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A TENSOR PRODUCT APPROACH TO THE ABSTRACT PARTIAL FOURIER TRANSFORMS OVER SEMI-DIRECT PRODUCT GROUPS

ALI AKBAR AREFIJAMAAL[1*] AND FAHIMEH ARABYANI NEYSHABURI[2]

Abstract. In this article, by using a partial on locally compact semi-direct product groups, we present a compatible extension of the Fourier transform. As a consequence, we extend the fundamental theorems of Abelian Fourier transform to non-Abelian case.

1. Introduction and preliminaries

The abstract notion of partial dual group (partial duality) for semi-direct product of locally compact groups with Abelian normal factor is introduced recently in [11, 13]. This notion has been applied in unifying the mathematical theory of partial Fourier analysis and partial Gabor analysis over semi-direct product of locally compact groups with Abelian normal factor [10, 12, 13].

The abstract theory of positive definite functions over locally compact groups plays a significant role in theoretical and applied modern harmonic analysis and representation theory, see [6, 7, 8, 9, 14, 17]. The abstract harmonic analysis of positive definite functions over locally compact groups is tightly connected to the representation theory and Fourier analysis. The representation theory and also harmonic analysis over semi-direct product groups have deep impact in recent developments of coherent states analysis, see [1-3].

Throughout this article, we present a unified group theoretical approach to the abstract harmonic analysis aspects of partial positive-definite functions over semi-direct product groups.

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* Corresponding author.

The Fourier analysis on locally compact Abelian (LCA) groups has been studied at depth by many authors, (see [9, 17] and references therein). Let K be an LCA group with the Haar measure m_K and the dual group $\widehat{K}$. The *Fourier transform* $\widehat{f}$ of any function $f \in L^1(K)$ is defined by

$$\widehat{f} : \widehat{K} \to \mathbb{C}; \quad \widehat{f}(\xi) = \int_K f(x)\overline{\xi(x)}dm_K(x).$$

The classical Fourier transform $f \longmapsto \widehat{f}$ is a $*$-homomorphism from $L^1(K)$ and its range is a dense subspace of $C_0(\widehat{K})$, the set of all continuous functions which vanishes at infinity. As usual we extend the Fourier transform to an isometry from $L^2(K)$ onto $L^2(\widehat{K})$, the so-called *Plancherel isomorphism.* By the Fourier inversion, we can recover a function from its Fourier transform. Several different Fourier inversion theorems exist, one of the most important of which states that if $f \in L^1(K)$ and $\widehat{f} \in L^1(\widehat{K})$, then

$$f(x) = \int_{\widehat{K}} \widehat{f}(\xi)\xi(x)dm_{\widehat{K}}(\xi). \tag{1.1}$$

This formula remains valid in spirit for all $f \in L^2(K)$, (see [9]). Many groups which appear in mathematical physics and quantum mechanics are non-Abelian, although they can be considered as semi-direct product of locally compact Abelian groups. The Fourier transform theory on LCA groups is based on the dual group. However, the dual of non-Abelian locally compact groups is considerably more intricate and consists of all classes of equivalence of irreducible representations of such groups (cf. [9, Section 7]). The theory of Fourier analysis can be extended to non-Abelian groups [4, 9, 16] and has been used in applications [18].

In this article, by using a new concept of dual group of semi-direct products, we present a compatible extension of the Fourier transform and extend the fundamental theorems of Abelian Fourier transform. In fact, this extension is based on classical Fourier transform and hence many algorithms can be utilized by Abelian Fourier transform tools for a large class of non-Abelian locally compact groups.

We end this section with a brief review of two norms on the tensor product of Banach spaces, (for more details see [19]). Let $X \otimes Y$ denote the algebraic tensor product of two Banach spaces X and Y. The projective norm $\|.\|_\pi$ on $X \otimes Y$ is defined by

$$\|x\|_\pi = \inf\{\sum_{i=1}^n \|x_i\|\|y_i\| : \quad x = \sum_{i=1}^n x_i \otimes y_i, n \in \mathbb{N}\}.$$

It can be shown that $\|.\|_\pi$ is a cross norm, that is, $\|x \otimes y\|_\pi = \|x\|\|y\|$. The completion of $X \otimes Y$ with respect to the projective norm is called the projective tensor product and is denoted by $X\hat{\otimes}Y$. Furthermore, the injective tensor product X and Y, denoted by $X\check{\otimes}Y$ is the completion of $X \otimes Y$ with the following injective (cross) norm

$$\|x\|_e = \sup\{|\langle x, u_1 \otimes u_2\rangle| : \quad u_1 \in ball(X^*), u_2 \in ball(Y^*)\}, \quad (x \in X \otimes Y).$$

Let f and g be two functions on locally compact Hausdorff spaces Ω_1 and Ω_2, respectively. The tensor product of f and g is defined by

$$(f \otimes g)(x, y) = f(x)g(y), \quad (x \in \Omega_1, y \in \Omega_2).$$

The following proposition will often be used in this article, (see section B.2 of [19]).

Proposition 1.1. *(i) Let Ω_1 and Ω_2 be two locally compact Hausdorff spaces. Then, $C_0(\Omega_1 \times \Omega_2)$ is isometrically and algebraically isomorphic to $C_0(\Omega_1)\check{\otimes}C_0(\Omega_2)$.*
(ii) Let G_1 and G_2 be locally compact groups. Then, $L^1(G_1 \times G_2)$ is isometrically and algebraically isomorphic to $L^1(G_1)\hat{\otimes}L^1(G_2)$.

2. Main Results

Throughout this article, we assume $G_\tau = H \times_\tau K$ is the semi-direct product group of locally compact group H and LCA group K. The mapping $h \mapsto \tau_h$ is a homomorphism of H into the group of automorphisms of K such that the mapping $(h, k) \longmapsto \tau_h(k)$ from $H \times K$ onto K is continuous. The group law is given by

$$(h, k).(h', k') = (hh', k\tau_h(k')), \quad ((h, k) \in G_\tau).$$

Then G_τ is a (not necessarily Abelian) locally compact group. Moreover, the left Haar measure of G_τ is $dm_{G_\tau}(h, k) = \delta(h)dm_H(h)dm_K(k)$, where the positive continuous homomorphism δ on H is given by

$$dm_K(k) = \delta(h)dm_K(\tau_h(k)). \tag{2.1}$$

2.1. Partial dual theory.

The notion of partial duality on locally compact semi-direct product groups introduced in [11]. Here we just briefly summarize building blocks of partial dual theory. It is still assumed that $h \mapsto \tau_h$ is a homomorphism of H into the group of automorphisms of K (i.e $Aut(K)$) such that the mapping $(h, k) \longmapsto \tau_h(k)$ from $H \times K$ onto K is continuous.

Define $\widehat{\tau} : H \to Aut(\widehat{K})$ via $h \mapsto \widehat{\tau}_h$, given by

$$\widehat{\tau}_h(\omega) := \omega_h = \omega \circ \tau_{h^{-1}}, \tag{2.2}$$

for all $\omega \in \widehat{K}$, where $\omega_h(k) = \omega(\tau_{h^{-1}}(k))$ for all $k \in K$, (see [11]). According to (2.2), for all $h \in H$, we have $\widehat{\tau}_h \in Aut(\widehat{K})$ and also $h \mapsto \widehat{\tau}_h$

is a homomorphism from H into $Aut(\widehat{K})$. Because if $h, t \in H$, then for all $\omega \in \widehat{K}$ and $k \in K$ we have

$$\begin{aligned}\widehat{\tau}_{th}(\omega)(k) &= \omega_{th}(k)\\ &= \omega(\tau_{h^{-1}}\tau_{t^{-1}}(k))\\ &= \omega_h(\tau_{t^{-1}}(k))\\ &= \widehat{\tau}_h(\omega)(\tau_{t^{-1}}(k))\\ &= \widehat{\tau}_t[\widehat{\tau}_h(\omega)](k).\end{aligned}$$

Now, we just state the following theorem.

Theorem 2.1. *Let H be a locally compact group and K be an LCA group. Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $\delta : H \to (0, \infty)$ be the positive continuous homomorphism satisfying (2.1). The semi-direct product*

$$G_{\widehat{\tau}} = H \times_{\widehat{\tau}} \widehat{K},$$

is a locally compact group with the left Haar measure

$$dm_{G_{\widehat{\tau}}}(h, \omega) = \delta(h)^{-1} dm_H(h) dm_{\widehat{K}}(\omega).$$

Proof. See Theorem 3.3 of [11]. □

Remark 2.2. If $H = (e)$ be the identity group, then $G_{\widehat{\tau}}$ coincides with $\widehat{K}$, the classical dual of Abelian group K. Moreover, if $G_\tau = H \times K$ be the direct product of LCA groups H and K (or equivalently $\tau_h = I$, for all $h \in H$), then $G_\tau = H \times K$ is Abelian and by informal notations we have

$$\begin{aligned}G_{\widehat{\tau}} &\simeq (\{e\} \times_\tau H \times K)^{\widehat{}}\\ &\simeq \{e\} \times_{\widehat{\tau}} (H \times K)^{\widehat{}}\\ &\simeq \{e\} \times_{\widehat{\tau}} \widehat{H} \times \widehat{K} \simeq \widehat{H} \times \widehat{K}.\end{aligned}$$

Therefore, τ-dual is a generalization of dual of LCA groups and can play a dual role.

It should be mentioned that, the function space $L^1(H)$ is an involutive Banach algebra with the following involution and convolution:

$$\begin{cases} f^*(h) = \Delta_H(h^{-1})\overline{f(h^{-1})}, \\ (f * g)(h) = \int_H f(a) g(a^{-1}h) dm_H(a), \end{cases} \tag{2.3}$$

where Δ_H is the modular function of H, (for more details see section 2.5 of [9]). Notice that when $p \geq 1$ and $F \in L^p(G_\tau)$, the mapping F_h defined on K via $F_h(k) = F(h, k)$ belongs to $L^p(K)$ for a.e. $h \in H$.

We introduce the tensor product $L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$ to analyze the properties of the Fourier transform on $L^1(G_\tau)$ where $L^1_\delta(H)$ is the set of all integrable functions on H with the measure $\delta(h)dm_H(h)$. The involution on $L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$ is defined by

$$\varphi^*(h,\omega) = \Delta_H(h^{-1})\overline{\varphi(h^{-1},\omega\circ\tau_h)}, \tag{2.4}$$

also its convolution can be given by

$$(\varphi * \psi)(h,\omega) = \delta(h^{-1})\int_H \varphi(a,\omega)\psi(a^{-1}h,\omega\circ\tau_a)dm_H(a). \tag{2.5}$$

The next theorem shows that the integral is absolutely convergent for almost every (h,ω). Obviously, (2.4) and (2.5) generalize the usual definitions of involution and convolution on the tensor product of Banach algebras [15].

Theorem 2.3. *Let $G_\tau = H \times_\tau K$. Then, the involution (2.4) and the convolution product (2.5) make $L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$ into a Banach $*$-algebra.*

Proof. We first show that the convolution (2.5) is well-defined. To see this, suppose $f_1 \otimes f_2$ and $g_1 \otimes g_2$ are two tensor product elements. Then

$$\begin{aligned}
&|(f_1\otimes f_2) * (g_1\otimes g_2)(h,\omega)| \\
&\quad = \delta(h^{-1})\left|\int_H f_1(a)g_1(a^{-1}h)f_2(\omega)g_2(\omega\circ\tau_a)dm_H(a)\right| \\
&\quad \le \left(\delta^{-1}|f_1| * \delta^{-1}|g_1|\right)(h)\|f_2\|_\infty\|g_2\|_\infty,
\end{aligned}$$

where $*$ in $\delta^{-1}|f_1| * \delta^{-1}|g_1|$ indicates the usual convolution on $L^1(H)$. Now, if

$$\varphi = \sum_{i=1}^n f_{1,i}\otimes f_{2,i}, \qquad \psi = \sum_{i=1}^m g_{1,i}\otimes g_{2,i},$$

are two elements of $L^1_\delta(H)\otimes C_0(\widehat{K})$, then

$$|(\varphi * \psi)(h,\omega)| \le \sum_{i=1}^n\sum_{j=1}^m \left(\delta^{-1}|f_{1,i}| * \delta^{-1}|g_{1,j}|\right)(h)\|f_{2,i}\|_\infty\|g_{2,j}\|_\infty.$$

In particular, $h \mapsto \varphi * \psi(h,\omega)$ belongs to $L^1(H)$ for a.e. $\omega\in\widehat{K}$ and

$$\begin{aligned}
\|(\varphi * \psi)(.,\omega)\|_1 &\le \sum_{i=1}^n\sum_{j=1}^m \left\|(\delta^{-1}|f_{1,i}| * \delta^{-1}|g_{1,j}|)\right\|_1 \|f_{2,i}\|_\infty\|g_{2,j}\|_\infty \\
&\le \sum_{i=1}^n \|f_{1,i}\|_{L^1_\delta(H)}\|f_{2,i}\|_\infty \sum_{j=1}^m \|g_{1,j}\|_{L^1_\delta(H)}\|g_{2,j}\|_\infty.
\end{aligned}$$

This easily follows that

$$\|(\varphi * \psi)(.,\omega)\|_1 \le \|\varphi\|_\pi\|\psi\|_\pi, \quad \omega\in\widehat{K}. \tag{2.6}$$

Now assume that Φ and Ψ are two elements of $L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$, then there exist sequences $\{\varphi_i\}_{i=1}^\infty$ and $\{\psi_i\}_{i=1}^\infty$ in $L^1_\delta(H)\otimes C_0(\widehat{K})$ such that $\|\varphi_i - \Phi\|_\pi \to 0$ and $\|\psi_i - \Psi\|_\pi \to 0$. Applying (2.6) we obtain

$$\begin{aligned}
&\|\left(\Phi * \Psi - \varphi_i * \psi_i\right)(.,\omega)\|_1 \\
&\qquad \leq \|\left(\Phi * (\psi_i - \psi)\right)(.,\omega)\|_1 + \|\left((\varphi_i - \mathit{\Phi}) * \Psi\right)(.,\omega)\|_1 \\
&\qquad \leq \|\Phi\|_\pi \|\psi_i - \Psi\| + \|\Psi\|_\pi \|\varphi_i - \Phi\|_\pi .
\end{aligned}$$

In particular, the integral in (2.5) is finite. Moreover, by using (2.6) we obtain

$$\|\Phi * \Psi\|_\pi \leq \|\Phi\|_\pi \|\Psi\|_\pi .$$

Furthermore, from Definition (2.5), it follows that

$$\begin{aligned}
&(\Phi * \Psi)^*(h,\omega) \\
&\quad = \Delta_H(h^{-1})\overline{(\Phi * \Psi)(h^{-1}, \omega \circ \tau_h)} \\
&\quad = \Delta_H(h^{-1})\delta(h^{-1}) \int_H \overline{\Phi(a, \omega \circ \tau_h)\Psi(a^{-1}h^{-1}, \omega \circ \tau_{ha})} dm_H(a) \\
&\quad = \delta(h^{-1}) \int_H \Delta_H(h^{-1})\overline{\Phi(h^{-1}a, \omega \circ \tau_{aa^{-1}h})\Psi(a^{-1}, \omega \circ \tau_a)} dm_H(a) \\
&\quad = \delta(h^{-1}) \int_H \Phi^*(a^{-1}h, \omega \circ \tau_a) Psi^*(a,\omega) dm_H(a) \\
&\quad = (\Psi^* * \Phi^*)(h,\omega).
\end{aligned}$$

All the other properties of involution and convolution are straightforward to check and left to the reader. □

Definition 2.4. Let $G_\tau = H \times_\tau K$. If $F \in L^1(G_\tau)$, its (generalized) Fourier transform $\widehat{F}$ on $H \times_{\widehat{\tau}} \widehat{K}$ is defined as

$$\widehat{F}(h,\omega) = \int_K F(h,k)\overline{\omega(k)}\delta(h) dm_K(k). \tag{2.7}$$

Proposition 2.5. *The Fourier transform (2.7) is a norm decreasing $*$-homomorphism from $L^1(G_\tau)$ to $L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$. Its range is dense in $L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$.*

Proof. Applying Proposition A.3.69 of [5] we have

$$\widehat{L^1(G_\tau)} = (L^1_\delta(H)\hat{\otimes}L^1(K))\widehat{\ } \subseteq L^1_\delta(H)\hat{\otimes}C_0(\widehat{K}).$$

To prove that the Fourier transform (2.7) is norm decreasing, it is enough to check it only on the tensor product elements. Let $F = \delta^{-1} f_1 \otimes f_2 \in L^1(G_\tau)$, where $f_1 \in L^1(H)$ and $f_2 \in L^1(K)$. Obviously $\widehat{F} = f_1 \otimes \widehat{f_2}$, and therefore by using the fact that the classical Fourier transform on $L^1(K)$ is norm decreasing, we have

$$\|\widehat{F}\|_\pi \leq \|f_1\|_1 \|\widehat{f_2}\|_\infty \leq \|f_1\|_1 \|f_2\|_1 = \|F\|_1.$$

We are going to show that $\widehat{L^1(G_\tau)}$ is dense in $L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$. Let $\Phi \in L^1_\delta(H)\hat{\otimes}C_0(\widehat{K})$ and $\epsilon > 0$ is given. Choose $\varphi \in L^1_\delta(H)\otimes C_0(\widehat{K})$ such that $\|\Phi - \varphi\| \leq \epsilon$ and $\varphi = \sum_{i=1}^n f_i \otimes u_i$, where $f_i \in L^1(H)$ and $u_i \in C_0(\widehat{K})$. Due to Proposition 4.13 of [9] there exist $g_1, g_2, \cdots, g_n \in L^1(K)$ such that $\|\widehat{g_i} - u_i\|_\infty \leq \frac{\epsilon}{nM}$, where $M = \max\{\|f_1\|, \|f_2\|, \cdots, \|f_n\|\}$. Putting $F = \sum_{i=1}^n f_i \otimes g_i$, then $F \in L^1(G_\tau)$ and

$$\begin{aligned}\|\widehat{F} - \Phi\|_\pi &\leq \|\widehat{F} - \varphi\|_\pi + \|\varphi - \Phi\|_\pi \\ &\leq \left\| \sum_{i=1}^n f_i \otimes (\widehat{g_i} - u_i) \right\|_\pi + \epsilon \leq 2\epsilon.\end{aligned}$$

To complete the proof, it is sufficient to show that the Fourier transform (2.7) is $*$-homomorphism. For each $F_1, F_2 \in L^1(G_\tau)$, by using (2.3), we obtain

$$\begin{aligned}&\widehat{F_1 * F_2}(h,\omega) \\ &= \delta(h)\int_K F_1 * F_2(h,x)\overline{\omega(x)}dm_K(x) \\ &= \delta(h)\int_K\int_{G_\tau} F_1(a,y)F_2(a^{-1}h, \tau_{a^{-1}}(y^{-1}x))\overline{\omega(x)}dm_{G_\tau}(a,y)dm_K(x) \\ &= \delta(h)\int_H\int_K\int_K F_1(a,y)F_2(a^{-1}h,\tau_{a^{-1}}(x))\overline{\omega(yx)}\delta(a)dm_K(x)dm_K(y)dm_H(a) \\ &= \delta(h)\int_H\int_K F_2(a^{-1}h,\tau_{a^{-1}}(x))\widehat{F_1}(a,\omega)\overline{\omega(x)}dm_K(x)dm_H(a) \\ &= \int_H \widehat{F_1}(a,\omega)\int_K F_2(a^{-1}h,x))\overline{\omega\circ\tau_a(x)}\delta(a^{-1}h)dm_K(x)dm_H(a) \\ &= \int_H \widehat{F_1}(a,\omega)\widehat{F_2}(a^{-1}h,\omega\circ\tau_a)dm_H(a) = \left(\widehat{F_1} * \widehat{F_2}\right)(h,\omega).\end{aligned}$$

Also,

$$\begin{aligned}\widehat{F^*}(h,\omega) &= \Delta(h^{-1})\int_K \overline{F(h^{-1},\tau_{h^{-1}}(x^{-1}))\omega(x)}dm_K(x) \\ &= \Delta(h^{-1})\int_K \overline{F(h^{-1},x)\omega\circ\tau_h(x)}\delta(h)dm_K(x) = (\widehat{F})^*(h,\omega).\end{aligned}$$

□

The benefit of using the Fourier transform (2.7) is that, the fundamental theorems of Abelian Fourier analysis can be extended to non-Abelian semi-direct product groups, (see also Theorems 4.2 and 4.3 of [11]).

Theorem 2.6 (Generalized Plancherel theorem)**.** *The (generalized) Fourier transform is a linear isometry from $L^2(G_\tau)$ onto $L^2(G_{\widehat{\tau}})$.*

Proof. If $F \in L^1(G_\tau) \cap L^2(G_\tau)$, then by using the classical Plancherel theorem we have

$$\begin{aligned}\|\widehat{F}\|_2^2 &= \int_H \int_{\widehat{K}} |\widehat{F}(h,\omega)|^2 \delta(h)^{-1} dm_{\widehat{K}}(\omega) dm_H(h) \\ &= \int_H \delta(h) \int_{\widehat{K}} |\widehat{F_h}(\omega)|^2 dm_{\widehat{K}}(\omega) dm_H(h) \\ &= \int_H \int_K \delta(h) |F_h(x)|^2 dm_K(x) dm_H(h) = \|f\|_2^2.\end{aligned}$$

Thus $F \mapsto \widehat{F}$ is an isometry in the L^2 norm, so it extends uniquely to an isometry from $L^2(G_\tau)$ into $L^2(G_{\widehat{\tau}})$. To show that it is surjective, let $\varphi \in L^2(G_{\widehat{\tau}})$ and $\langle \varphi, \widehat{F} \rangle = 0$ for all $F \in L^2(G_\tau)$. Putting $F(h,x) = \delta(h)^{\frac{-1}{2}} f_1(h) f_2(x)$, where $f_1 \in L^2(H)$ and $f_2 \in L^2(K)$. It is easy to see that $F \in L^2(G_\tau)$ and $\widehat{F}(h,\omega) = \delta(h)^{\frac{1}{2}} f_1(h) \hat{f}_2(\omega)$. Thus,

$$\begin{aligned}0 &= \langle \varphi, \widehat{f} \rangle \\ &= \int_H \int_{\widehat{K}} \varphi(h,\omega) \delta(h)^{\frac{1}{2}} f_1(h) \widehat{f_2}(\omega) dm_{\widehat{K}}(\omega) dm_H(h) \\ &= \int_H \langle \varphi_h, \widehat{f_2} \rangle \delta(h)^{\frac{1}{2}} f_1(h) dm_H(h).\end{aligned}$$

This follows that $\varphi_h = 0$ a.e. and therefore $\varphi = 0$. □

Similarly, we can derive the generalized inversion theorem as follows.

Theorem 2.7. *Let $G_\tau = H \times_\tau K$. If $F \in L^1(G_\tau)$ and $\widehat{F} \in L^1(G_{\widehat{\tau}})$, then*

$$F(h,k) = \delta(h^{-1}) \int_{\widehat{K}} \widehat{F}(h,\omega) \omega(k) dm_{\widehat{K}}(\omega), \quad (a.e.\ (h,k) \in G_{\widehat{\tau}}).$$

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[1] Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.

E-mail address: arefijamaal@hsu.ac.ir

[2] Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, Iran.

E-mail address: arabyanif@hsu.ac.ir