Sahand Communications in Mathematical Analysis (SCMA) Vol. 2 No. 2 (2015), 23-44
http://scma.maragheh.ac.ir

ABSTRACT STRUCTURE OF PARTIAL FUNCTION $*$-ALGEBRAS OVER SEMI-DIRECT PRODUCT OF LOCALLY COMPACT GROUPS

ARASH GHAANI FARASHAHI[1*] AND RAJAB ALI KAMYABI GOL[2]

ABSTRACT. This article presents a unified approach to the abstract notions of partial convolution and involution in L^p-function spaces over semi-direct product of locally compact groups. Let H and K be locally compact groups and $\tau : H \to Aut(K)$ be a continuous homomorphism. Let $G_\tau = H \ltimes_\tau K$ be the semi-direct product of H and K with respect to τ. We define left and right τ-convolution on $L^1(G_\tau)$ and we show that, with respect to each of them, the function space $L^1(G_\tau)$ is a Banach algebra. We define τ-convolution as a linear combination of the left and right τ-convolution and we show that the τ-convolution is commutative if and only if K is abelian. We prove that there is a τ-involution on $L^1(G_\tau)$ such that with respect to the τ-involution and τ-convolution, $L^1(G_\tau)$ is a non-associative Banach $*$-algebra. It is also shown that when K is abelian, the τ-involution and τ-convolution make $L^1(G_\tau)$ into a Jordan Banach $*$-algebra. Finally, we also present the generalized notation of τ-convolution for other L^p-spaces with $p > 1$.

1. Introduction

In classical harmonic analysis, convolution is an integral which interprets the quantity of overlap of a function as it is shifted over another function, (see [11, 21, 22] and references therein). The standard convolution on the finite cyclic groups or the group of integer numbers, which sometimes called circular convolution, has many applications in the mathematical theory of signal processing and filtering [14, 18]. Classical generalizations for the concept of standard convolution on (finite or

---

2010 *Mathematics Subject Classification.* 43A15, 43A85.
Key words and phrases. Semi-direct products of groups, Left τ-convolution (τ_l-convolution), Right τ-convolution (τ_r-convolution), τ-convolution, τ-involution, τ-approximate identity.
Received: 09 March 2015, Accepted: 25 July 2015.
* Corresponding author.

infinite) discrete groups to the set up of integrable functions on the real line, play important roles in the theory of partial differential equations and other mathematical theories in theoretical physics [11, 20, 22]. The group theoretical based extensions for the theory of convolution [7], both in numerical and abstract aspects has many interesting applications in variant and vast branches of science, such as, physics, mechanical engineering and communication technologies [6]. The principal role played by convolutions and involutions in classical harmonic analysis, is in evidence throughout [5, 8, 24, 30].

Non-commutative algebraic structures, such as locally compact non-abelian groups or homogeneous spaces of locally compact non-abelain groups [26-29] play, important role in mathematical physics, computational sciences and engineering, (see [6, 11, 13, 16, 19] and references therein). In (numerical) harmonic analysis, many non-abelian groups, can be considered as a semi-direct product of groups. Over the last decades, aspects of coherent states transforms over locally compact groups, have achieved significant popularity in modern coherent states analysis (see [1, 2, 3, 4, 15, 16, 17, 25] and references therein). In this paper, we present a unified approach to the notion of convolution and involution over the semi-direct products of groups and we study basic properties of this approach.

Throughout this article, which contains 4 sections, we assume that H and K are locally compact topological groups, $\tau : H \to Aut(K)$ is a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ is the semi-direct product of H and K with respect to τ. Section 2 is devoted to fix notations and also a brief summary on the classic properties of the semi-direct products of groups. In section 3, first we define left and right τ-convolution on $L^1(G_\tau)$, which makes $L^1(G_\tau)$ into a Banach algebra. Then we define the τ-convolution as a linear combination of left and right convolution and also we define a τ-involution on $L^1(G_\tau)$ and we show that $L^1(G_\tau)$ is a non-associative Banach $*$-algebra with respect to the τ-convolution and the τ-involution. We prove that the τ-convolution is commutative if and only if K is abelian. Finally, in section 4, as an application for $p > 1$, we make $L^p(G_\tau)$ into a left Banach $L^1_{\tau_l}(G_\tau)$-module.

2. Preliminaries and notations

A non-associative algebra $\mathcal{B}$ is a linear space $\mathcal{B}$ over field of complex or real numbers, endowed with a bilinear map $(\alpha, \beta) \mapsto \alpha\beta$ from $\mathcal{B} \times \mathcal{B}$ into $\mathcal{B}$, and also a Jordan algebra is a commutative non-associative algebra $\mathcal{B}$ whose product satisfies the Jordan identity $(\alpha\beta)\alpha^2 = \alpha(\beta\alpha^2)$, for all $\alpha, \beta \in \mathcal{B}$. Note that the term non-associative stands for not necessarily associative. More precisely we use the term non-associative algebra in order to emphasize that the associativity of the product is not being assumed. A non-associative Banach algebra is a non-associative algebra $\mathcal{B}$ over the field of complex or real numbers, whose underlying linear space is a Banach space with respect to a norm $\|.\|$ satisfying $\|\alpha\beta\| \leq \|\alpha\|\|\beta\|$ for all $\alpha, \beta \in \mathcal{B}$.

Let X be a locally compact Hausdorff space. By $\mathcal{C}_c(X)$, we mean the space of all continuous complex valued functions on X with compact support. If μ is a positive Radon measure on X, for each $1 \leq p < \infty$, the Banach space of equivalence classes of μ-measurable complex valued functions $f : X \to \mathbb{C}$, such that

$$\|f\|_p = \left(\int_X |f(x)|^p d\mu(x) \right)^{1/p} < \infty,$$

is denoted by $L^p(X, \mu)$, which contains dense subspace $\mathcal{C}_c(X)$. When G is a locally compact group with left Haar measure dx and modular function Δ_G, for each $p \geq 1$, we mean by $L^p(G)$, the Banach space $L^p(G, dx)$. If $p = 1$, standard convolution of $f, g \in L^1(G)$ is defined via

$$f * g(x) = \int_G f(y)g(y^{-1}x)dy, \tag{2.1}$$

and also, the standard involution of $f \in L^1(G)$, is defined by

$$f^*(x) = \Delta_G(x^{-1})\overline{f(x)}, \tag{2.2}$$

which makes $L^1(G)$ into a Banach $*$-algebra. We recall that the Banach $*$-algebra $L^1(G)$ with respect to the standard convolution given in (2.1), has an approximate identity (see Proposition 2.42 of [11]) and also the Banach $*$-algebra $L^1(G)$ is commutative if and only if G is abelian, (see [9-11] and standard references therein).

Let H and K be locally compact groups with identity elements e_H and e_K respectively and left Haar measures dh and dk respectively and also let $\tau : H \to Aut(K)$ be a homomorphism such that, the map $(h, k) \mapsto \tau_h(k)$ is continuous from $H \times K$ to K. In this case, the homomorphism $\tau : H \to Aut(K)$ is called continuous. The semidirect product $G_\tau = H \ltimes_\tau K$ is a locally compact topological group with underlying set $H \times K$ which is equipped with the product topology and the group operation is defined by

$$(h, k) \ltimes_\tau (h', k') = (hh', k\tau_h(k')), \tag{2.3}$$

and

$$(h, k)^{-1} = (h^{-1}, \tau_{h^{-1}}(k^{-1})). \tag{2.4}$$

The left Haar measure of G_τ is

$$d\mu_{G_\tau}(h, k) = \delta(h)dhdk,$$

and the modular function of G_τ is

$$\Delta_{G_\tau}(h,k) = \delta(h)\Delta_H(h)\Delta_K(k),$$

where the positive continuous homomorphism $\delta : H \to (0,\infty)$ satisfies

$$dk = \delta(h)d(\tau_h(k)),$$

and Δ_H and Δ_K are modular functions of locally compact groups H and K respectively, (for more details see [22, 23]).

3. τ-convolution and τ-involution

Throughout this article, we assume that H and K are locally compact groups and $\tau : H \to Aut(K)$ is a continuous homomorphism.

For $f \in L^1(G_\tau)$, let $\widetilde{f} \in L^1(K)$ be given by

$$\widetilde{f}(k) := \int_H f_t(k)\delta(t)dt, \tag{3.1}$$

where for each $t \in H$, the function f_t is defined for a.e. k in K via $f_t(k) = f(t,k)$. Then the integral defined in (3.1) converges. In fact, we have

$$\begin{aligned}
\|\widetilde{f}\|_{L^1(K)} &= \int_K |\widetilde{f}(k)|dk \\
&= \int_K \left| \int_H f_t(k)\delta(t)dt \right| dk \\
&= \int_K \left| \int_H f(t,k)\delta(t)dt \right| dk \\
&\leq \int_K \left(\int_H |f(t,k)|\delta(t)dt \right) dk \\
&\leq \int_H \int_K |f(t,k)|\delta(t)dtdk \\
&= \|f\|_{L^1(G_\tau)}.
\end{aligned}$$

For $f, g \in L^1(G_\tau)$, we define the right τ-convolution on $L^1(G_\tau)$ by

$$f \overset{\tau_r}{*} g(h,k) := \int_H f_h * g_t(k)\delta(t)dt. \tag{3.2}$$

where $f_h * g_t$ is the standard convolution on $L^1(K)$. Recall that according to the Fubini-Toneli theorem, for each $f \in L^1(G_\tau)$, we have $f_h \in L^1(K)$ for a.e. h in H. The integral defined in (3.2) converges and also for each $f, g \in L^1(G_\tau)$ we have

$$\|f \overset{\tau_r}{*} g\|_{L^1(G_\tau)} \leq \|f\|_{L^1(G_\tau)} \|g\|_{L^1(G_\tau)}.$$

Indeed, by using Fubini's theorem and Proposition 2.39 of [11] we get

$$\begin{aligned}
\|f \stackrel{\tau_r}{*} g\|_{L^1(G_\tau)} &= \int_H \int_K \left| \int_H f_h * g_t(k)\delta(t)dt \right| \delta(h)dkdh \\
&\leq \int_H \int_K \int_H |f_h * g_t(k)|\delta(h)\delta(t)dtdkdh \\
&\leq \int_H \int_H \left(\int_K |f_h * g_t(k)|dk \right) \delta(h)\delta(t)dtdh \\
&= \int_H \int_H \|f_h * g_t\|_{L^1(K)}\delta(h)\delta(t)dtdh \\
&\leq \int_H \int_H \|f_h\|_{L^1(K)}\|g_t\|_{L^1(K)}\delta(h)\delta(t)dtdh \\
&= \|f\|_{L^1(G_\tau)}\|g\|_{L^1(G_\tau)}.
\end{aligned}$$

The right τ-convolution for a.e $(h,k) \in G_\tau$ satisfies

$$f \stackrel{\tau_r}{*} g(h,k) = f_h * \widetilde{g}(k).$$

This is because, for a.e. $(h,k) \in G_\tau$ we have

$$\begin{aligned}
f \stackrel{\tau_r}{*} g(h,k) &= \int_H f_h * g_t(k)\delta(t)dt \\
&= \int_H \left(\int_K f(h,s)g(t,s^{-1}k)ds \right) \delta(t)dt \\
&= \int_K f(h,s) \left(\int_H g(t,s^{-1}k)\delta(t)dt \right) ds \\
&= f_h * \widetilde{g}(k).
\end{aligned}$$

Similarly the left τ-convolution on $L^1(G_\tau)$ is defined by

$$f \stackrel{\tau_l}{*} g(h,k) := \int_H f_t * g_h(k)\delta(t)dt, \tag{3.3}$$

which analogy for a.e. $(h,k) \in G_\tau$ satisfies

$$f \stackrel{\tau_l}{*} g(h,k) = \widetilde{f} * g_h(k),$$

and also

$$\|f \stackrel{\tau_l}{*} g\|_{L^1(G_\tau)} \leq \|f\|_{L^1(G_\tau)}\|g\|_{L^1(G_\tau)}.$$

Then, the τ-convolution of $f, g \in L^1(G_\tau)$ is defined as

$$f \stackrel{\tau}{*} g = 2^{-1} \left(f \stackrel{\tau_r}{*} g + f \stackrel{\tau_l}{*} g \right). \tag{3.4}$$

Then, readily $f * g \in L^1(G_\tau)$ and we have

$$\begin{aligned}\|f \overset{\tau}{*} g\|_{L^1(G_\tau)} &= 2^{-1}\|f \overset{\tau_r}{*} g + f \overset{\tau_l}{*} g\|_{L^1(G_\tau)} \\ &\leq 2^{-1}\left(\|f \overset{\tau_r}{*} g\|_{L^1(G_\tau)} + \|f \overset{\tau_l}{*} g\|_{L^1(G_\tau)}\right) \\ &\leq \|f\|_{L^1(G_\tau)}\|g\|_{L^1(G_\tau)}.\end{aligned}$$

It should be mentioned that, the τ-convolution of $f, g \in L^1_\tau(G_\tau)$ defined in (3.4), for a.e. $(h,k) \in G_\tau$ can be rewritten in the following form

$$f \overset{\tau}{*} g(h,k) = 2^{-1}\left(f_h * \widetilde{g}(k) + \widetilde{f} * g_h(k)\right). \tag{3.5}$$

The following theorem states that the right τ-convolution makes $L^1(G_\tau)$ into a Banach algebra.

Theorem 3.1. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. The right τ-convolution defined in (3.2) makes $L^1(G_\tau)$ into a Banach algebra.*

Proof. It is enough to prove the associativity of the right τ-convolution. To this end, let $f, g, u \in L^1(G_\tau)$. Then using the associativity of the standard convolution on $L^1(K)$, for a.e. $(h,k) \in G_\tau$, we have

$$\begin{aligned}(f \overset{\tau_r}{*} g) \overset{\tau_r}{*} u(h,k) &= (f \overset{\tau_r}{*} g)_h * \widetilde{u}(k) \\ &= \int_K (f \overset{\tau_r}{*} g)_h(s)\widetilde{u}(s^{-1}k)ds \\ &= \int_K \left(\int_H f_h * g_t(s)\delta(t)dt\right)\widetilde{u}(s^{-1}k)ds \\ &= \int_H \left(\int_K f_h * g_t(s)\widetilde{u}(s^{-1}k)ds\right)\delta(t)dt \\ &= \int_H (f_h * g_t) * \widetilde{u}(k)\delta(t)dt \\ &= \int_H f_h * (g_t * \widetilde{u})(k)\delta(t)dt \\ &= \int_H \left(\int_K f_h(s)(g_t * \widetilde{u})(s^{-1}k)ds\right)\delta(t)dt \\ &= \int_H \left(\int_K f_h(s)(g \overset{\tau_r}{*} u)_t(s^{-1}k)ds\right)\delta(t)dt \\ &= \int_H f_h * (g \overset{\tau_r}{*} u)_t(k)\delta(t)dt \\ &= f \overset{\tau_r}{*} (g \overset{\tau_r}{*} u)(h,k).\end{aligned}$$

□

Next theorem states a similar result for the left τ-convolution.

Theorem 3.2. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. The left τ-convolution defined in (3.3) makes $L^1(G_\tau)$ into a Banach algebra.*

Proof. Similarly, it is enough to show that the left τ-convolution is associative. Let $f, g, u \in L^1(G_\tau)$. Then, for a.e. $(h,k) \in G_\tau$, we have

$$\begin{aligned}(f \overset{\tau_l}{*} g) \overset{\tau_l}{*} u(h,k) &= \int_H (f \overset{\tau_l}{*} g)_t * u_h(k)\delta(t)dt \\ &= \int_H (\widetilde{f} * g_t) * u_h(k)\delta(t)dt \\ &= \int_H \widetilde{f} * (g_t * u_h)(k)\delta(t)dt \\ &= f \overset{\tau_l}{*} (g \overset{\tau_l}{*} u)(h,k).\end{aligned}$$

□

The following proposition guarantees that the τ-convolution is not associative in general.

Proposition 3.3. *The τ-convolution defined in (3.4), for each $f, g, u \in L^1(G_\tau)$, satisfies*

$$(f \overset{\tau}{*} g) \overset{\tau}{*} u - f \overset{\tau}{*} (g \overset{\tau}{*} u) = f \overset{\tau_r}{*} g \overset{\tau_r}{*} u - f \overset{\tau_l}{*} g \overset{\tau_l}{*} u. \tag{3.6}$$

Proof. Let $f, g, u \in L^1(G_\tau)$. Using Theorem 3.1 and Theorem 3.2, left and right τ-convolutions are associative. Thus, it can be easily checked that, for a.e. $(h,k) \in G_\tau$ we have

$$f \overset{\tau_r}{*} (g \overset{\tau_l}{*} u)(h,k) = f \overset{\tau_r}{*} (g \overset{\tau_r}{*} u)(h,k) = (f \overset{\tau_r}{*} g) \overset{\tau_r}{*} u(h,k), \tag{3.7}$$

and

$$(f \overset{\tau_r}{*} g) \overset{\tau_l}{*} u(h,k) = f \overset{\tau_l}{*} (g \overset{\tau_l}{*} u)(h,k) = (f \overset{\tau_l}{*} g) \overset{\tau_l}{*} u(h,k). \tag{3.8}$$

Using definition of the τ-convolution, (3.7) and (3.8) we get

$$\begin{aligned}&(f \overset{\tau}{*} g) \overset{\tau}{*} u \\ &\quad = 2^{-1}\left((f \overset{\tau}{*} g) \overset{\tau_r}{*} u + (f \overset{\tau}{*} g) \overset{\tau_l}{*} u\right) \\ &\quad = 2^{-2}\left((f \overset{\tau_r}{*} g + f \overset{\tau_l}{*} g) \overset{\tau_r}{*} u + (f \overset{\tau_r}{*} g + f \overset{\tau_l}{*} g) \overset{\tau_l}{*} u\right) \\ &\quad = 2^{-2}\left((f \overset{\tau_r}{*} g) \overset{\tau_r}{*} u + (f \overset{\tau_l}{*} g) \overset{\tau_r}{*} u + (f \overset{\tau_r}{*} g) \overset{\tau_l}{*} u + (f \overset{\tau_r}{*} g) \overset{\tau_r}{*} u\right),\end{aligned}$$

and similarly

$$\begin{aligned}
&f \overset{\tau}{*} (g \overset{\tau}{*} u) \\
&= 2^{-1} \Big(f \overset{\tau_r}{*} (g \overset{\tau}{*} u) + f \overset{\tau_l}{*} (g \overset{\tau}{*} u) \Big) \\
&= 2^{-2} \Big(f \overset{\tau_r}{*} (g \overset{\tau_r}{*} u + g \overset{\tau_l}{*} u) + f \overset{\tau_l}{*} (g \overset{\tau_r}{*} u + g \overset{\tau_l}{*} u) \Big) \\
&= 2^{-2} \Big(f \overset{\tau_r}{*} (g \overset{\tau_r}{*} u) + f \overset{\tau_r}{*} (g \overset{\tau_l}{*} u) + f \overset{\tau_l}{*} (g \overset{\tau_r}{*} u) + f \overset{\tau_l}{*} (g \overset{\tau_l}{*} u) \Big).
\end{aligned}$$

Then a straightforward calculation implies (3.6). □

For $f \in L^1(G_\tau)$, let the τ-involution be denoted by f^{*^τ} and defined for a.e. $(h,k) \in G_\tau$ by

$$f^{*^\tau}(h,k) := (f_h)^*(k), \tag{3.9}$$

where $(f_h)^*$ is the standard involution of f_h in $L^1(K)$ given by

$$(f_h)^*(k) = \Delta_K(k^{-1})\overline{f_h(k^{-1})},$$

which satisfies $\|(f_h)^*\|_{L^1(K)} = \|f_h\|_{L^1(K)}$ and $(f_h)^{**} = f_h$. Then, we have $f^{*^\tau} \in L^1(G_\tau)$. More precisely the linear map $^{*^\tau} : L^1(G_\tau) \to L^1(G_\tau)$ is an isometry, because,

$$\begin{aligned}
\|f^{*^\tau}\|_{L^1(G_\tau)} &= \int_H \int_K |f^{*^\tau}(h,k)|\delta(h)dkdh \\
&= \int_H \left(\int_K |f_h^*(k)|dk \right) \delta(h)dh \\
&= \int_H \|f_h\|_{L^1(K)}\delta(h)dh \\
&= \|f\|_{L^1(G_\tau)}.
\end{aligned}$$

Thus, we can prove the following theorem.

Theorem 3.4. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. The Banach function space $L^1(G_\tau)$ is a non-associative Banach $*$-algebra with respect to the τ-convolution defined in (3.4) and the τ-involution defined in (3.9).*

Proof. It remains to show that the τ-involution is an anti-homomorphism. Let $f, g \in L^1(G_\tau)$. Using the anti-homomorphism property of the involution on $L^1(K)$, for a.e. $(h,k) \in G_\tau$ we have

$$\begin{aligned}
(f \overset{\tau}{*} g)^{*_\tau}(h,k) &= (f \overset{\tau_r}{*} g)^{*_\tau}(h,k) + (f \overset{\tau_l}{*} g)^{*_\tau}(h,k) \\
&= \{(f \overset{\tau_r}{*} g)_h\}^*(k) + \{(f \overset{\tau_l}{*} g)_h\}^*(k) \\
&= (f_h * \widetilde{g})^*(k) + (\widetilde{f} * g_h)^*(k) \\
&= (\widetilde{g})^* * f_h^*(k) + g_h^* * (\widetilde{f})^*(k) \\
&= g^{*_\tau} \overset{\tau}{*} f^{*_\tau}(h,k).
\end{aligned}$$

□

Remark 3.5. From now on, the notation $L^1_\tau(G_\tau)$ stands for the non-associative Banach ∗-algebra mentioned in Theorem 3.4. We also use the notations $L^1_{\tau_l}(G_\tau)$ (resp. $L^1_{\tau_r}(G_\tau)$) for the Banach algebra mentioned in Theorem 3.1 (resp. Theorem 3.2).

In the following, we introduce a canonical tool which transfers elements of $L^1(K)$ into the elements of $L^1(G_\tau)$.

For $\Phi \in L^1(G_\tau)$ and $\psi \in L^1(K)$, let

$$\Phi(\psi)(h,k) := \psi(k) \int_K \Phi(h,s)ds. \tag{3.10}$$

Then $\Phi(\psi) \in L^1_\tau(G_\tau)$ and

$$\|\Phi(\psi)\|_{L^1_\tau(G_\tau)} \le \|\Phi\|_{L^1_\tau(G_\tau)} \|\psi\|_{L^1(K)}.$$

Indeed, we have

$$\begin{aligned}
\int_H \int_K |\Phi(\psi)(h,k)|\delta(h)dkdh &= \int_H \int_K \left|\psi(k) \int_K \Phi(h,s)ds\right| dk\delta(h)dh \\
&\le \int_H \int_K \int_K |\psi(k)||\Phi(h,s)|dsdk\delta(h)dh \\
&= \left(\int_K |\psi(k)|dk\right)\left(\int_H \int_K |\Phi(h,s)|ds\delta(h)dh\right) \\
&= \|\Phi\|_{L^1_\tau(G_\tau)} \|\psi\|_{L^1(K)}.
\end{aligned}$$

Let L^+_1 be the set of all nonnegative Φ in $L^1(G_\tau)$ with $\|\Phi\|_{L^1(G_\tau)} = 1$.

Proposition 3.6. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. Then, for each $\Phi \in L^+_1$, $f \in L^1(G_\tau)$ and $\psi \in L^1(K)$ we have*

(i) $\widetilde{f} * \psi(k) = \int_H f_t * \psi(k)\delta(t)dt,$

(ii) $\psi * \widetilde{f}(k) = \int_H \psi * f_t(k)\delta(t)dt,$

(iii) $\Phi(\psi) \overset{\tau}{*} f(h,k) = 2^{-1}\left(\|\Phi_h\|_{L^1(K)}\psi * \widetilde{f}(k) + \psi * f_h(k)\right),$

(iv) $f \overset{\tau}{*} \Phi(\psi)(h,k) = 2^{-1}\left(f_h * \psi(k) + \|\Phi_h\|_{L^1(K)}\widetilde{f} * \psi(k)\right).$

Proof. (i) Let $f \in L^1(G_\tau)$ and $\psi \in L^1(K)$. Then, for a.e. $k \in K$, we have

$$\begin{aligned}\widetilde{f} * \psi(k) &= \int_K \widetilde{f}(s)\psi(s^{-1}k)ds \\ &= \int_K \left(\int_H f(t,s)\delta(t)dt\right)\psi(s^{-1}k)ds \\ &= \int_H \left(\int_K f(t,s)\psi(s^{-1}k)ds\right)\delta(t)dt \\ &= \int_H f_t * \psi(k)\delta(t)dt.\end{aligned}$$

(ii) Similar to (i), for a.e $k \in K$, we have

$$\psi * \widetilde{f}(k) = \int_H \psi * f_t(k)\delta(t)dt.$$

(iii) Let $\Phi, f \in L^1(G_\tau)$ and $\psi \in L^1(K)$. Using (3.5), for a.e. $(h,k) \in G_\tau$, we have

$$\begin{aligned}\Phi(\psi) \overset{\tau}{*} f(h,k) &= 2^{-1}\left((\Phi(\psi))_h * \widetilde{f} + \widetilde{\Phi(\psi)} * f_h(k)\right) \\ &= 2^{-1}\left(\psi * \widetilde{f}(k)\|\Phi_h\|_{L^1(K)} + \psi * f_h(k)\right).\end{aligned}$$

(iv) Similar to (iii), for a.e $(h,k) \in G_\tau$, we have

$$f \overset{\tau}{*} \Phi(\psi)(h,k) = 2^{-1}\left(\widetilde{f} * \psi(k)\|\Phi_h\|_{L^1(K)} + f_h * \psi(k)\right).$$

□

The next theorem shows that each $\Phi \in L_1^+$ determines a $*$-homomorphism from $L^1(K)$ into $L_\tau^1(G_\tau)$.

Theorem 3.7. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism, $G_\tau = H \ltimes_\tau K$, and $\Phi \in L_1^+$. The linear map $\lambda_\Phi : L^1(K) \to L^1(G_\tau)$ defined by $\psi \mapsto \lambda_\Phi(\psi) := \Phi(\psi)$ is an isometric $*$-homomorphism.*

Proof. According to (3.4) and (3.10), if $\Phi \in L_1^+$ then for each $\psi \in L^1(K)$ we have

$$\begin{aligned}\|\Phi(\psi)\|_{L^1(G_\tau)} &= \|\Phi\|_{L^1(G_\tau)}\|\psi\|_{L^1(K)} \\ &= \|\psi\|_{L^1(K)}.\end{aligned}$$

Let $\Phi \in L_1^+$ and also $\psi, \phi \in L^1(K)$. Then, for a.e. $(h,k) \in G_\tau$, we have

$$\Phi(\psi) \overset{\tau}{*} \Phi(\phi)(h,k) = \psi * \phi(k)\|\Phi_h\|_{L^1(K)}, \tag{3.11}$$

because

$$\begin{aligned}
&\Phi(\psi) \overset{\tau_r}{*} \Phi(\phi)(h,k) \\
&= \int_H \Phi(\psi)_h * \Phi(\phi)_t(k)\delta(t)dt \\
&= \int_H \left(\int_K \Phi(\psi)(h,s)\Phi(\phi)(t,s^{-1}k)ds\right)\delta(t)dt \\
&= \int_H \left(\int_K \psi(s)\phi(s^{-1}k)\left(\int_K \Phi(h,s')ds'\right)\left(\int_K \Phi(t,s'')ds''\right)ds\right)\delta(t)dt \\
&= \int_H \left(\int_K \Phi(h,s')ds'\right)\left(\int_K \Phi(t,s'')ds''\right)\left(\int_K \psi(s)\phi(s^{-1}k)ds\right)\delta(t)dt \\
&= \psi * \phi(k)\|\Phi_h\|_{L^1(K)} \int_H \left(\int_K \Phi(t,s'')ds''\right)\delta(t)dt \\
&= \psi * \phi(k)\|\Phi_h\|_{L^1(K)}.
\end{aligned}$$

Similarly, for a.e. $(h,k) \in G_\tau$, we have

$$\Phi(\psi) \overset{\tau_l}{*} \Phi(\phi)(h,k) = \psi * \phi(k)\|\Phi_h\|_{L^1(K)},$$

which implies (3.11). Now (3.11) implies that, for each $\psi, \phi \in L^1(K)$ and also for a.e. $(h,k) \in G_\tau$, we have

$$\begin{aligned}
\Phi(\psi * \phi)(h,k) &= \psi * \phi(k) \int_K \Phi(h,s)ds \\
&= \Phi(\psi) \overset{\tau}{*} \Phi(\phi)(h,k).
\end{aligned}$$

If $\psi \in L^1(K)$, then for $(h,k) \in G_\tau$ we have

$$\begin{aligned}
\Phi(\psi^*)(h,k) &= \psi^*(k) \int_K \Phi(h,s)ds \\
&= \left(\Phi(\psi)_h\right)^*(k) \\
&= \Phi(\psi)^{*_\tau}(h,k).
\end{aligned}$$

□

The next theorem shows that $f \mapsto \widetilde{f}$ is a norm decreasing ∗-homomorphism from $L^1_\tau(G_\tau)$ onto $L^1(K)$.

Theorem 3.8. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. The map defined by $f \mapsto S^\tau_K(f) = \widetilde{f}$ is a norm decreasing ∗-homomorphism from $L^1_\tau(G_\tau)$ onto $L^1(K)$.*

Proof. Let $f, g \in L^1_\tau(G_\tau)$. Using Fubini's theorem, for a.e. k in K, we have

$$\begin{aligned}
\widetilde{f \overset{\tau_r}{*} g}(k) &= \int_H f *_\tau g(h,k)\delta(h)dh \\
&= \int_H \left(\int_H f_h * g_t(k)\delta(t)dt\right)\delta(h)dh \\
&= \int_H \int_H \left(\int_K f(h,s)g(t,s^{-1}k)ds\right)\delta(h)\delta(t)dtdh \\
&= \int_K \left(\int_H f(h,s)\delta(h)dh\right)\left(\int_H g(t,s^{-1}k)\delta(t)dt\right)ds \\
&= \int_K \widetilde{f}(s)\widetilde{g}(s^{-1}k)ds \\
&= \widetilde{f} * \widetilde{g}(k).
\end{aligned}$$

Similarly, for a.e. $k \in K$, we have

$$\widetilde{f \overset{\tau_l}{*} g}(k) = \widetilde{f} * \widetilde{g}(k).$$

Thus, we have

$$\widetilde{f \overset{\tau}{*} g} = 2^{-1}\left(\widetilde{f \overset{\tau_r}{*} g} + \widetilde{f \overset{\tau_l}{*} g}\right) = \widetilde{f} * \widetilde{g}.$$

Also, for a.e. $k \in K$, we have

$$\begin{aligned}
\widetilde{f^{*_\tau}}(k) &= \int_H f^{*_\tau}(t,k)\delta(t)dt \\
&= \int_H f_t^*(k)\delta(t)dt \\
&= \Delta_K(k^{-1})\int_H \overline{f_t(k^{-1})}\delta(t)dt \\
&= \left(\widetilde{f}\right)^*(k).
\end{aligned}$$

Note that since $\|\widetilde{f}\|_{L^1(K)} \leq \|f\|_{L^1(G_\tau)}$, we deduce that the linear map $f \mapsto \widetilde{f}$ is norm-decreasing. Now let $\Phi \in L_1^+$ and $\psi \in L^1(K)$ be arbitrary.

Then, for a.e. $k \in K$, we have

$$\begin{aligned}\widetilde{\Phi(\psi)}(k) &= \int_H \Phi(\psi)(t,k)\delta(t)dt \\ &= \int_H \psi(k)\left(\int_K \Phi(t,s)ds\right)\delta(t)dt \\ &= \psi(k)\int_H\int_K \Phi(t,s)ds\delta(t)dt \\ &= \psi(k).\end{aligned}$$

Thus, we get that $\widetilde{L^1_\tau(G_\tau)} = \{\widetilde{f} : f \in L^1_\tau(G_\tau)\} = L^1(K)$. □

Then we state the following corollary. It should be mentioned that the same result holds for $L^1_{\tau_l}(G_\tau)$.

Corollary 3.9. *The map $f \mapsto S^\tau_K(f) = \widetilde{f}$ is a norm decreasing $*$-homomorphism from $L^1_{\tau_r}(G_\tau)$ onto $L^1(K)$.*

The following corollary shows that, if $L^1_\tau(G_\tau)$ has an identity, then K should be discrete.

Proposition 3.10. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. If the Banach algebra $L^1_{\tau_r}(G_\tau)$ or $L^1_{\tau_l}(G_\tau)$ has identity, then K is discrete.*

Proof. It is enough to prove the result for the left τ-convolution. Let ϵ_{τ_l} be an identity for the left τ-convolution. Using Theorem 3.8, for each $f \in L^1_{\tau_l}(G_\tau)$ and also for a.e. $k \in K$, we have

$$\begin{aligned}\widetilde{f} * \widetilde{e_\tau}(k) &= \widetilde{f * (e_\tau)}(k) \\ &= \int_H f \overset{\tau}{*} e_\tau(h,k)\delta(h)dh \\ &= \int_H f(h,k)\delta(h)dh \\ &= \widetilde{f}(k).\end{aligned}$$

Since the linear map $S^\tau_K : L^1_\tau(G_\tau) \to L^1(K)$ is surjective, we deduce that $\widetilde{e_\tau}$ is an identity for $L^1(K)$. Thus, Theorem 19.19 and Theorem 19.20 of [22], imply that K is discrete. □

Corollary 3.11. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. If τ-convolution has identity, then K is discrete.*

Next proposition guarantees that the linear map

$$S^\tau_K : L^1_\tau(G_\tau) \to L^1(K),$$

given by $S^\tau_K(f) = \widetilde{f}$ is not injective in general.

Proposition 3.12. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. The linear map $S_K^\tau : L^1_\tau(G_\tau) \to L^1(K)$ is injective if and only if H be the trivial group.*

Proof. If H be the trivial group then clearly $\widetilde{f} = 0$ implies $f = 0$ and so the linear map S_K^τ is injective. To prove the converse, note that for each nonnegative function $\varphi \in L^1(H)$ with $\|\varphi\|_{L^1(H)} = 1$ and for each $\psi \in L^1(K)$, we can define $\psi_\varphi(h,k) := \delta(h^{-1})\varphi(h)\psi(k)$ for a.e $(h,k) \in G_\tau$. Then ψ_φ belongs to $L^1_\tau(G_\tau)$ and also, for a.e. $k \in K$ we have $\widetilde{\psi_\varphi}(k) = \psi(k)$. Now if S_K^τ is injective and also H is not the trivial group, then for a fixed $\psi \in L^1(K)$, we have

$$\begin{aligned}\{0\} &\subset \{\psi_\varphi - \psi_{\varphi'} : \varphi, \varphi' \in \mathcal{C}_c^+(H), \|\varphi\|_{L^1(H)} = \|\varphi'\|_{L^1(H)} = 1, \varphi \neq \varphi'\}\\ &\subseteq \mathcal{J}^1_\tau,\end{aligned}$$

which contradicts injectivity of the linear map S_K^τ. □

As an application of Proposition 3.12, we can prove that $L^1_\tau(G_\tau)$ is an associative Banach $*$-algebra if and only if H be the trivial group.

Corollary 3.13. *Let K be second countable and $\tau : H \to Aut(K)$ be a continuous homomorphism and let $G_\tau = H \ltimes_\tau K$. The τ-convolution defined in (3.4) is associative if and only if H be the trivial group.*

Proof. Obviously, when H is the trivial group, the τ-convolution is associative. Conversely, suppose the τ-convolution is associative and $\widetilde{f} = 0$. Let $\{\psi_n\}_{n=1}^\infty$ be a sequence approximate identity for $L^1(K)$ according to Proposition 2.42 of [11], and $\Phi \in L_1^+$. Using associativity of the τ-convolution for a.e. $(h,k) \in G_\tau$ we have

$$\begin{aligned}f(h,k) &= \lim_n f_h * \psi_n(k)\\ &= \lim_n \lim_m f_h * (\psi_n * \psi_m)(k)\\ &= \lim_n \lim_m f_h * \left(\widetilde{\Phi(\psi_n)} * \widetilde{\Phi(\psi_m)}\right)(k)\\ &= \lim_n \lim_m f \overset{\tau_r}{*} \Phi(\psi_n) \overset{\tau_r}{*} \Phi(\psi_m)(h,k)\\ &= \lim_n \lim_m f \overset{\tau_l}{*} \Phi(\psi_n) \overset{\tau_l}{*} \Phi(\psi_m)(h,k)\\ &= \lim_n \lim_m \widetilde{f} * \widetilde{\Phi(\psi_n)} * (\psi_m)_h(k)\\ &= 0.\end{aligned}$$

Now Proposition 3.12 guarantees that H is the trivial group. □

Next corollary shows that when K is second countable, the right τ-convolution is commutative if and only if K is abelian and H is the trivial group.

Corollary 3.14. *Let K be second countable and $\tau : H \to Aut(K)$ be a continuous homomorphism and let $G_\tau = H \ltimes_\tau K$. The right τ-convolution is commutative if and only if K be abelian and H be the trivial group.*

Proof. Clearly when H is the trivial group the right τ-convolution coincides with the standard convolution on $L^1(K)$ and so if K is abelian, the left τ-convolution is commutative. Now let right τ-convolution be commutative and $f, g \in L^1_{\tau_r}(G_\tau)$. Using Proposition 3.8 for a.e. $k \in K$ we have

$$\begin{aligned}\widetilde{f} * \widetilde{g}(k) &= \widetilde{f \overset{\tau_r}{*} g}(k)\\ &= \widetilde{g \overset{\tau_r}{*} f}(k)\\ &= \widetilde{g} * \widetilde{f}(k).\end{aligned}$$

Now since $\{\widetilde{f} : f \in L^1_{\tau_r}(G_\tau)\} = L^1(K)$, we get that $L^1(K)$ is commutative and so, K is abelian. Also, since the right τ-convolution is commutative, using (3.5) for each $f, g \in L^1_{\tau_r}(G_\tau)$ and a.e. $(h, k) \in G_\tau$ we have

$$f_h * \widetilde{g}(k) = g_h * \widetilde{f}(k).$$

To show that H is the trivial group, assume that $\{\psi_n\}_{n=1}^\infty$ be an approximate identity for $L^1(K)$ and also $\widetilde{f} = 0$. Then if $\Phi \in L_1^+$ for a.e. $(h, k) \in G_\tau$ we have

$$\begin{aligned}f(h, k) &= \lim_n f_h * \psi_n(k)\\ &= \lim_n f_h * \widetilde{\Phi(\psi_n)}(k)\\ &= \lim_n \Phi(\psi_n)_h * \widetilde{f}(k)\\ &= 0.\end{aligned}$$

□

The same result for the left τ-convolution can be obtained by the similar argument.

In the next theorem we show that the τ-convolution is commutative whenever K is abelian and also, we prove that the converse is true.

Theorem 3.15. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. The τ-convolution is commutative if and only if K is abelian.*

Proof. When the τ-convolution is commutative, similar method as we used in Corollary 3.14, works and implies that K is abelian. Now let K be an abelian group. Thus, the Banach $*$-algebra $L^1(K)$ is commutative and so according to the definition of the τ-convolution, for each $f, g \in L^1_\tau(G_\tau)$ and for a.e. $(h,k) \in G_\tau$ we have

$$\begin{aligned} f \overset{\tau}{*} g(h,k) &= 2^{-1}\int_H (f_h * g_t(k) + f_t * g_h(k))\delta(t)dt \\ &= 2^{-1}\int_H (g_t * f_h(k) + g_h * f_t(k))\delta(t)dt \\ &= g \overset{\tau}{*} f(h,k). \end{aligned}$$

□

As an immediate consequence of Theorem 3.15, we show that when K is abelian, the τ-convolution and τ-involution make $L^1(G_\tau)$ into a Jordan Banach $*$-algebra.

Corollary 3.16. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$ with K abelian. The Banach function space $L^1(G_\tau)$ with respect to the τ-convolution and τ-involution is a Jordan Banach $*$-algebra.*

Proof. If K is abelian, Theorem 3.15 guarantees that τ-convolution is commutative. According to Theorem 3.4, it is sufficient to show that the τ-convolution satisfies the Jordan identity. Now let $f, g \in L^1(G_\tau)$. Using Proposition 3.3 and also, commutativity and associativity of the standard convolution on $L^1(K)$ for a.e $(h,k) \in G_\tau$, we have

$$\begin{aligned} &(f \overset{\tau}{*} g) \overset{\tau}{*} (f \overset{\tau}{*} f)(h,k) - f \overset{\tau}{*} \left(g \overset{\tau}{*} (f \overset{\tau}{*} f)\right)(h,k) \\ &= (f \overset{\tau_r}{*} g) \overset{\tau_r}{*} (f \overset{\tau_r}{*} f)(h,k) - f \overset{\tau_l}{*} \left(g \overset{\tau_l}{*} (f \overset{\tau_l}{*} f)\right)(h,k) \\ &= (f \overset{\tau_r}{*} g)_h * \widetilde{(f \overset{\tau_r}{*} f)}(k) - \widetilde{f} * \left(g \overset{\tau_l}{*} (f \overset{\tau_l}{*} f)\right)_h(k) \\ &= (f \overset{\tau_r}{*} g)_h * \left(\widetilde{f} * \widetilde{f}\right)(k) - \widetilde{f} * \left(\widetilde{g} * (f \overset{\tau_l}{*} f)_h\right)(k) \\ &= (f_h * \widetilde{g}) * \left(\widetilde{f} * \widetilde{f}\right)(k) - \widetilde{f} * \left(\widetilde{g} * (\widetilde{f} * f_h)\right)(k) \\ &= 0. \end{aligned}$$

□

A sequence $\{u_n\}_{n=1}^\infty$ in $L^1(G_\tau)$ is called a *τ-sequence approximate identity* for $L^1(G_\tau)$, if for each $f \in L^1(G_\tau)$ we have

$$\lim_n \|u_n \overset{\tau}{*} f - f\|_{L^1(G_\tau)} = \lim_n \|f \overset{\tau}{*} u_n - f\|_{L^1(G_\tau)} = 0. \tag{3.12}$$

The next theorem shows that the non-associative Banach ∗-algebra $L^1_\tau(G_\tau)$ has a τ-sequence approximate identity if and only if H be the trivial group.

Theorem 3.17. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$. The τ-convolution admits a τ-sequence approximate identity if and only if H be the trivial group.*

Proof. If H is the trivial group $\{e\}$, then for each continuous homomorphism $\tau : H \to Aut(K)$ we have

$$G_\tau = \{e\} \ltimes_\tau K = K.$$

which implies $L^1(G_\tau) = L^1(K)$. Thus the τ-convolution coincides with the standard convolution of $L^1(K)$ and so, any standard approximate identity for $L^1(K)$ is a τ-sequence approximate identity for $L^1(G_\tau)$. Conversely, assume that the τ-convolution admits a τ-sequence approximate identity $\{u_n\}_{n=1}^\infty$. Let $f \in \mathcal{J}^1_\tau$, we show that $f(h,k) = 0$ for a.e. $(h,k) \in G_\tau$. By Theorem 3.8, $\{\widetilde{u_n}\}_{n=1}^\infty$ is a sequence of approximate identity for $L^1(K)$. Using (3.5), for a.e. $(h,k) \in G_\tau$ we have

$$\begin{aligned} f(h,k) &= \lim_n f \overset{\tau}{*} u_n(h,k) \\ &= 2^{-1} \lim_n \Big(f_h * \widetilde{u_n}(k) + \widetilde{f} * (u_n)_h(k) \Big) \\ &= 2^{-1} \lim_n f_h * \widetilde{u_n}(k) \\ &= 2^{-1} f(h,k). \end{aligned}$$

Thus we get $f(h,k) = 0$ for a.e. $(h,k) \in G_\tau$ and so, we have $\mathcal{J}^1_\tau = \{0\}$. Now, Proposition 3.12 implies that H is the trivial group. □

As an immediate consequence of Theorem 3.17, we deduce that, when H and K are second countable locally compact groups, the τ-convolution coincides with the standard convolution of $L^1(G_\tau)$ if and only if H be the trivial group.

Corollary 3.18. *Let H and K be second countable locally compact groups, $\tau : H \to Aut(K)$ be a continuous homomorphism and let $G_\tau = H \ltimes_\tau K$. The τ-convolution defined in (3.4) coincides with the standard convolution of $L^1(G_\tau)$ if and only if H be the trivial group.*

Proof. Clearly if H is the trivial group the τ-convolution coincides with the standard convolution of $L^1(G_\tau)$. Conversely, if the τ-convolution coincides with the standard convolution of $L^1(G_\tau)$ for each $f, g \in L^1(G_\tau)$, we have $f \overset{\tau}{*} g = f * g$. Since the underlying topological space of G_τ is second countable, the standard convolution on $L^1(G_\tau)$ possess a sequence approximate identity, thus the τ-convolution admits a τ-sequence approximate identity and using Theorem 3.17, the result holds. □

4. Banach $L^1_{\tau_l}(G_\tau)$-module structure over $L^p(G_\tau)$

The left τ-convolution defined in (3.3) can be extended from $L^1(G_\tau)$ to other $L^p(G_\tau)$ spaces with $1 \leq p \leq \infty$. In this section we make $L^p(G_\tau)$ into a left Banach $L^1(G_\tau)$-module. First we define a module action.

Let the left module action $\overset{\tau_l}{*}_{(p)}$: $L^1_{\tau_l}(G_\tau) \times L^p(G_\tau) \to L^p(G_\tau)$ defined via $(f,u) \mapsto f \overset{\tau_l}{*}_{(p)} u$ where $f \overset{\tau_l}{*}_{(p)} u$ is given by

$$f \overset{\tau_l}{*}_{(p)} u(h,k) := \int_H f_t * u_h(k)\delta(t)dt. \tag{4.1}$$

The left module action defined in (4.1), for a.e. $(h,k) \in G_\tau$ can be written in the form

$$f \overset{\tau}{*}_{(p)} u(h,k) = \widetilde{f} * u_h(k). \tag{4.2}$$

Because using Fubini's theorem for a.e. $(h,k) \in G_\tau$, we have

$$\begin{aligned} f \overset{\tau}{*}_{(p)} u(h,k) &= \int_H f_t * u_h(k)\delta(t)dt \\ &= \int_H \left(\int_K f(t,s)u(h,s^{-1}k)ds\right)\delta(h)dh \\ &= \int_K \left(\int_H f(t,s)\delta(t)dt\right)u(h,s^{-1})kds = \widetilde{f} * u_h(k). \end{aligned}$$

The module action defined in (4.1) is converges and also belongs to $L^p(G_\tau)$. Because using Fubini's theorem and also Proposition 2.39 of [11], we have

$$\begin{aligned} \|f \overset{\tau_l}{*}_{(p)} u\|^p_{L^p(G_\tau)} &= \int_K \int_H |f \overset{\tau_l}{*}_{(p)} u(h,k)|^p \delta(h)dhdk \\ &= \int_K \int_H |\widetilde{f} * u_h(k)|^p \delta(h)dhdk \\ &= \int_H \left(\int_K |\widetilde{f} * u_h(k)|^p dk\right)\delta(h)dh \\ &= \int_H \|\widetilde{f} * u_h\|^p_{L^1(K)}\delta(h)dh \\ &\leq \|\widetilde{f}\|^p_{L^1(K)} \int_H \|u_h\|^p_{L^p(K)}\delta(h)dh \\ &= \|f\|^p_{L^1_{\tau_l}(G_\tau)}\|u\|^p_{L^p(G_\tau)}. \end{aligned}$$

Remark 4.1. When $p = 1$, the left module action defined in (4.1) coincides with the left τ-convolution defined in (3.3). If H is the trivial group, the left module action defined in (4.1) coincides with the left module action defined on $L^p(K)$ via Proposition 2.39 of [11].

The next theorem shows that the left module action defined in (4.1) is associative and makes $L^p(G_\tau)$ into a left Banach $L^1_{\tau_l}(G_\tau)$-module.

Theorem 4.2. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism, $G_\tau = H \ltimes_\tau K$ and $1 \le p \le \infty$. The Banach space $L^p(G_\tau)$ with respect to the left module action $\overset{\tau}{*}_{(p)}$ defined in (4.1) is a left Banach $L^1_{\tau_l}(G_\tau)$-module.*

Proof. It is sufficient to show that the left module action (4.1) is associative. Let $f, g \in L^1_{\tau_l}(G_\tau)$ and $u \in L^p(G_\tau)$. Using Theorem 3.8 and also (4.2), for a.e. $(h,k) \in G_\tau$ we have

$$\begin{aligned}\left(f \overset{\tau_l}{*} g\right) \overset{\tau}{*}_{(p)} u(h,k) &= \widetilde{(f \overset{\tau_l}{*} g)} * u_h(k)\\ &= (\widetilde{f} * \widetilde{g}) * u_h(k)\\ &= \widetilde{f} * (\widetilde{g} * u_h)(k)\\ &= \widetilde{f} * (g \overset{\tau_l}{*}_{(p)} u)_h(k)\\ &= f \overset{\tau_l}{*}_{(p)} \left(g \overset{\tau_l}{*}_{(p)} u\right)(h,k).\end{aligned}$$

□

A sequence $\{f_n\}_{n=1}^\infty$ in $L^1_{\tau_l}(G_\tau)$ is called a left τ_l-sequence approximate identity, if for each $p \ge 1$ and $u \in L^p(G_\tau)$ satisfies

$$\lim_n \|f_n \overset{\tau_l}{*} u - u\|_{L^p(G_\tau)} = 0. \tag{4.3}$$

In the following theorem we show that when K is a second countable locally compact group, the left Banach $L^1_{\tau_l}(G_\tau)$-module $L^p(G_\tau)$ admits a left τ_l-sequence approximate identity.

Theorem 4.3. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$, with K second countable. The Banach algebra $L^1_{\tau_l}(G_\tau)$ possesses a left τ_l-sequence approximate identity.*

Proof. Let $\{\psi_n\}_{n=1}^\infty$ be an approximate identity for $L^1(K)$ according to Proposition 2.42 of [11]. Thus for each $p \ge 1$ and $\varphi \in L^p(K)$, we have

$$\lim_n \|\psi_n * \varphi - \varphi\|_{L^p(G_\tau)} = 0. \tag{4.4}$$

Let $\Phi \in L_1^+$ and for all n put $f_n(h,k) := \Phi(\psi_n)(h,k)$, for a.e. $(h,k) \in G_\tau$. We show that $\{f_n\}_{n=1}^\infty$ is a left τ_l-sequence approximate identity. Let $p \ge 1$ and $u \in L^p(G_\tau)$. Using (4.1), (4.4), Theorem 3.8 and The

Dominated Convergence Theorem, we achieve

$$\begin{aligned}
&\lim_n \|f_n \overset{\tau_l}{*} u - u\|^p_{L^p(G_\tau)} \\
&\qquad = \lim_n \int_H \left(\int_K |f_n \overset{\tau_l}{*} u(h,k) - u(h,k)|^p dk \right) \delta(h) dh \\
&\qquad = \lim_n \int_H \left(\int_K |\Phi(\psi_n) \overset{\tau_l}{*} u(h,k) - u(h,k)|^p dk \right) \delta(h) dh \\
&\qquad = \lim_n \int_H \left(\int_K |\widetilde{\Phi(\psi_n)} * u_h(k) - u_h(k)|^p dk \right) \delta(h) dh \\
&\qquad = \lim_n \int_H \left(\int_K |\psi_n * u_h(k) - u_h(k)|^p dk \right) \delta(h) dh \\
&\qquad = \lim_n \int_H \left(\|\psi_n * u_h - u_h\|^p_{L^p(K)} \right) \delta(h) dh \\
&\qquad = \int_H \left(\lim_n \|\psi_n * u_h - u_h\|^p_{L^p(K)} \right) \delta(h) dh \\
&\qquad = 0.
\end{aligned}$$

□

As an immediate consequence of Theorem 4.3 we deduce the following corollary.

Corollary 4.4. *Let $\tau : H \to Aut(K)$ be a continuous homomorphism and $G_\tau = H \ltimes_\tau K$, with K second countable. The Banach algebra $L^1_{\tau_l}(G_\tau)$ possesses a τ_l-sequence approximate identity.*

Acknowledgments. The authors would like to thank the referees for their valuable comments and remarks. The first author would like to express his deepest gratitude to Prof. Hans G. Feichtinger for reading this article carefully and also his valuable comments. Thanks are also due to Prof. Massoud Amini for stimulating discussions and pointing out various references.

References

1. A.A. Arefijamaal, *The continuous Zak transform and generalized Gabor frames*, Mediterr. J. Math. 10 (2013), No. 1, 353-365.
2. A.A. Arefijamaal and A. Ghaani Farashahi, *Zak transform for semidirect product of locally compact groups*, Anal. Math. Phys. 3 (2013), No. 3, 263-276.
3. A.A. Arefijamaal and R.A. Kamyabi-Gol, *On the square integrability of quasi regular representation on semidirect product groups,* J. Geom. Anal. 19 (2009), No. 3, 541-552.
4. A.A. Arefijamaal and R.A. Kamyabi-Gol, *On construction of coherent states associated with semidirect products*, Int. J. Wavelets Multiresolut. Inf. Process. 6 (2008), No. 5, 749-759.

5. W.R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter Studies in Mathematics, 20, Walter de Gruyter (1995).
6. G. Chirikjian and A. Kyatkin, *Engineering Applications of Noncommutative Harmonic Analysis With Emphasis on Rotation and Motion Groups*, Boca Raton, FL: CRC Press. xxii, 2001.
7. A. Derighetti, *Convolution operators on groups*, Lecture Notes of the Unione Matematica Italiana, 11. Springer, Heidelberg; UMI, Bologna, 2011. xii+171 pp. ISBN: 978-3-642-20655-9.
8. J. Dixmier, *C^*-Algebras*, North-Holland and Publishing company, 1977.
9. J. Fell and R. Doran, *Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles*, Pure and Applied Mathematics, Vol. 1, Academic Press, 1998.
10. J. Fell and R. Doran, *Representations of $*$-Algebras, Locally Compact Groups, and Banach $*$-Algebraic Bundles*, Pure and Applied Mathematics, Vol. 2, Academic Press, 1998.
11. G.B. Folland, *A course in Abstract Harmonic Analysis*, CRC press, 1995.
12. A. Ghaani Farashahi, *Continuous partial Gabor transform for semi-direct product of locally compact groups*, Bull. Malays. Math. Sci. Soc. 38 (2015), No. 2, 779-803.
13. A. Ghaani Farashahi, *A unified group theoretical method for the partial Fourier analysis on semi-direct product of locally compact groups*, Results Math. 67 (2015), No. 1-2, 235-251.
14. A. Ghaani Farashahi, *Cyclic wave packet transform on finite Abelian groups of prime order*, Int. J. Wavelets Multiresolut. Inf. Process. 12 (2014), No. 6, 1450041, 14 pp.
15. A. Ghaani Farashahi, *Generalized Weyl-Heisenberg (GWH) groups*, Anal. Math. Phys. 4 (2014), No. 3, 187-197.
16. A. Ghaani Farashahi, *Convolution and involution on function spaces of homogeneous spaces*, Bull. Malays. Math. Sci. Soc., (2) 36 (2013), No. 4, 1109-1122.
17. A. Ghaani Farashahi, *Abstract Non-Commutative Harmonic Analysis of Coherent State Transforms*, Ferdowsi University of Mashhad (FUM) (2012) PhD Thesis.
18. A. Ghaani Farashahi and M. Mohammad-Pour, *A unified theoretical harmonic analysis approach to the cyclic wavelet transform (CWT) for periodic signals of prime dimensions*, Sahand Commun. Math. Anal. Vol. 1, No. 2, 1-17 (2014).
19. A. Ghaani Farashahi and R.A. Kamyabi-Gol, *Frames and homogeneous spaces*, J. Sci. Islam. Repub. Iran., 22 (2011), No. 4, 355-361, 372.
20. S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, San Francisco, London, 1978.
21. E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis*, Vol 2, 1970.
22. E. Hewitt and K.A. Ross, *Absrtact Harmonic Analysis*, Vol 1, 1963.
23. G. Hochschild, *The Structure of Lie Groups*, Hpolden-day, San Francisco, 1965.
24. R.I. Jewett, *Spaces with an abstract convolution of measures*, Advances in Math., 18 (1975), 1-101.
25. R.A. Kamyabi-Gol and N. Tavallaei, *Wavelet transforms via generalized quasi-regular representations*, Appl. Comput. Harmon. Anal., 26 (2009), No. 3, 291-300.
26. V. Kisil, *Calculus of operators: covariant transform and relative convolutions*, Banach J. Math. Anal. 8 (2014), No. 2, 156-184.

27. V. Kisil, *Geometry of Möbius transformations. Elliptic, parabolic and hyperbolic actions of $SL_2(\mathbb{R})$*, Imperial College Press, London, 2012.
28. V. Kisil, *Relative convolutions. I. Properties and applications*, Adv. Math. 147 (1999), No. 1, 35-73.
29. V. Kisil, *Connection between two-sided and one-sided convolution type operators on non-commutative groups*, Integral Equations Operator Theory 22 (1995), No. 3, 317-332.
30. H. Reiter and J.D. Stegeman, *Classical Harmonic Analysis*, 2nd Ed, Oxford University Press, New York, 2000.

---

[1] Numerical Harmonic Analysis Group (NuHAG), Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, A-1090 Wien, Vienna, Austria.

E-mail address: arash.ghaani.farashahi@univie.ac.at
E-mail address: ghaanifarashahi@hotmail.com

[2] Department of Pure Mathematics, Ferdowsi University of Mashhad, Center of Excellence in Analysis on Algebraic Structures (CEAAS), P. O. Box 1159-91775, Mashhad, Iran.

E-mail address: kamyabi@ferdowsi.um.ac.ir