Sahand Communications in Mathematical Analysis (SCMA) Vol. 2 No. 2 (2015), 45-59
http://scma.maragheh.ac.ir

GENERALIZED CONCEPT OF J-BASIS

TOFIG NAJAFOV[1]*

Abstract. A generalization of Schauder basis associated with the concept of generalized analytic functions is introduced. Corresponding concepts of density, completeness, biorthogonality and basicity are defined. Also, corresponding concept of the space of coefficients is introduced. Under certain conditions for the corresponding operators, some properties of the space of coefficients and basicity criterion are considered.

1. Introduction

One of the methods for solving partial differential equations of elliptic type is a functional analytical method. Known long ago, it is based on the fact that the solution of the Laplace equation in the domain of complex plane $\mathbb{C}$ is an analytic function. Of course, the solution has the properties of analytic functions, and in particular, it can be expanded in power series in the neighborhood of any point in domain. Finding such an approach to the solution of other elliptic equations requires the construction of various generalizations of the Cauchy theory. Perhaps, I.N. Vekua [23], L. Bers [1] and A. Douglis [6] were the first to use these ideas which have been further developed in [4, 11, 24]. They are also mentioned in [2]. Lately, there arose a great interest in using these ideas in various problems of mathematical physics and mechanics. For example, A.P. Soldatov [17-21] used them for solving high order elliptic systems. In [8, 12, 13, 22] similar ideas helped to solve nonclassical Sobolev type differential equations. The core in these works is to build the analogues of classical Cauchy theory of analytic functions. Generalized functions as well as the solutions of above equations can be

2010 *Mathematics Subject Classification.* 30G20, 41A65, 46B15.
Key words and phrases. J-completeness, J-biorthogonality, J-basicity, The space of coefficients.
Received: 19 May 2015, Accepted: 21 September 2015.
* Corresponding author.

expanded in series with respect to generalized exponential functions. Let us demonstrate the previously mentioned ideas in the following simple example.

Let $J:\mathbb{R}^2\to\mathbb{R}^2$ be some matrix, where $\mathbb{R}$ is a real axis. Consider the vector function

$$\vec{\Phi}\left(\vec{x}\right)=\left(\Phi_1\left(\vec{x}\right),\Phi_2\left(\vec{x}\right)\right),\qquad \vec{x}=\left(x_1;x_2\right)\in\mathbb{R}^2.$$

$\vec{\Phi}$ is called J- analytic function at a point $\vec{x}_0=\left(x_1^0;x_2^0\right)$, if it is defined in a neighborhood of $\vec{x}_0$, then

$$\frac{\partial\vec{\Phi}}{\partial x_2}-J\frac{\partial\vec{\Phi}}{\partial x_1}=0,\quad \text{for}\quad \vec{x}=\vec{x}_0.$$

Set

$$\left[\vec{x}\right]_J=x_1I+x_2J,\qquad \vec{\Phi}^{(n)}=\frac{\partial^n\vec{\Phi}}{\partial x_1^n},$$

where $I:\mathbb{R}^2\to\mathbb{R}^2$ is the identity matrix. If the spectrum of J belongs to upper half plane, i.e. $\sigma\left(J\right)\subset\{Imz>0\}$, then J-analytic function $\vec{\Phi}$ at the neighborhood of the point $\vec{x}_0$ can be expanded in series

$$\vec{\Phi}\left(\vec{x}\right)=\sum_{n=0}^{\infty}\frac{\left[x-x_0\right]_J^n}{n!}\vec{\Phi}^{(n)}\left(\vec{x}_0\right).$$

As J, we take the following matrix

$$J=\begin{pmatrix} i & 1\\ 0 & i\end{pmatrix},$$

where i is the imaginary unit.

This approach allows us to determine the appropriate Cauchy integral, and subsequently construct an analogue of the Cauchy theory of analytic functions in this direction. And this in turn allows us to consider the various Riemann boundary value problems for J-analytic functions. More details about these and other facts can be found in e.g. [8, 12, 13, 17, 18, 19, 20, 21, 22]. The latter, of course, requires the introduction of corresponding concept of basis. This makes possible to enlarge the concept of the solution of corresponding differential equations and, moreover, enables to treat other types of differential equations, where the Fourier method of separation of variables can be applied to these differential equations.

This work is dedicated to the above mentioned matters concerning basis. Some needful concepts are introduced such as J-independence, J-completeness, J-minimality, J-basicity and J-space of coefficients. Some of their properties are stated. J-basicity criterion is established. To avoid some special cases, we will use an arbitrary Banach space.

Let us note that earlier the idea of such a generalized basis was considered in [3]. More details about various generalizations of basis and problems concerning the concept of the space of coefficients, can be found in monographs I. Singer [15, 16], R. Young [25], O. Heil [10], and O. Christensen [5].

It should be noted that the results of [5, 7, 9, 14], relating to the generalized concept of frames, is closely related to this work. In these works similar ideas are applied to issues of frames in Banach spaces.

2. Needful notations and concepts

Let X and Y be some Banach spaces. Let $L(X;Y)$ −be the Banach space of bounded operators from X to Y and $L(X) \equiv L(X;X)$. Let $\Gamma \subset \mathbb{C}$ be the piecewise smooth curve. By $L_p(\Gamma; L(X))$, $(L_p(\Gamma; X))$, $p \geq 1$, we denote a Lebesgue space of $L(X)$-valued (respectively X-valued) functions on Γ with the norm

$$\|f\|_p = \left(\int_\Gamma \|f(t)\|^p |dt| \right)^{\frac{1}{p}},$$

where $f : \Gamma \to L(X)$ $(f : \Gamma \to X)$. Let $M \subset L_p(\Gamma; L(X))$. As in [8-12], the generalized concepts will be denoted using symbol "$J-$".

$J-$span of a set M in $L_p(\Gamma; L(X))$ is defined as

$$L_J[M] \equiv \left\{ x(\cdot) : \quad x(\cdot) = \sum_{k=1}^{n} T_k(\cdot) x_k, \{T_k(\cdot)\} \subset M, \{x_k\} \subset X \right\} \subset L_p(\Gamma; X).$$

Usual completeness of systems in Banach spaces, takes the following form.

Definition 2.1. A system $\{T_n\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$ is called J-complete in $L_p(\Gamma; X)$, if $\overline{L_J\left[\{T_n\}_{n\in\mathbb{N}}\right]} \equiv L_p(\Gamma; X)$, where $\left(\bar{\cdot}\right)$ – is the closure in $L_p(\Gamma; X)$.

We will also need the generalized concept of independence.

Definition 2.2. A system

$$\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$$

is called $\omega J-$independent, if

$$\sum_{n=1}^{\infty} T_n(\cdot) f_n = 0,$$

in $L_p(\Gamma; X)$, implies $\bar{f} = 0$, where $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \subset X$.

A biorthogonal system is defined in our case as follows.

Definition 2.3. A system

$$\{T_n^*(\cdot)\}_{n\in\mathbb{N}} \subset L(L_p(\Gamma; X); X),$$

is called $J-$biorthogonal to the system

$$\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X)),$$

if

$$T_n^*(T_k(\cdot)\, x) = \delta_{nk} x, \quad \forall x \in X, \quad \forall n, k \in \mathbb{N},$$

where $\delta_{nk}-$ is the Kronecker symbol.

Now we introduce the significant concept of the generalized basicity.

Definition 2.4. A system $\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$ is called $J-$basis in $L_p(\Gamma; X)$, if

$$\forall f \in L_p(\Gamma; X), \quad \exists!\, \{f_n\}_{n\in\mathbb{N}} \subset X : f = \sum_{n=1}^{\infty} T_n f_n,$$

in $L_p(\Gamma; X)$, i.e.

$$\int_\Gamma \left\| f(\tau) - \sum_{n=1}^{m} T_n(\tau)\, f_n \right\|^p |d\tau| \to 0, \quad \text{as} \quad m \to \infty.$$

$\{f_n\}_{n\in\mathbb{N}}$ are called coordinate coefficients.

To obtain our main results we will often use the following major property.

Property 2.5. *The system $\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$ satisfies condition: $\exists T_n^{-1}(t)$, a.e. $t \in \Gamma$, $\forall n \in \mathbb{N}$ and $\left\{T_n^{-1}(\cdot)\right\}_{n\in\mathbb{N}} \subset L_q(\Gamma; L(X))$, where $\frac{1}{p} + \frac{1}{q} = 1$.*

Remark 2.6. It should be noted that if instead of X, we take the complex field $\mathbb{C}$, then all the above mentioned concepts will be the generalizations of the corresponding concepts with respect to the basis from a system of functions in functional spaces.

3. Main results

3.1. **Space of coefficients.** We consider the spaces $L_p(\Gamma; L(X))$ and $L_p(\Gamma; X)$, $p \geq 1$. Let $S_{\bar{T}} \equiv \{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$ be some system. Define

$$\mathscr{K}_{\bar{T}} \equiv \left\{ \{f_n\}_{n\in\mathbb{N}} \subset X : \sum_{n=1}^{\infty} T_n(\cdot)\, f_n \quad \text{converges in} \quad L_p(\Gamma; X) \right\}.$$

It is easy to see that $\mathscr{K}_{\bar{T}}$ is a linear space with regard to the usual operations of addition and multiplication by a complex number. Assume that the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ has the Property 2.5. Introduce

$$\|\cdot\|_{\mathscr{K}_{\bar{T}}} : \left\|\bar{f}\right\|_{\mathscr{K}_{\bar{T}}} = \sup_m \left\|\sum_{n=1}^{m} T_n(\cdot) f_n\right\|_{L_p(\Gamma,X)},$$

where $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{T}}$. Evidently we have

$$\left\|\lambda\bar{f}\right\|_{\mathscr{K}_{\bar{T}}} = |\lambda| \left\|\bar{f}\right\|_{\mathscr{K}_{\bar{T}}}, \qquad \forall\lambda \in \mathbb{C};$$

$$\left\|\bar{f}+\bar{g}\right\|_{\mathscr{K}_{\bar{T}}} \le \left\|\bar{f}\right\|_{\mathscr{K}_{\bar{T}}} + \|\bar{g}\|_{\mathscr{K}_{\bar{T}}}, \qquad \forall\bar{f},\bar{g} \in \mathscr{K}_{\bar{T}}.$$

Let $\left\|\bar{f}\right\|_{\mathscr{K}_{\bar{T}}} = 0$ and $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}}$. Set $n_0 = \inf\{n : f_k = 0, \quad \forall k \le n-1\}$. It is clear that $f_n = T_n^{-1}(t)\, T_n(t)\, f_n$, a.e. $t \in \Gamma$. Consequently, $\|f_n\|_X \le \left\|T_n^{-1}(t)\right\| \|T_n(t) f_n\|$, a.e. $t \in \Gamma$. Integrating this inequality over Γ and applying Holder inequality we obtain

$$|\Gamma| \, \|f_n\|_X \le \left(\int_\Gamma \left\|T_n^{-1}(t)\right\|^q |dt|\right)^{\frac{1}{q}} \left(\int_\Gamma \|T_n(t) f_n\|^p |dt|\right)^{\frac{1}{p}},$$

where $|\Gamma|$ is the length of the curve Γ. It is evident that

$$\int_\Gamma \left\|T_n^{-1}(t)\right\|^q |dt| \ne 0,$$

because, if not, we would have $T_n^{-1}(t) = 0$ a.e. $t \in \Gamma$, which is contrary to the existence of $T_n^{-1}(t)$ a.e. $t \in \Gamma$. So we have

$$\|T_n(\cdot) f_n\|_{L_p(\Gamma;X)} \ge \frac{|\Gamma|}{\left\|T_n^{-1}\right\|_{L_q(\Gamma;L(X))}} \|f_n\|_X. \tag{3.1}$$

Let $n_0 < +\infty$. The following relation holds

$$\begin{aligned}\sup_m \left\|\sum_{n=1}^{m} T(\cdot) f_n\right\|_{L_p(\Gamma;X)} &\ge \left\|\sum_{n=1}^{n_0} T_n(\cdot) f_n\right\|_{L_p(\Gamma;X)} \\ &= \|T_{n_0} f_{n_0}\|_{L_p(\Gamma;X)} \\ &\ge \frac{|\Gamma|}{\left\|T_{n_0}^{-1}\right\|_{L_q(\Gamma;L(X))}} \|f_{n_0}\|_X \\ &> 0.\end{aligned}$$

We now have a contradiction. So, $n_0 = +\infty$, i.e. $\bar{f} = 0$. As a result of this, we get that $\left(\mathscr{K}_{\bar{T}}; \|\cdot\|_{\mathscr{K}_{\bar{T}}}\right)$ is a normed space. Let's show that this space is complete. Let $\left\{\bar{f}_n\right\}_{n\in\mathbb{N}} \subset \mathscr{K}_{\bar{T}}$ be some fundamental sequence, where $\bar{f}_n \equiv \left\{ f_k^{(n)} \right\}_{k\in\mathbb{N}} \subset X$. Fix $k \in \mathbb{N}$. Taking into account inequality (3.1), we have

$$\begin{aligned}
\left\| f_k^{(n)} - f_k^{(n+p)} \right\|_X &\le \frac{\left\| T_k^{-1} \right\|_{L_q(\Gamma;L(X))}}{|\Gamma|} \left\| T_k(\cdot) \left[f_k^{(n)} - f_k^{(n+p)} \right] \right\|_{L_p(\Gamma;X)} \\
&= \frac{\left\| T_k^{-1} \right\|_{L_q(\Gamma;L(X))}}{|\Gamma|} \times \left\| \sum_{i=1}^{k} T_i(\cdot) \left[f_i^{(n)} - f_i^{(n+p)} \right] \right. \\
&\quad \left. - \sum_{i=1}^{k-1} T_i(\cdot) \left[f_i^{(n)} - f_i^{(n+p)} \right] \right\|_{L_p(\Gamma;X)} \\
&\le \frac{2}{|\Gamma|} \left\| T_k^{-1} \right\|_{L_q(\Gamma;L(X))} \left\| \bar{f}_n - \bar{f}_{n+p} \right\|_{\mathscr{K}_{\bar{T}}} \\
&\to 0, \quad \text{as} \quad n, p \to \infty.
\end{aligned}$$

Consequently, for every fixed $k \in \mathbb{N}$, the sequence $\left\{ f_k^{(n)} \right\}_{n\in\mathbb{N}}$ is fundamental in X. Let $f_k^{(n)} \to f_k$, $n \to \infty$. Take $\varepsilon > 0$ arbitrary. Evidently, $\exists n_0 : \forall n \ge n_0, \forall p \in \mathbb{N}$ we have $\left\| \bar{f}_n - \bar{f}_{n+p} \right\|_{\mathscr{K}_{\bar{T}}} < \varepsilon$. Thus

$$\left\| \sum_{k=1}^{m} T_k(\cdot) \left[f_k^{(n)} - f_k^{(n+p)} \right] \right\|_{L_p(\Gamma;X)} < \varepsilon, \quad \forall n \ge n_0, \quad \forall p, m \in \mathbb{N}.$$

Passing to the limit in this inequality as $p \to \infty$ yields

$$\left\| \sum_{k=1}^{m} T_k(\cdot) \left[f_k^{(n)} - f_k \right] \right\|_{L_p(\Gamma;X)} \le \varepsilon, \quad \forall n \ge n_0, \quad \forall m \in \mathbb{N}. \tag{3.2}$$

In fact, the latter inequality follows from the obvious relation

$$\| T_k(\cdot) f \|_{L_p(\Gamma;X)} \le \| T_k(\cdot) \|_{L_p(\Gamma;L(X))} \| f \|_X, \quad \forall f \in X, \quad \forall k \in \mathbb{N}.$$

It is evident that

$$\left\| \sum_{k=m}^{m+p} T_k(\cdot) \left[f_k^{(n)} - f_k \right] \right\|_{L_p(\Gamma;X)} \le 2\varepsilon, \quad \forall n \ge n_0, \quad \forall m, p \in \mathbb{N}.$$

The convergence of the series

$$\sum_{k=1}^{\infty} T_k(\cdot) f_k^{(n)},$$

in $L_p(\Gamma; X)$ implies that

$$\exists m_0^{(n)} : \forall m \ge m_0^{(n)}, \forall p \in \mathbb{N},$$

we have

$$\left\| \sum_{k=m}^{m+p} T_k\left(\cdot\right) f_k^{(n)} \right\|_{L_p(\Gamma;X)} < 2\varepsilon.$$

Also, we have

$$\begin{aligned}\left\| \sum_{k=m}^{m+p} T_k\left(\cdot\right) f_k \right\|_{L_p(\Gamma;X)} &\leq \left\| \sum_{k=m}^{m+p} T_k\left(\cdot\right) \left[f_k^{(n)} - f_k \right] \right\|_{L_p(\Gamma;X)} \\ &\quad + \left\| \sum_{k=m}^{m+p} T_k\left(\cdot\right) f_k^{(n)} \right\|_{L_p(\Gamma;X)} \\ &\leq 3\varepsilon, \quad \forall m \geq m_0^{(n)}, \quad \forall p \in \mathbb{N}.\end{aligned}$$

It follows directly from this inequality that the series

$$\sum_{k=1}^{\infty} T_k\left(\cdot\right) f_k,$$

converges in $L_p\left(\Gamma; X\right)$, i.e. $\bar{f} \equiv \{f_k\}_{k\in\mathbb{N}} \in \mathscr{K}_{\bar{T}}$. It follows from (3.2) that $\left\|\bar{f}_n - \bar{f}\right\|_{\mathscr{K}_{\bar{T}}} \to 0$, $n \to \infty$. As a result of this, we get that $\mathscr{K}_{\bar{T}}$ is a Banach space. Take arbitrary $\bar{f} \in \mathscr{K}_{\bar{T}}$ and consider operator $K : \mathscr{K}_{\bar{T}} \to L_p\left(\Gamma; X\right)$:

$$K\bar{f} = \sum_{n=1}^{\infty} T\left(\cdot\right) f_n, \qquad \bar{f} \equiv \{f_n\}_{n\in\mathbb{N}}\,. \tag{3.3}$$

It is evident that K is a linear operator. Let $f = K\bar{f}$, then we have

$$\begin{aligned}\left\|K\bar{f}\right\|_{L_p(\Gamma;X)} &= \|f\|_{L_p(\Gamma;X)} \\ &= \left\| \sum_{n=1}^{\infty} T_n\left(\cdot\right) f_n \right\|_{L_p(\Gamma;X)} \\ &\leq \sup_m \left\| \sum_{n=1}^{m} T_n\left(\cdot\right) f_n \right\|_{L_p(\Gamma;X)} \\ &= \|f\|_{\mathscr{K}_{\bar{T}}}\,.\end{aligned}$$

Consequently, $K \in L\left(\mathscr{K}_{\bar{T}}; L_p\left(\Gamma; X\right)\right)$ and, moreover, $\|K\| \leq 1$. Let $\bar{f}_0 \equiv \{x; 0; \ldots\}$, $x \in X$. It is evident that $\left\|K\bar{f}_0\right\|_{L_p(\Gamma;X)} = \left\|\bar{f}_0\right\|_{\mathscr{K}_{\bar{T}}}$, and follows directly from this relation that $\|K\| = 1$.

It is clear that if the system $\{T_n\left(\cdot\right)\}_{n\in\mathbb{N}} \subset L_p\left(\Gamma; L\left(X\right)\right)$ is $\omega J-$independent, then $\ker K = \{0\}$. In this case

$$\exists K^{-1} : L_p\left(\Gamma; X\right) \to \mathscr{K}_{\bar{T}}.$$

If ImK be closed, then, according to Banach's theorem on an inverse operator we get that $K^{-1} \in L(ImK; \mathscr{K}_{\bar{T}})$. It is easy to see that the same reasoning is also true in case when the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ has a J-biorthogonal system. We call the above defined operator J coefficient operator. So the following theorem holds.

Theorem 3.1. *Let the system*

$$S_{\bar{T}} \equiv \{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X)), \quad p \geq 1,$$

have Property 2.5. Then a Banach space of coefficients $\mathscr{K}_{\bar{T}}$ and the coefficient operator $K \in L(\mathscr{K}_{\bar{T}}; L_p(\Gamma; X))$, $\|K\| = 1$, defined by the expression (3.3) *corresponds to this system. If in addition, the system $S_{\bar{T}}$ is ωJ-independent or has a $J-$biorthogonal system, then $\exists K^{-1}$. Moreover, if ImK is closed, then $K^{-1} \in L(ImK; \mathscr{K}_{\bar{T}})$.*

In what follows, we will also need the concept of $J-$basis in space $\mathscr{K}_{\bar{T}}$.

Definition 3.2. A system $\{A_n\}_{n\in\mathbb{N}} \subset L(X; \mathscr{K}_{\bar{T}})$ is called $J-$basis in $\mathscr{K}_{\bar{T}}$, if

$$\forall \bar{f} \in \mathscr{K}_{\bar{T}}, \quad \exists! \{f_n\}_{n\in\mathbb{N}} \subset X : \bar{f} = \sum_{n=1}^{\infty} A_n f_n$$

(convergence is meant in $\mathscr{K}_{\bar{T}}$).

Consider operators

$$E_n : X \to \mathscr{K}_{\bar{T}}, \quad n \in \mathbb{N} : E_n x = \{\underbrace{0; \ldots; 0}_{n-1}; x; 0; \ldots\}, \quad \forall x \in X.$$

We have

$$\begin{aligned} \|E_n x\|_{\mathscr{K}_{\bar{T}}} &= \|T_n(\cdot)\, x\|_{L_p(\Gamma;X)} \\ &= \left(\int_\Gamma \|T_n(t)\, x\|^p \,|dt|\right)^{\frac{1}{p}} \\ &\leq \left(\int_\Gamma \|T_n(t)\|^p \,|dt|\right)^{\frac{1}{p}} \|x\| . \end{aligned}$$

Consequently, $E_n \in L(X; \mathscr{K}_{\bar{T}})$, for all $n \in \mathbb{N}$. Let $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{T}}$, it's easy to see that $\bar{f} = \sum_{n=1}^{\infty} E_n f_n$. Consider operators $P_n : \mathscr{K}_{\bar{T}} \to X$, for all $n \in \mathbb{N}$, such that $P_n \bar{f} = f_n$, where $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}}$. Let's show that

$\{P_n\}_{n\in\mathbb{N}} \subset L(\mathscr{K}_{\bar{T}}; X)$. Taking into account the inequality (3.1) we get

$$\begin{aligned}\left\|P_n \bar{f}\right\|_X &= \|f_n\|_X \\ &\le \frac{c_n}{|\Gamma|}\|T(\cdot) f_n\|_{L_p(\Gamma;X)} \\ &\le \frac{c_n}{|\Gamma|}\sup_m \left\|\sum_{k=1}^m T_n(\cdot) f_k\right\|_{L_p(\Gamma;X)} \\ &= \frac{c_n}{|\Gamma|}\left\|\bar{f}\right\|_{\mathscr{K}_{\bar{T}}},\end{aligned}$$

where $c_n = \left\|T_n^{-1}\right\|_{L_q(\Gamma;L(X))}$. Consequently, $P_n \in L(\mathscr{K}_{\bar{T}}; X)$. Let's show that the expansion

$$\bar{f} = \sum_{n=1}^{\infty} E_n f_n,$$

is unique. Let

$$\sum_{n=1}^{\infty} E_n f_n = 0.$$

We have

$$P_k\left(\sum_{n=1}^{\infty} E_n f_n\right) = \sum_{n=1}^{\infty} P_k(E_n f_n) = f_k = 0, \quad \forall k \in \mathbb{N}.$$

As a result of this we get that the system $\{E_n\}_{n\in\mathbb{N}}$ forms a $J-$basis for $\mathscr{K}_{\bar{T}}$. We will call it $J-$canonical system. So we have just proved the following

Theorem 3.3. *Let the system*

$$S_{\bar{T}} \equiv \{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X)), \quad p \ge 1,$$

have Property 2.5. Then the J-canonical system $\{E_n\}_{n\in\mathbb{N}} \subset L(X; \mathscr{K}_{\bar{T}})$ forms J-basis for space of coefficients $\mathscr{K}_{\bar{T}}$.

Now we suppose that the system

$$\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X)), \quad p \ge 1,$$

has Property 2.5 and forms J-basis for $L_p(\Gamma; L(X))$. Consider the coefficient operator $K : \mathscr{K}_{\bar{T}} \to L_p(\Gamma; X)$. It follows directly from the definition of J-basicity that for all $f \in L_p(\Gamma; X)$ the equation $K\bar{f} = f$ has a solution $\bar{f} \in \mathscr{K}_{\bar{T}}$. As J-basicity implies ωJ-independence, then it is clear that $\ker K = \{0\}$. Then it follows from Theorem 3.1 and Banach theorem, that $K^{-1} \in L(L_p(\Gamma; X); \mathscr{K}_{\bar{T}})$. Therefore, spaces $\mathscr{K}_{\bar{T}}$ and $L_p(\Gamma; X)$ are isometrically isomorphic, i.e. K performs isomorphism between $\mathscr{K}_{\bar{T}}$ and $L_p(\Gamma; X)$.

Conversely, let the system $\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$, be given with Property 2.5. Suppose the coefficient operator K is an isomorphism between $\mathscr{K}_{\bar{T}}$ and $L_p(\Gamma; X)$. Take arbitrary $f \in L_p(\Gamma; X)$. It is clear that

$$\exists \bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{T}} : K\bar{f} = f, \quad \text{i.e.} \quad f(\cdot) = \sum_{n=1}^{\infty} T_n(\cdot) f_n,$$

in $L_p(\Gamma; X)$. Consequently, $f(\cdot)$ can be expanded with respect to the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ in $L_p(\Gamma; X)$. Let's show that such an expansion is unique. Let

$$\sum_{n=1}^{\infty} T_n(\cdot) g_n = 0,$$

in $L_p(\Gamma; X)$, where $\{g_n\}_{n\in\mathbb{N}} \subset X$. It follows directly from the definition of space $\mathscr{K}_{\bar{T}}$, that $\bar{g} \equiv \{g_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{T}}$ and therefore, $K\bar{g} = 0$. $\ker K = \{0\}$ implies $\bar{g} = 0$. Thus, the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ forms a J-basis for $L_p(\Gamma; X)$. So the following theorem hplds.

Theorem 3.4. *Let the system*

$$\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X)), \quad p \geq 1,$$

satisfies the Property 2.5, and K be the corresponding coefficient operator $K : \mathscr{K}_{\bar{T}} \to L_p(\Gamma; X)$. Then it forms a J-basis for $L_p(\Gamma; X)$ if and only if when K be an isomorphism in $L(\mathscr{K}_{\bar{T}}; L_p(\Gamma; X))$.

4. Basicity criterion

Let the systems $\{T_n^*(\cdot)\}_{n\in\mathbb{N}} \subset L(L_p(\Gamma; X); X)$ and $\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$ be J-biorthogonal. Take arbitrary $f \in L_p(\Gamma; X)$ and consider partial sums

$$[S_n f](t) = \sum_{k=1}^{n} T_k(t) f_k, n \in \mathbb{N},$$

where $f_k = T_k^*[f]$. We have

$$\begin{aligned} S_n(S_m f)(\cdot) &= \sum_{k=1}^{n} T_k(\cdot)\left(T_k^*[S_m f]\right) \\ &= \sum_{k=1}^{n} T_k(\cdot)\left(T_k^*\left[\sum_{i=1}^{m} T_i(\cdot) f_i\right]\right) \\ &= \sum_{k=1}^{n} T_k(\cdot)\left(\sum_{i=1}^{m} \delta_{ki} f_i\right) \\ &= \sum_{k=1}^{n}\sum_{i=1}^{m} \delta_{ki} T_k(\cdot) f_i \\ &= \sum_{k=1}^{\min\{n;m\}} T_k(\cdot) f_k \\ &= \left[S_{\min\{n;m\}} f\right](\cdot). \end{aligned}$$

It follows $S_n^2 = S_n$, for all $n \in \mathbb{N}$, i.e. S_n is a projector in $L_p(\Gamma; X)$, for all $n \in \mathbb{N}$. Assume that the system $\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X))$ forms a J-basis for $L_p(\Gamma; X)$. Then, any $f \in L_p(\Gamma; X)$ has a unique expansion

$$f(t) = \sum_{n=1}^{\infty} T_n(t) f_n,$$

in $L_p(\Gamma; X)$. Relation $f(t) \to f_n$ will be denoted by P_n, i.e. $P_n[f] = f_n$. It is clear that $P_n : L_p(\Gamma; X) \to X$ is a linear operator. Let the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ has Property 2.5. It is evident that $\|f_n\|_X = \left\|T_n^{-1}(t) T_n(t) f_n\right\|_X$, a.e. $t \in \Gamma$. With integrating and applying Holder inequality, we obtain

$$|\Gamma| \, \|P_n[f]\|_X \le c_n \left(\int_\Gamma \|T_n(t) f_n\|_X^p \, |dt|\right)^{\frac{1}{p}}, \quad n \in \mathbb{N}.$$

We have

$$\begin{aligned} \|P_n[f]\|_X &\le \frac{c_n}{|\Gamma|} \|T_n(\cdot) f_n\|_{L_p(\Gamma;X)} \\ &\le \frac{c_n}{|\Gamma|} \sup_m \left\|\sum_{k=1}^{m} T_n(\cdot) f_n\right\|_{L_p(\Gamma;X)} \\ &= \frac{c_n}{|\Gamma|} \left\|\bar{f}\right\|_{\mathscr{K}_{\bar{T}}}, \end{aligned}$$

where $\mathscr{K}_{\bar{T}}$ is the space of coefficients correspond to the basis $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ and $\bar{f} \equiv \{f_n\}_{n\in\mathbb{N}} \in \mathscr{K}_{\bar{T}}$. Let $K : \mathscr{K}_{\bar{T}} \to L_p(\Gamma; X)$ be the coefficient operator of the basis $\{T_n(\cdot)\}_{n\in\mathbb{N}}$. It follows from Theorem 3.4 that K is an isomorphism in $L(\mathscr{K}_{\bar{T}}; L_p(\Gamma; X))$. Taking into account this fact, we obtain

$$\|P_n[f]\|_X \le \frac{c_n}{|\Gamma|}\left\|K^{-1}\right\| \|f\|_{L_p(\Gamma;X)}.$$

Consequently,

$$P_n \in L(L_p(\Gamma; X); X), \quad \forall n \in \mathbb{N}.$$

Take arbitrary $x \in X$ and $f(\cdot) = T_k(\cdot)x$. It is easy to see that $f \in L_p(\Gamma; X)$. From the uniqueness of expansion we derive

$$\|P_n[f]\| = P_n[T_k(\cdot)x] = \delta_{nk}x, \quad \forall n, k \in \mathbb{N}.$$

As a result of this, we obtain that $\{P_n\}_{n\in\mathbb{N}}$ is J-biorthogonal to $\{T_n(\cdot)\}_{n\in\mathbb{N}}$. Take arbitrary $f \in L_p(\Gamma; X)$ and consider partial sums

$$S_m[f] = \sum_{n=1}^{m} T_n(\cdot)P_n[f]. \tag{4.1}$$

Let us prove the following

Theorem 4.1. *Let the system $\{T_n(\cdot)\}_{n\in\mathbb{N}} \subset L_p(\Gamma; L(X)), p \ge 1$, satisfies Property 2.5. Then it forms a J-basis for $L_p(\Gamma; X)$ if and only if the following conditions hold:*

(i) *it is J-complete in $L_p(\Gamma; X)$;*
(ii) *it has a J-biorthogonal system;*
(iii) *family of projectors $\{S_m\}_{m\in\mathbb{N}}$, defined by (4.1), is uniformly bounded.*

Proof. As $\lim_{m\to\infty}[S_m f]$ exists in $L_p(\Gamma; X)$ for all $f \in L_p(\Gamma; X)$, then it follows from Banach-Steinhaus theorem that the relation

$$\|S_m[f]\|_{L_p(\Gamma;X)} \le c\|f\|_{L_p(\Gamma;X)}$$

holds, where c is an absolute constant, i.e.

$$\sup_m \|S_m\| < +\infty. \tag{4.2}$$

It is clear that the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ is J-complete. Thus, if the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ forms a J-basis for $L_p(\Gamma; X)$, then it is J-complete, has a J-biorthogonal system and a family of projectors $\{S_m\}_{m\in\mathbb{N}}$, defined by (4.1), is uniformly bounded.

Conversely, let the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$ be J-complete and has a J-biorthogonal system $\{P_n\}_{n\in\mathbb{N}}$. Consider the family of projectors (4.1) and assume that (4.2) holds. Take arbitrary $f \in L_p(\Gamma; X)$ and consider $\{S_m[f]\}_{m\in\mathbb{N}}$. Obviously

$$\|S_m[f]\|_{L_p(;X)} \leq c\,\|f\|_{L_p(\Gamma;X)}, \quad \forall m \in \mathbb{N}.$$

Take arbitrary $\varepsilon > 0$. It is clear that $\exists \{g_k\}_{k=1}^{m_0} \subset X$ such that

$$\|f(\cdot) - F(\cdot)\|_{L_p(\Gamma;X)} < \varepsilon,$$

where

$$F(\cdot) = \sum_{k=1}^{m_0} T_k(\cdot)\, g_k.$$

We have

$$\begin{aligned} P_n[F] &= \sum_{k=1}^{m_0} P_n[T_k(\cdot)\, g_k] \\ &= \sum_{k=1}^{m_0} \delta_{nk} g_k \\ &= g_n, \quad 1 \leq n \leq m_0; \end{aligned}$$

$$P_n[F] = 0, \quad \forall n > m_0.$$

Consequently, for $m \geq m_0$ we obtain

$$\begin{aligned} \|f - S_m[f]\|_{L_p(\Gamma;X)} &\leq \|f - F\|_{L_p(\Gamma;X)} + \|F - S_m[f]\|_{L_p(\Gamma;X)} \\ &\leq \varepsilon + \left\| \sum_{n=1}^{m} T_n(\cdot)\, P_n[f-F] \right\|_{L_p(\Gamma;X)} \\ &\leq \varepsilon + c\,\|f - F\|_{L_p(\Gamma;X)} \\ &\leq (c+1)\,\varepsilon. \end{aligned}$$

It follows directly that $\lim_{m\to\infty} S_m[f] = f$ in $L_p(\Gamma; X)$. Thus, every $f \in L_p(\Gamma; X)$ can be expanded with respect to the system $\{T_n(\cdot)\}_{n\in\mathbb{N}}$. The existence of J-biorthogonal system implies the uniqueness of the expansion. □

Remark 4.2. By using the concept of J-basis in $L_p(\Gamma; X)$ one can determine the appropriate Hardy space $H_p(X)$ J-analytic functions in the unit disk and study their main properties of associated classical case.

Acknowledgments. The author would like to express his deepest gratitude to Professor Bilal T. Bilalov for his encouragement and valuable guidance throughout this research. Author is also grateful to the referees for their valuable suggestions and comments.

References

1. L. Bers, *Theory of pseudo-analytic functions*, Lecture Notes, New York University, 1953.
2. L. Bers, F. John, and M. Schechter, *Partial differential equations*, Lectures in Applied Mathematics III. Interscience Publishers, New York, 1964.
3. B.T. Bilalov, *Bases and tensor product*, Trans. of NAS of Az., XXV(4) (2005), 15-20.
4. A.V. Bitsadze, *Boundary value problems for second order elliptic equations*, Moscow, Nauka, 1966, 204 p.
5. O. Christensen, *An introduction to frames and Riesz bases*, Springer, 2002.
6. A. Douglis, *A function theoretical approach to elliptic systems of equations in two variables*, Comm. Pure Math., VI (1953) 259-289.
7. H.G. Feichtinger and K.H. Gröchenig, *Banach spaces related to integrable group representations and their atomic decompositions*, Part II. Monatshefte fur Mathematik, 108(2-3) (1989), 129-148.
8. S.A. Gabov and A.G. Sveshnikov, *Linear problems of the theory of nonstationary internal waves*, M., Nauka, 1990.
9. K.H. Gröchenig, *Describing functions: atomic decompositions versus frames*, Monatshefte fur Mathematik, 112(1) (1991), 1-42.
10. Ch. Heil, *A Basis Theory Primer*, Springer, 2011.
11. G.N. Hile, *Elliptic systems in the plane with order terms and constant coefficients*, Comm. in Part. Diff. Equat., 3(10) (1978), 949-977.
12. Yu.D. Pletner, *Representation of solutions of the two-dimensional gravitational-gyroscopic wave equation by generalized Taylor and Laurent series*, Zh. Vychisl. Mat. Mat. Fiz., 30(11) (1990), 1728-1740.
13. Yu.D. Pletner, *Representation of the solutions of two-dimensional analogues of the Sobolev equation by generalized Taylor and Laurent series*, Comput. Math. Math. Phys., 32(1) (1992), 51-60.
14. A. Rahimi, *Frames and Their Generalizations in Hilbert and Banach Spaces*, Lambert Academic Publishing. 2011.
15. I. Singer, *Bases in Banach spaces*, v.2, Springer, 1981.
16. I. Singer, *Bases in Banach spaces*, v. 1, Springer, 1970.
17. A.P. Soldatov, *Second-order elliptic systems in the half-plane*, Izvestiya: Mathematics, 70(6) (2006), 161-192.
18. A.P. Soldatov, *A function theorety method in elliptic problems in the plane, II. The piecewise smooth case*, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 56(3) (1992), 566-604.
19. A.P. Soldatov, *A function theory method in boundary value problems in the plane, I. The smooth case*, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya, 55(5) (1991), 1070-1100.
20. A.P. Soldatov, *Boundary properties of integrals of Cauchy type*, Differential Equations, 26(1) (1990), 116-120.
21. A.P. Soldatov, *High-order elliptic-systems*, Differential Equations, 25(1) (1989), 109-115.
22. A.G. Sveshnikov, A.B. Al'shin, M.O. Korpusov, and Yu.D. Pletner, *Linear and nonlinear equations of Sobolev type*, Fizmatlit, Moscow, 2007.
23. I.N. Vekua, *New methods of solution of elliptic equations*, Moscow, Gostekhizdat, 1948.
24. I.N. Vekua, *Generalized Analytic Functions*, Moscow, Fizmatqiz, 1959.

25. R.M. Young, *An Introduction to Nonharmonic Fourier series*, Springer, 1980.

---

[1] Nakhchivan State University, University campus, AZ7012 Nakhchivan, Azerbaijan.

E-mail address: tofiq-necefov@mail.ru