all $x, y \in A$.

Proof. (i) It follows from $\Delta_m f(0, 0) = 0$ that $f(0) = 0$ and setting $y = 0$ in (1.1), we get $f(x) = \frac{1}{m} f(mx)$ for all $x \in A$. On the other hand, $\Delta_m f(mx, my) = 0$ for all $x, y \in A$ and hence f is additive. The converse is obvious.

(ii) It follows from $\Delta_2 f(x, 0) = 0$ that $f(x) = \frac{1}{2} f(2x)$ for all $x \in A$. Since $\Delta_2 f(2x, 2y) = 0$ for all $x, y \in A$, we conclude that f is additive. The converse is obvious. \square

We now establish the conditions providing the superstability of m -additive maps on complete non-Archimedean spaces.

Theorem 2.2. *Let $f : A \rightarrow B$ be a map for which there exists a map $\varphi : A \rightarrow [0, \infty)$ such that*

$$(2.1) \quad \|\Delta_m f(x, y)\| \leq \varphi(x)$$

for all $x, y \in A$ and each integer m greater than 2. If there exists a constant $0 < L < 1$ such that

$$(2.2) \quad \varphi(mx) \leq |m|L\varphi(x)$$

for all $x \in A$, then f is m -additive.

Proof. It follows from (2.2) that

$$(2.3) \quad \lim_{n \rightarrow \infty} \frac{1}{|m|^n} \varphi(m^n x) = 0$$

for all $x \in A$ and so $\varphi(0) = 0$. Letting $x = y = 0$ in (2.1), we get $\|(m-2)f(0)\| \leq \varphi(0) = 0$ and hence $f(0) = 0$. Let Ω be the set of all maps $h : A \rightarrow B$ and introduce a generalized metric on Ω as follows:

$$d(g, h) = \inf\{t \in (0, \infty) : \|g(x) - h(x)\| \leq t\varphi(x), \quad \forall x \in A\}.$$

The space (Ω, d) is a generalized complete metric space [7, 5]. Consider the map $\Theta : \Omega \rightarrow \Omega$ defined by $(\Theta h)(x) = \frac{1}{m}h(mx)$ for all $x \in A$ and all $h \in \Omega$. Let $d(g, h) < t$ for $g, h \in \Omega$ and $t \in (0, \infty)$. Then,

$$(2.4) \quad \|g(x) - h(x)\| \leq t\varphi(x)$$

for all $x \in A$. Replace x by mx in (2.4) to find that

$$\|g(mx) - h(mx)\| \leq t\varphi(mx)$$

for all $x \in A$. Apply (2.2) to get

$$\left\| \frac{1}{m}g(mx) - \frac{1}{m}h(mx) \right\| \leq Lt\varphi(x)$$

for all $x \in A$. This entails that $d(\Theta g, \Theta h) \leq Lt$ and so $d(\Theta g, \Theta h) \leq Ld(g, h)$ for all $g, h \in \Omega$, that is, Θ is a strictly contractive map of Ω with the Lipschitz constant L . Putting $y = 0$ in (2.1) ensures that

$$(2.5) \quad \left\| mf\left(\frac{x}{m}\right) - f(x) \right\| \leq \varphi(x)$$

for all $x \in A$. Replace x by mx and use (2.2) to derive that

$$\begin{aligned} \left\| f(x) - \frac{1}{m}f(mx) \right\| &\leq \frac{1}{|m|}\varphi(mx) \\ &\leq L\varphi(x) \end{aligned}$$

for all $x \in A$, that is, $d(f, \Theta f) \leq L < \infty$.

It follows from the fixed point alternative Theorem 1.1 that there exists a fixed point f^* of Θ in Ω such that $\lim_{n \rightarrow \infty} d(\Theta^n f, f^*) = 0$ and so

$$(2.6) \quad f^*(x) = \lim_{n \rightarrow \infty} \frac{1}{m^n} f(m^n x)$$

for all $x \in A$. On the other hand, (2.1), (2.3) and (2.6) ensure that

$$\begin{aligned} \|\Delta_m f^*(x, y)\| &= \lim_{n \rightarrow \infty} \frac{1}{|m|^n} \|\Delta_m f(m^n x, m^n y)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{|m|^n} \varphi(m^n x) \\ &= 0. \end{aligned}$$

Thus, $\Delta_m f^*(x, y) = 0$ for all $x, y \in A$, that is, f^* is m -additive.

Set $x = 0$ in (2.1) to find that

$$\|mf\left(\frac{y}{m}\right) - f(y)\| \leq \varphi(0) = 0$$

and so $f(y) = \frac{1}{m}f(my)$ for all $y \in A$. According to the fixed point alternative Theorem 1.1, f^* is the unique fixed point of Θ in the set $Y = \{h \in \Omega : d(f, h) < \infty\}$ and hence $f^* = f$. This means that f is m -additive. \square

The proof of the following theorem is similar to that of Theorem 2.2. It is enough to define the map $\Theta : \Omega \rightarrow \Omega$ by $(\Theta h)(x) = mh\left(\frac{x}{m}\right)$ for all $x \in A$ and all $h \in \Omega$.

Theorem 2.3. *Let $f : A \rightarrow B$ be a map for which there exists a map $\varphi : A \rightarrow [0, \infty)$ such that $\varphi(0) = 0$ and satisfying (2.1). If there exists a constant $0 < L < 1$ such that*

$$(2.7) \quad \varphi\left(\frac{x}{m}\right) \leq \frac{L}{|m|} \varphi(x)$$

for all $x \in A$, then f is m -additive.

Corollary 2.4. *Let α and $p \neq 1$ be non-negative real numbers and $f : A \rightarrow B$ be a map such that $f(0) = 0$ and*

$$\|\Delta_m f(x, y)\| \leq \alpha \|x\|^p,$$

for all $x \in A$, where $m > 1$ is a positive integer. Then, f is m -additive.

Proof. Define $\varphi(x) := \alpha \|x\|^p$ for all $x \in A$. For $p > 1$ choose $L = |m|^{p-1}$ and apply Theorem 2.2. For the case where $p < 1$ the assertion follows from Theorem 2.3 by letting $L = |m|^{1-p}$. \square

The following two corollaries are straightforward consequence of Lemma 2.1, Theorem 2.2, and Theorem 2.3.

Corollary 2.5. *Let $f : A \rightarrow B$ be a map for which there exists a map $\varphi : A \rightarrow [0, \infty)$ satisfying (2.1). If there exists a constant $0 < L < 1$ satisfying (2.2), then f is additive.*

Corollary 2.6. *Let $f : A \rightarrow B$ be a map for which there exists a map $\varphi : A \rightarrow [0, \infty)$ such that $\varphi(0) = 0$ and satisfying (2.1). If there exists a constant $0 < L < 1$ satisfying (2.7), then f is additive.*

Remark 2.7. Under the hypotheses of Corollary 2.4 and applying Lemma 2.1 f is additive.

Remark 2.8. A map $f : A \rightarrow B$ defined on complete non-Archimedean spaces A and B associated with the functional equation $\Delta_2 f(x, y) = 0$ is superstable with an additional assumption $f(0) = 0$.

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