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NUMERICAL SOLUTION OF A CLASS OF NONLINEAR TWO-DIMENSIONAL INTEGRAL EQUATIONS USING BERNOULLI POLYNOMIALS

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Abstract. In this study, the Bernoulli polynomials are used to obtain an approximate solution of a class of nonlinear two-dimensional integral equations. To this aim, the operational matrices of integration and the product for Bernoulli polynomials are derived and utilized to reduce the considered problem to a system of nonlinear algebraic equations. Some examples are presented to illustrate the efficiency and accuracy of the method.

1. Introduction

Let $I = [0, 1]$ and define

$$\begin{aligned} I^r &= \underbrace{I \times I \times \cdots \times I}_{r \text{ times}} \\ &= \{(x_1, \dots, x_r) | x_i \in I, i = 1, \dots, r\}, \end{aligned}$$

as the Cartesian power of I of order r. We consider the numerical solution of a class of nonlinear two-dimensional Volterra-Fredholm-Hammerstein (2DVFH) integral equations of the form

$$\begin{aligned} u(x,t) = f(x,t) &+ \int_0^t \int_0^x k_1(x,t,y,z) N_1\big(u(y,z)\big) dydz \\ &+ \int_0^1 \int_0^1 k_2(x,t,y,z) N_2\big(u(y,z)\big) dydz, \quad (x,t) \in I^2. \end{aligned} \tag{1.1}$$

In Eq. (1.1), $u(x,t)$ is an unknown real valued function, $f(x,t)$ and $k_i(x,t,y,z)$, $i = 1, 2$, are known functions defined, respectively on I^2

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and I^4, and $N_i\big(u(x,t)\big)$, $i=1,2$, are polynomials of $u(x,t)$ with constant coefficients.

The numerical solution of two-dimensional integral equations has been subject of several studies. For example, in [5] Guoqiang et al. studied the numerical solution of two-dimensional nonlinear Volterra integral equations by collocation and iterated collocation methods. A method based on the use of rationalized Haar functions for the numerical solution of a class of nonlinear two-dimensional Volterra and Fredholm integral equations is proposed in [1]. Tari et al. [12] applied the differential transform method to obtain the solution of a class of linear and nonlinear two-dimensional Volterra integral equations. Nemati et al. [10] used shifted Legendre functions to get the solution of a class of nonlinear two-dimensional Volterra integral equations and two methods based on the use of operational matrices of block-pulse and shifted Legendre functions, are respectively proposed in [9] and [11].

In this paper, we use Bernoulli polynomials to approximate the solution of (1.1). For convenience, we assume that

$$N_i\big(u(x,t)\big) = u^{m_i}(x,t), \quad i=1,2, \tag{1.2}$$

where m_i, $i=1,2$, are positive integers, but the method can be easily extended and applied to any nonlinear 2DVFH integral equation of the form (1.1), where $N_i\big(u(x,t)\big)$, $i=1,2$, are polynomials of $u(x,t)$ with constant coefficients.

The reminder of this paper is organized as follows. We give some background on Bernoulli polynomials in Subsections 2.1 and 2.2. Bernoulli operational matrices of integration and the product are constructed in Subsection 2.3. In Section 3, the collocation method together with the Bernoulli operational matrices, which are derived in Section 2.3, are used to reduce the solution of (1.1) to the solution of a nonlinear system of algebraic equations. Some numerical examples are given in Section 4 to demonstrate the efficiency and accuracy of the method. Conclusions of the work are given in Section 5.

2. Bernoulli polynomials

In this section, we provide some background on the Bernoulli polynomials.

2.1. **Definition and some properties.**

The generalized Bernoulli polynomials $B_k^{(a)}(x)$ of degree k can be defined by the generating formula [8]

$$\frac{t^a e^{xt}}{(e^t-1)^a} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k^{(a)}(x), \quad |t| \le 2\pi.$$

If $a = 1$, we have the Bernoulli polynomials $B_k^{(1)}(x) \equiv B_k(x)$, and if, further, $x = 0$, we have the Bernoulli numbers $B_k(0) = B_k$. The Bernoulli polynomials $B_k(x)$ satisfy the familiar expansion [4]

$$\sum_{r=0}^{k-1} \binom{k}{r} B_r(x) = kx^{k-1}, \quad k = 1, 2, \ldots. \tag{2.1}$$

The Bernoulli polynomials also satisfy relations [4]

$$\begin{aligned} &B_k'(x) = kB_{k-1}(x), && k \geq 1,\\ &\int_0^1 B_k(x)dx = 0, && k \geq 1,\\ &B_k(x+1) - B_k(x) = kx^{k-1}, && k \geq 1,\\ &B_k(x) = \sum_{r=0}^{k} \binom{k}{r} B_r x^{k-r}, && k \geq 1. \end{aligned} \tag{2.2}$$

By relation (2.1), the Bernoulli polynomial vector

$$B(x) = [B_0(x), B_1(x), \ldots, B_N(x)]^T, \tag{2.3}$$

can be written in the form

$$B(x) = D^{-1} T_N(x), \tag{2.4}$$

where

$$T_N(x) = \left[1, x, x^2, \ldots, x^N\right]^T, \tag{2.5}$$

and D is a lower triangular matrix of the form

$$D = [d_{ij}]_{i,j=0}^N, \qquad d_{ij} = \begin{cases} \frac{1}{i+1}\binom{i+1}{j}, & 0 \leq j \leq i,\\ 0, & i < j \leq N. \end{cases}$$

Also, by varying k from 0 to N in the third part of Eq. (2.2), we get

$$B(x) = \widehat{D} T_N(x), \tag{2.6}$$

where $\widehat{D}$ is a lower triangular matrix as

$$\widehat{D} = [\hat{d}_{ij}]_{i,j=0}^N, \qquad \hat{d}_{ij} = \begin{cases} \binom{i}{i-j} B_{i-j}, & 0 \leq j \leq i,\\ 0, & i < j \leq N, \end{cases} \tag{2.7}$$

and $T_N(x)$ is the vector defined by Eq. (2.5). The dual matrix of $B(x)$ is defined by

$$Q = \int_0^1 B(x)B^T(x)dx = \widehat{D}H\widehat{D}^T, \tag{2.8}$$

where $\widehat{D}$ is the matrix defined in (2.7) and H is the Hilbert matrix

$$H = \int_0^1 T_N(x)T_N^T(x)dx = \left[\frac{1}{i+j+1}\right]_{i,j=0}^{N}.$$

2.2. **Function approximation and error analysis.** Let $\mathcal{C}^{n_1,\dots,n_r}(I^r)$ be the space of functions $f : I^r \to \mathbb{R}$ with continuous derivatives

$$f^{(i_1,\dots,i_r)}(x_1,\dots,x_r) = \frac{\partial^{i_1+\dots+i_r}}{\partial x^{i_1}\cdots\partial x^{i_r}} f(x_1,\dots,x_r), \quad (x_1,\dots,x_r) \in I^r,$$

for all $(i_1,\dots,i_r)$ such that $0 \le i_j \le n_j$, $0 \le j \le r$. Also, as usual, consider the supremum norm on I^r as

$$\|f\|_\infty = \sup_{x\in I^r} |f(x)|.$$

Suppose that $\mathcal{H} = \mathfrak{L}^2(I^2)$ be the space of square integrable functions with respect to the product Lebesgue measure on the unit square I^2. The inner product in this space is defined by

$$\langle f, g\rangle = \int_0^1\int_0^1 f(x,t)g(x,t)dxdt, \tag{2.9}$$

and the norm is as follows

$$\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}} = \Big(\int_0^1\int_0^1 f^2(x,t)dxdt\Big)^{\frac{1}{2}}. \tag{2.10}$$

We define the bivariate Bernoulli functions on I^2 as

$$B_{ij}(x,t) = B_i(x)B_j(t), \quad i,j = 0,1,...,$$

and put

$$\mathcal{H}_N = \operatorname{span}\{B_{ij}(x,t) : i,j = 0,1,\dots,N\}. \tag{2.11}$$

Since $\mathcal{H}_N$ is a finite dimensional subspace of $\mathcal{H}$, then it is closed [6] and for every given $g \in \mathcal{H}$ there exists a unique best approximation $\bar{g} \in \mathcal{H}_N$ [6] such that

$$\|g - \bar{g}\|_2 \le \|g - f\|_2, \quad \forall f \in \mathcal{H}_N. \tag{2.12}$$

Since $\bar{g} \in \mathcal{H}_N$, there exist unique coefficients g_{ij}, $i, j = 0, 1, \ldots, N$, such that

$$\begin{aligned} g(x,t) &\simeq \bar{g}(x,t) \\ &= \sum_{i,j=0}^{N} g_{ij} B_{ij}(x,t) \\ &= B^T(x,t)G, \end{aligned} \tag{2.13}$$

where

$$B(x,t) = B(x) \otimes B(t), \tag{2.14}$$

and G is the Bernoulli coefficient vector defined as

$$G = [g_{00}, \ldots, g_{0N}, g_{10}, \ldots, g_{1N}, \ldots, g_{N0}, \ldots, g_{NN}].$$

In (2.14), the notation $\otimes$ denotes the Kronecker product defined for two arbitrary matrices A and B as [7]

$$A \otimes B = (a_{ij}B).$$

Corollary 2.1 ([3])**.** *Suppose that $g(x,t) \in \mathcal{C}^{N,N}(I^2)$ is approximated by the two variable truncated Bernoulli series*

$$P_N[g](x,t) = \sum_{i,j=0}^{N} g_{ij} B_{ij}(x,t).$$

Then we have

$$g_{ij} = \frac{1}{i!j!} \int_0^1 \int_0^1 \frac{\partial^{i+j}}{\partial x^i \partial t^j} g(x,t) dx dt, \quad i, j = 0, 1, \ldots, N.$$

In the next theorem, an error term is provided for the approximation presented in Corollary 2.1.

Theorem 2.2 ([13])**.** *Let $g(x,t) \in \mathcal{C}^{N,N}(I^2)$ and $P_N[g](x,t)$ be its approximation in terms of Bernoulli polynomials. Then*

$$\begin{aligned} E(g) &= \|g(x,t) - P_N[g](x,t)\|_\infty \\ &\le C\hat{G}N(2\pi)^{-N}, \end{aligned}$$

where C is a positive constant independent of N, and $\hat{G}$ is such that

$$\left\| \frac{\partial^{i+j}}{\partial x^i \partial t^j} g(x,t) \right\|_\infty \le \hat{G}, \quad i, j = 0, 1, \ldots, N.$$

Let $k(x,t,y,z) \in \mathfrak{L}^2(I^4)$. By a similar argument to that has been used for (2.13), we can show that

$$k(x,t,y,z) \simeq \sum_{i,j,l,m=0}^{N} k_{ijlm} B_{ij}(x,t) B_{lm}(y,z) = B^T(x,t) K B(y,z), \tag{2.15}$$

where K is a block matrix of the form

$$K = [K^{(i,l)}]_{i,l=0}^N,$$

in which

$$K^{(i,l)} = [k_{ijlm}]_{j,m=0}^N, \quad i,l = 0,1,\ldots,N.$$

Also, Corollary 2.1 and Theorem 2.2 can be extended to the case of four variable functions as follows.

Corollary 2.3. *Assume that the function $k(x,t,y,z) \in \mathcal{C}^{N,N,N,N}(I^4)$ is approximated by Bernoulli polynomials as*

$$P_N[k](x,t,y,z) = B^T(x,t) K B(y,z),$$

then the coefficients k_{ijlm}, for all $i,j,l,m = 0,1,\ldots,N$, can be calculated from the following relation

$$k_{ijlm} = \frac{1}{i!j!l!m!} \int_0^1 \int_0^1 \int_0^1 \int_0^1 \frac{\partial^{i+j+l+m}}{\partial x^i \partial t^j \partial y^l \partial z^m} k(x,t,y,z) dx dt dy dz.$$

Proof. The proof proceeds in a similar manner as the one of Corollary 2.1. □

Theorem 2.4. *Suppose $k(x,t,y,z) \in \mathcal{C}^{N,N,N,N}(I^4)$ and $P_N[k](x,t,y,z)$ be its approximation in terms of Bernoulli polynomials. Then the error bound would be obtained as follows*

$$E(k) = \|k(x,t,y,z) - P_N[k](x,t,y,z)\|_\infty \leq C \hat{K} N (2\pi)^{-N},$$

where C is a positive constant independent of N, and $\hat{K}$ is such that

$$\left\| \frac{\partial^{i+j+l+m}}{\partial x^i \partial t^j \partial y^l \partial z^m} k(x,t,y,z) \right\|_\infty \leq \hat{K}, \quad i,j,l,m = 0,1,\ldots,N.$$

Proof. The proof proceeds in a similar manner as the one of Theorem 2.2. □

2.3. **Operational matrices.** It is shown in [2] that the integration of the vector $B(x)$ defined by (2.3) can be approximately obtained as

$$\int_0^x B(t)dt \simeq PB(x), \tag{2.16}$$

where

$$P = \begin{pmatrix} -B_1(0) & 1 & 0 & \dots & 0 \\ \frac{-B_2(0)}{2} & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{-B_N(0)}{N} & 0 & 0 & \dots & \frac{1}{N} \\ \frac{-B_{N+1}(0)}{N+1} & 0 & 0 & \dots & 0 \end{pmatrix}. \tag{2.17}$$

We call P as the one-dimensional Bernoulli operational matrix of integration.

Also, similar to Eq. (2.13), we can approximate the functions $x^k B_i(x)$ in terms of bernoulli polynomials as

$$x^k B_i(x) \simeq B^T(x) e_{k,i}, \quad i, k = 0, 1, \dots, n.$$

If, for $k = 0, 1, \dots, n$, we define the matrices E_k as

$$E_k = [e_{k,0}, e_{k,1}, \dots, e_{k,N}],$$

then for any arbitrary vector $C = [c_0, c_1, \dots, c_N]^T$ in $\mathbb{R}^{N+1}$ we can write [2]

$$B(x)B^T(x)C \simeq \hat{C}B(x), \tag{2.18}$$

where

$$\hat{C} = \hat{D}\tilde{C}^T,$$

in which $\hat{D}$ is the matrix defined by (2.7) and

$$\tilde{C} = \left[\tilde{E}_0, \tilde{E}_1, \dots, \tilde{E}_N\right],$$
$$\tilde{E}_k = E_k C, \quad k = 0, 1, \dots, N.$$

We call $\hat{C}$ as the one-dimensional Bernoulli operational matrix of the product.

The above results can be extended to the two-dimensional case by the following two theorems. We state these theorems without proof since the proofs are easy.

Theorem 2.5. *Assume that $B(x, t)$ is the Bernoulli vector defined in* (2.14)*. Then we have*

$$\int_0^t \int_0^x B(y, z)dydz \simeq P_0 B(x, t), \tag{2.19}$$

where

$$P_0 = P \otimes P. \tag{2.20}$$

We call P_0 as the two-dimensional Bernoulli operational matrix of integration which is of order $(N+1)^2 \times (N+1)^2$.

Theorem 2.6. *For any arbitrary vector C in $\mathbb{R}^{(N+1)^2}$ we have*

$$B(x,t)B^T(x,t)C \simeq \hat{C}B(x,t), \tag{2.21}$$

where

$$\hat{C} = (\hat{D} \otimes \hat{D})\tilde{C}^T,$$

in which $\tilde{C}$ is an square matrix of order $(N+1)^2$ of the form

$$\tilde{C} = \left[\tilde{B}_{0,0}, \dots, \tilde{B}_{0,N}, \dots, \tilde{B}_{N,0}, \dots, \tilde{B}_{N,N}\right],$$

where

$$\tilde{B}_{i,j} = (E_i \otimes E_j)C, \quad i,j = 0,1,\dots,N.$$

We call $\hat{C}$ as the two-dimensional Bernoulli operational matrix of the product which is of order $(N+1)^2 \times (N+1)^2$.

3. Numerical solution of nonlinear 2DVFH integral equations

In this section, we use the Bernoulli operational matrices and the collocation method to solve problem(1.1) with assumption (1.2) numerically. So, we consider the following integral equation

$$\begin{aligned} u(x,t) = f(x,t) &+ \int_0^t \int_0^x k_1(x,t,y,z)u^{m_1}(y,z)dydz \\ &+ \int_0^1 \int_0^1 k_2(x,t,y,z)u^{m_2}(y,z)dydz, \quad (x,t) \in I^2. \end{aligned} \tag{3.1}$$

If we approximate functions $f(x,t)$, $u(x,t)$, $u^{m_i}(x,t)$ and $k_i(x,t,y,z)$ in terms of Bernoulli polynomials, as described by equations (2.13) and (2.15), then we obtain

$$\begin{aligned} f(x,t) &\simeq B^T(x,t)F, \\ u(x,t) &\simeq B^T(x,t)U, \\ u^{m_i}(x,t) &\simeq B^T(x,t)U^{(m_i)}, \quad i = 1,2, \\ k_i(x,t,y,z) &\simeq B^T(x,t)K_iB(y,z), \quad i = 1,2, \end{aligned} \tag{3.2}$$

where the vectors U, $U^{(m_i)}$ and matrices K_i are Bernoulli polynomial coefficients of $u(x,t)$, $u^{m_i}(x,t)$ and $k_i(x,t,y,z)$ respectively.

For the numerical implementation of the presented method, we need to express the components of vectors $U^{(m_i)}$, $i = 1, 2$, as nonlinear functions of the elements of the vector U. To do this, we state the following lemma.

Lemma 3.1. *Let m_i, $i = 1, 2$, be positive integers and U and $U^{(m_i)}$, $i = 1, 2$, be the Bernoulli coefficient vectors of functions $u(x, t)$ and $u^{m_i}(x, t)$, $i = 1, 2$, respectively. Then, we have*

$$U^{(m_i)} \simeq (Q \otimes Q)^{-1} \hat{U}^{m_i} (e_1 \otimes e_1), \quad i = 1, 2, \tag{3.3}$$

where e_1 denotes the first standard unit vector of order $(N + 1)$.

Proof. With the aid of relations (2.14) and (2.8), we have

$$\int_0^1 \int_0^1 B(x, t) B^T(x, t) dx dt = Q \otimes Q. \tag{3.4}$$

Therefore, for $i = 1, 2$, we can write

$$(Q \otimes Q) U^{(m_i)} = \int_0^1 \int_0^1 B(x, t) B^T(x, t) U^{(m_i)} dx dt.$$

Also, using (3.2), (2.21) and (2.2), we can write

$$\begin{aligned}
\int_0^1 \int_0^1 B(x, t) B^T(x, t) U^{(m_i)} dx dt &\simeq \int_0^1 \int_0^1 B(x, t) u^{m_i}(x, t) dx dt \\
&\simeq \int_0^1 \int_0^1 B(x, t) \Big(B^T(x, t) U\Big)^{m_i} dx dt \\
&\simeq \hat{U} \int_0^1 \int_0^1 B(x, t) \Big(B^T(x, t) U\Big)^{m_i - 1} dx dt \\
&\simeq \hat{U}^2 \int_0^1 \int_0^1 B(x, t) \Big(B^T(x, t) U\Big)^{m_i - 2} dx dt \\
&\vdots \\
&\simeq \hat{U}^{m_i} \int_0^1 \int_0^1 B(x, t) dx dt \\
&= \hat{U}^{m_i} \left(\int_0^1 B(x) dx\right) \otimes \left(\int_0^1 B(t) dt\right) \\
&= \hat{U}^{m_i} (e_1 \otimes e_1), \quad i = 1, 2.
\end{aligned}$$

Since Q is invertible, so is $Q \otimes Q$ and we obtain (3.3). $\square$

Using relations (3.2), (2.19), (2.21) and (3.4), the Volterra and the Fredholm parts of Eq. (3.1) may be written respectively as

(3.5)
$$\begin{aligned}\int_0^t\int_0^x k_1(x,t,y,z)u^{m_1}(y,z)dydz &\simeq B^T(x,t)K_1\int_0^t\int_0^x B(y,z)B^T(y,z)U^{(m_1)}dydz\\ &\simeq B^T(x,t)K_1\int_0^t\int_0^x \widehat{U^{(m_1)}}B(y,z)dydz\\ &= B^T(x,t)K_1\widehat{U^{(m_1)}}\int_0^t\int_0^x B(y,z)dydz\\ &\simeq B^T(x,t)K_1\widehat{U^{(m_1)}}P_0B(x,t),\end{aligned}$$

and

(3.6)
$$\begin{aligned}\int_0^1\int_0^1 k_2(x,t,y,z)u^{m_2}(y,z)dydz &\simeq B^T(x,t)K_2\Big(\int_0^1\int_0^1 B(y,z)B^T(y,z)dydz\Big)U^{(m_2)}\\ &= B^T(x,t)K_2(Q\otimes Q)U^{(m_2)},\end{aligned}$$

where P_0 is the two-dimensional Bernoulli operational matrix of integration introduced in (2.20) and Q is the matrix given in (2.8).

With substituting approximations (3.5), (3.6) and (3.2) into (3.1) and then collocating the resulting equation at the Newton-Cotes nodes

$$(x_l,t_j)=\left(\frac{2l+1}{2(N+1)},\frac{2j+1}{2(N+1)}\right),\quad l,j=0,1,\dots,N, \tag{3.7}$$

we obtain

$$\begin{aligned}B^T(x_l,t_j)U \simeq B^T(x_l,t_j)F + B^T(x_l,t_j)K_1\widehat{U^{(m_1)}}P_0B(x_l,t_j)\\ + B^T(x_l,t_j)K_2(Q\otimes Q)U^{(m_2)}.\end{aligned} \tag{3.8}$$

Since $U^{(m_i)}$, $i=1,2$, are column vectors whose elements are nonlinear functions of the elements of the unknown vector $U=[u_{00},u_{01},\dots,u_{NN}]$, then equation (3.8) is a set of $(N+1)^2$ nonlinear algebraic equations with $(N+1)^2$ unknowns u_{00}, u_{01}, …, u_{NN}. This nonlinear system of algebraic equations can be solved by numerical methods such as Newton's iterative method. If $\bar{U}$ be an approximate solution of this system, then $\bar{u}_N(x,t)=B^T(x,t)\bar{U}$ is an approximate solution of equation (3.1).

4. Numerical results

In order to analyze the error of the method, we introduce notation

$$\bar{e}_N(x,t)=u(x,t)-\bar{u}_N(x,t),$$

where $\bar{u}_N(x,t)$ denotes the approximate solution of order N, obtained by the method described in Section 3, and $u(x,t)$ is the exact solution of integral equation.

Experiments were performed on a personal computer using a 2.50 GHz processor and the codes were written in Mathematica 9. In the considered examples, we have solved system (3.8) using the Mathematica function FindRoot, which uses Newton's method as the default method.

Example 4.1. Consider the following nonlinear 2D Volterra integral equation [1]

$$u(x,t) = f(x,t) + \int_0^t \int_0^x \left(xy^2 + \cos(z)\right) u^2(y,z)dydz, \quad (x,t) \in I^2,$$

where

$$f(x,t) = x\sin(t)\left(1 - \frac{1}{9}x^2\sin^2(t)\right) + \frac{1}{10}x^6\left(\frac{1}{2}\sin(2t) - t\right).$$

The exact solution is $u(x,t) = x\sin(t)$. In Table 1 the numerical results for this example with $N = 2, 4$ are displayed together with the results obtained in [1] using 2D rationalized Haar functions. The error graph of $|\bar{e}_N(x,t)|$ is plotted in Fig. 1 for $N = 7$.

TABLE 1. Numerical results for Example 4.1.

$(x,t) = (\frac{1}{2^i}, \frac{1}{2^i})(i)$	Method of [1]		Present method	
	$m = 16$	$m = 32$	$N = 2$	$N = 4$
0	–	–	1.7×10^{-3}	2.8×10^{-4}
1	2.9×10^{-2}	1.4×10^{-2}	3.6×10^{-5}	1.0×10^{-6}
2	1.6×10^{-2}	7.9×10^{-3}	5.8×10^{-5}	9.9×10^{-6}
3	8.7×10^{-3}	4.1×10^{-3}	3.4×10^{-5}	2.9×10^{-6}
4	4.9×10^{-3}	2.2×10^{-3}	1.7×10^{-5}	8.9×10^{-7}
5	1.6×10^{-7}	1.2×10^{-3}	5.3×10^{-6}	2.5×10^{-7}
6	7.3×10^{-4}	9.3×10^{-9}	1.5×10^{-6}	6.6×10^{-8}

Example 4.2. Consider the following nonlinear 2D Fredholm integral equation [1]

$$u(x,t) = f(x,t) + \int_0^1 \int_0^1 \frac{x}{1+t}(1+y+z)^2 u^2(y,z)dydz, \quad (x,t) \in I^2,$$

where

$$f(x,t) = \frac{1}{(1+x+t)^2} - \frac{x}{6(1+t)},$$

and the exact solution is $u(x,t) = \frac{1}{(1+x+t)^2}$. In Table 2 the numerical results for this example with $N = 2, 4$ are displayed together with the

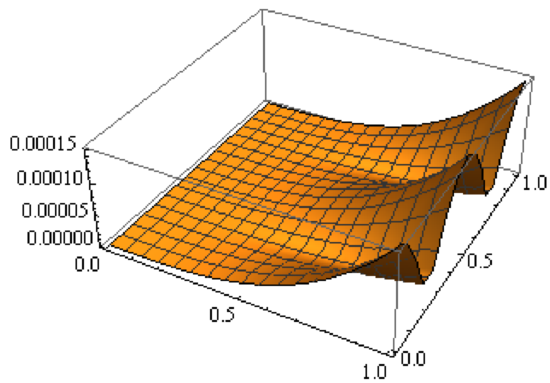

FIGURE 1. Graph of $|\bar{e}_N(x,t)|$ for Example 4.1 with $N = 7$.

results obtained in [1] using 2D rationalized Haar functions. The error graph of $|\bar{e}_N(x,t)|$ is plotted in Fig. 2 for $N = 7$.

TABLE 2. Numerical results for Example 4.2.

$(x,t) = (\frac{1}{2^i}, \frac{1}{2^i})(i)$	Method of [1]		Present method	
	$m = 16$	$m = 32$	$N = 2$	$N = 4$
0	–	–	3.4×10^{-3}	9.1×10^{-4}
1	1.2×10^{-2}	7.9×10^{-3}	1.2×10^{-3}	1.9×10^{-4}
2	2.1×10^{-2}	1.2×10^{-2}	1.4×10^{-3}	3.4×10^{-4}
3	2.4×10^{-2}	1.3×10^{-2}	8.3×10^{-4}	2.0×10^{-4}
4	2.4×10^{-2}	1.4×10^{-2}	4.1×10^{-4}	9.8×10^{-5}
5	6.6×10^{-3}	1.4×10^{-2}	2.0×10^{-4}	4.7×10^{-5}
6	9.0×10^{-3}	1.9×10^{-3}	9.6×10^{-5}	2.3×10^{-5}

Example 4.3. Consider the following nonlinear 2D Volterra-Fredholm-Hammerstein integral equation

$$u(x,t) = f(x,t) + \int_0^t \int_0^x xzu^2(y,z)dydz + \int_0^1 \int_0^1 (y-t)u(y,z)dydz, \quad (x,t) \in I^2. \tag{4.1}$$

The function $f(x,t)$ was chosen so that the analytical solution of (4.1) is $u(x,t) = \cos(x+t)$. Table 3 shows the numerical results for this example. The error graph of $|\bar{e}_N(x,t)|$ is plotted in Fig. 3 for $N = 7$.

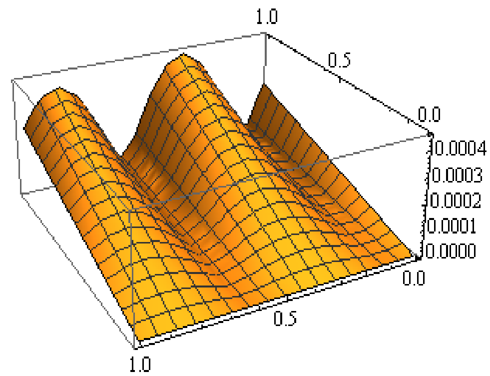

FIGURE 2. Graph of $|\bar{e}_N(x,t)|$ for Example 4.2 with $N = 7$.

TABLE 3. Numerical results for Example 4.3.

$(x,t) = (\frac{1}{2^i}, \frac{1}{2^i})(i)$	$N = 2$	$N = 4$	$N = 6$
0	1.6×10^{-3}	3.4×10^{-6}	3.6×10^{-6}
1	1.4×10^{-3}	8.8×10^{-5}	1.1×10^{-5}
2	3.3×10^{-3}	3.3×10^{-4}	4.5×10^{-5}
3	2.7×10^{-3}	1.7×10^{-4}	2.2×10^{-5}
4	2.6×10^{-3}	1.5×10^{-4}	1.8×10^{-5}
5	2.6×10^{-3}	1.5×10^{-4}	1.8×10^{-5}
6	2.6×10^{-3}	1.5×10^{-4}	1.8×10^{-5}

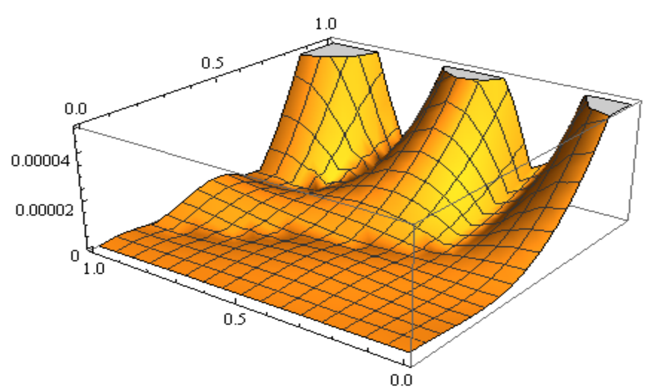

FIGURE 3. Graph of $|\bar{e}_N(x,t)|$ for Example 4.3 with $N = 7$.

5. Conclusion

In the present method, the Bernoulli operational matrices of integration and the product together with the collocation method are used to solve problem (1.1) with assumption (1.2) numerically. This approach transformed the considered problem to a nonlinear system of algebraic equations with unknown Bernoulli coefficients of the exact solution. In the numerical examples presented in the paper, we have solved this system using the Mathematica function FindRoot, which uses Newton's method as the default method. As it is illustrated by the presented examples, high accuracy results can be achieved only using a small number of basis functions.

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