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A NOTE ON "GENERALIZED BIVARIATE COPULAS AND THEIR PROPERTIES"

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ABSTRACT. In 2004, Rodríguez-Lallena and Úbeda-Flores have introduced a class of bivariate copulas which generalizes some known families such as the Farlie-Gumbel-Morgenstern distributions. In 2006, Dolati and Úbeda-Flores presented multivariate generalizations of this class. Then in 2011, Kim et al. generalized Rodríguez-Lallena and Úbeda-Flores' study to any given copula family. But there are some inaccuracies in the study by Kim et al. We mean to consider the interval for the parameter proposed by Kim et al. and show that it is inaccurate.

1. Introduction

A copula is a function $C : [0,1]^2 \to [0,1]$ which satisfies:

(a) for every u and v in $[0,1]$, $C(u,0) = 0 = C(0,v)$, $C(u,1) = u$ and $C(1,v) = v$;

(b) for every u_1, u_2, v_1 and v_2 in [0, 1] such that $u_1 \leq u_2$ and $v_1 \leq v_2$, $V_C(R) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$ (in other words, for all rectangles $R = [u_1,u_2] \times [v_1,v_2]$ whose vertices lie in $[0,1]^2$, $V_C(R) \geq 0$).

Copulas are multivariate distributions in modeling the dependence structure between variables, irrespective of their marginal distributions. Obviously with a wide range of copulas we are able to capture more miscellaneous dependence structures. Hence there is widely effort on constructions of copulas in the literature (see for instance, [1, 2, 3, 9, 10, 13]). Also Nelsen [11] summarized different methods of constructing copulas.

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Rodríguez-Lallena and Úbeda-Flores [12] introduced a class of bivariate copulas of the form:

$$C_\lambda(u,v) = uv + \lambda f(u)g(v), \quad (u,v) \in [0,1]^2, \tag{1.1}$$

where f and g are two non-zero absolutely continuous functions such that $f(0) = f(1) = g(0) = g(1) = 0$ and the admissible range of the parameter λ is

$$\frac{-1}{\max(\alpha\gamma, \beta\delta)} \leq \lambda \leq \frac{-1}{\min(\alpha\delta, \beta\gamma)},$$

where
(1.2)

$$\begin{array}{ll} \alpha = \inf\{f^{'}(u) : u \in A\} < 0, & \beta = \sup\{f^{'}(u) : u \in A\} > 0, \\ \gamma = \inf\{g^{'}(v) : v \in B\} < 0, & \delta = \sup\{g^{'}(u) : u \in B\} > 0, \\ A = \{u \in [0,1] : f^{'}(u) \text{ exists}\}, & B = \{v \in [0,1] : g^{'}(u) \text{ exists}\}. \end{array}$$

This class of copulas provides a method for constructing bivariate distributions with a variety of dependence structures and generalizes several known families such as the Farlie-Gumble-Morgenstern (FGM) distributions. Dolati and Úbeda-Flores [6] provided procedures to construct parametric families of multivariate distributions which generalize (1.1).

Kim et al. [7] generalized Rodríguez-Lallena and Úbeda-Flores' study to any given copula family. They presented an extension for any given copula family C as below

$$C^*_\lambda(u,v) = C(u,v) + \lambda f(u)g(v), \quad (u,v) \in [0,1]^2 \tag{1.3}$$

and

$$\frac{-V_C(R)}{\triangle \times \max(\alpha\gamma, \beta\delta)} \leq \lambda \leq \frac{-V_C(R)}{\triangle \times \min(\alpha\gamma, \beta\delta)}, \tag{1.4}$$

where α, β, γ and δ are same as (1.2), $\triangle = (u_2 - u_1)(v_2 - v_1)$, u_1, u_2, v_1 and v_2 are in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$ and f and g are two non-zero absolutely continuous functions defined on $[0,1]$ such that $f(0) = f(1) = g(0) = g(1) = 0$.

Kim et al.'s results had been discussed by Mesiar et al. in [8] and also by Bekrizadeh et al. in [4, 5]. In this study, we concentrate on the interval of the λ parameter given by (1.4) and we comment on inaccuracy of this interval.

2. Comments on inaccuracy of the interval

In this section we concentrate on the bounds in (1.4) to specify some comments about this relation. We impose on C^* in (1.3) the property of being 2-increasing. For every u_1, u_2, v_1 and v_2 in $[0, 1]$ such that $u_1 < u_2$ and $v_1 < v_2$ and $R = [u_1, u_2] \times [v_1, v_2]$, with simple calculation, we get

$$V_{C^*}(R) = V_C(R) + \lambda(f(u_2) - f(u_1))(g(v_2) - g(v_1)) \geq 0,$$

then

$$\lambda(f(u_2) - f(u_1))(g(v_2) - g(v_1)) \geq -V_C(R). \tag{2.1}$$

By multiplying both sides of the relation (2.1) by $1/((u_2 - u_1)(v_2 - v_1))$ we have

$$\lambda \frac{f(u_2) - f(u_1)}{(u_2 - u_1)} \frac{g(v_2) - g(v_1)}{(v_2 - v_1)} \geq \frac{-V_C(R)}{(u_2 - u_1)(v_2 - v_1)}.$$

If f and g are absolutely continuous functions, as in Rodríguez-Lallena and Úbeda-Flores' study we get

$$\frac{-V_C(R)}{\max(\alpha\gamma, \beta\delta)(u_2 - u_1)(v_2 - v_1)} \leq \lambda \leq \frac{-V_C(R)}{\min(\alpha\delta, \beta\gamma)(u_2 - u_1)(v_2 - v_1)},$$

and to get the optimal interval we have

$$\sup(\frac{-V_C(R)}{\triangle \times \max(\alpha\gamma, \beta\delta)}) \leq \lambda \leq \inf(\frac{-V_C(R)}{\triangle \times \min(\alpha\delta, \beta\gamma)}), \tag{2.2}$$

where $\triangle, \alpha, \beta, \gamma$ and δ are same as (1.2). From the comparison of (2.2) and (1.4), the inaccuracy of the interval by (1.4) is clear. Moreover we show inaccuracy of the mentioned interval in (1.4) by several counter-examples:

I. Let f and g be of the form $f(u) = u(1-u)$ and $g(v) = v(1-v)$ then $\alpha = -1$, $\beta = 1$, $\gamma = -1$, $\delta = 1$ and $\max(\alpha\gamma, \beta\delta) = \min(\alpha\gamma, \beta\delta) = 1$. For any rectangle R we have

$$\lambda = \frac{-V_C(R)}{(u_2 - u_1)(v_2 - v_1)},$$

and then there does not exist any λ satisfying the above equalities whenever C is different from the product copula. Moreover, for the product copula, there is the unique solution $\lambda = -1$, i.e. this fact also contradicts the result of Rodríguez-Lallena and Úbeda-Flores' study if we let $C(u, v) = uv$.

II. Since $\max(\alpha\gamma, \beta\delta)$ and $\min(\alpha\gamma, \beta\delta)$ are positive numbers then λ in (1.4) is restricted by negative bounds, hence we are not able to extend the lower Fréchet-Hoeffding bound copula, $W(u, v) = \max(u + v - 1, 0)$.

III. Let $C(u, v) = \min(u, v)$ and f and g for all u and v in $[0, 1]$ are non-zero absolutely continuous functions as below

$$f(u) = u^2(1 - u), \qquad g(v) = v(1 - v),$$

then $\alpha = -0.3333$, $\beta = 1$, $\gamma = -1$ and $\delta = 1$ and hence

$$\max(\alpha\gamma, \beta\delta) = 1, \qquad \min(\alpha\gamma, \beta\delta) = 0.3333.$$

With numerical calculations we get $-10 \leq \lambda \leq -1$. Obviously for some values of λ, $V_{C^*}(R)$ is negative for some values of u_1, u_2, v_1 and v_2 in $[0,1]$. As an example, for $\lambda = -4$ and $u_2 = 0.3, u_1 = 0.1, v_2 = 0.1$ and $v_1 = 0$, we have $V_{C^*}(B) = -0.0238$. It means that the extended new family does not satisfy in the 2-increasing condition of copulas.

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