Sahand Communications in Mathematical Analysis (SCMA) Vol. 5 No. 1 (2017), 61-67
http://scma.maragheh.ac.ir

OZAKI'S CONDITIONS FOR GENERAL INTEGRAL OPERATOR

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Abstract. Assume that $\mathbb{D}$ is the open unit disk. Applying Ozaki's conditions, we consider two classes of locally univalent, which denote by $\mathcal{G}(\alpha)$ and $\mathcal{F}(\mu)$ as follows

$$\mathcal{G}(\alpha) := \left\{ f \in \mathcal{A} : \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}, \quad 0 < \alpha \leq 1 \right\},$$

and

$$\mathcal{F}(\alpha) := \left\{ f \in \mathcal{A} : \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{1}{2} - \mu, \quad -1/2 < \mu \leq 1 \right\},$$

respectively, where $z \in \mathbb{D}$ and $\mathcal{A}$ is class of normalized functions. In this paper, we study the mapping properties of this classes under general integral operator. We also, obtain some conditions for integral operator to be convex or starlike function.

1. Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The subclass of $\mathcal{A}$ consisting of all univalent functions $f(z)$ in $\mathbb{D}$ is denoted by $\mathcal{U}$. A function $f \in \mathcal{U}$ is called starlike (with respect to 0), denoted by $f \in \mathcal{S}^*$, if $tw \in f(\mathbb{D})$ whenever $w \in f(\mathbb{D})$ and $t \in [0, 1]$. Also, a function $f \in \mathcal{U}$ that maps $\mathbb{D}$ onto a convex domain, denoted by $f \in \mathcal{K}$,

2010 *Mathematics Subject Classification.* 30C45.
Key words and phrases. Starlike function, Convex function, Locally univalent, Integral operator, Ozaki's conditions.
Received: 09 November 2015, Accepted: 03 January 2016.
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is called a convex function. The class $\mathcal{S}^*(\gamma)$ a of starlike functions of order $0 \leq \gamma < 1$ may be defined as

$$\mathcal{S}^*(\gamma) := \left\{ f \in \mathcal{A} : \mathfrak{Re}\left(\frac{zf'(z)}{f(z)}\right) > \gamma, \quad z \in \mathbb{D} \right\}.$$

The class $\mathcal{S}^*(\gamma)$ and the class $\mathcal{K}(\gamma)$ of convex functions of order $0 \leq \gamma < 1$

$$\mathcal{K}(\gamma) := \left\{ f \in \mathcal{A} : \mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \gamma, \quad z \in \mathbb{D} \right\},$$

were introduced by Robertson in [13]. If $\gamma \in [0,1)$, then a function in either of these sets is univalent. In particular we denote $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{K}(0) \equiv \mathcal{K}$. The univalence or the starlikeness, or more, the convexity in one direction of functions convex of negative order was considered in the past by Ozaki [10], Umezawa [11, 14] and others. In this work we return to Ozaki's conditions.

In [10], Ozaki proved that if $f(z)$ given by (1.1) is analytic in $\mathbb{D}$, with $f(z)f'(z)/z \neq 0$, there, and if either

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \geq -\frac{1}{2}, \qquad z \in \mathbb{D},$$

or

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) \leq \frac{3}{2}, \qquad z \in \mathbb{D},$$

holds throughout $\mathbb{D}$, then f is univalent and convex in at least one direction in $\mathbb{D}$. The following interesting result was published in a minor journal and so, it was not well known in the public of univalent function theory. It shows that the constants $-1/2$ and $3/2$ are, in a certain sense, the best possible.

Following [9], for $\alpha \in \mathbb{R}$ we consider the class $\mathcal{G}(\alpha)$ consisting of locally univalent functions $f \in \mathcal{A}$ which satisfy the condition

$$\mathfrak{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}, \qquad z \in \mathbb{D}. \tag{1.2}$$

It is easy to see that $\mathcal{G}(\alpha) \neq \emptyset$ iff $\alpha > 0$, and we will make this assumption on α in the sequel. Since $\mathcal{G}(\alpha) \subset \mathcal{G}(\alpha')$ whenever $\alpha < \alpha'$, it readily follows that the class $\mathcal{G}(\alpha)$ is included in $\mathcal{S}^*$, whenever $\alpha \in (0,1]$, so in particular the functions in $\mathcal{G}(\alpha)$ are univalent functions. It can be easily seen that functions in $\mathcal{G}(\alpha)$ are not necessarily univalent in $\mathbb{D}$ if $\alpha > 1$, as shown by the example below.

Example 1.1. Consider $a > 0$ and let $f : \mathbb{D} \to \mathbb{C}$ be defined by $f(z) = \frac{1}{a}\left(1 - e^{-az}\right)$. It is easily seen that the function f belongs to the class $\mathcal{A}$

and it is locally univalent. Since

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)=\mathfrak{Re}\,(1-az)<1+a,\qquad z\in\mathbb{D},$$

it follows that $f\in\mathcal{G}\,(2a)$. If $a>\pi$ the function f is not univalent in $\mathbb{D}$, and therefore it follows that for $\alpha>2\pi=6.28...$ the class $\mathcal{G}\,(\alpha)$ does not consist entirely of univalent functions. It is an open question whether $\mathcal{G}\,(\alpha)\subset\mathcal{U}$ for $1\leq\alpha\leq 2\pi$.

Let $\mathcal{F}(\mu)$ denote the class of locally univalent normalized analytic functions f in the unit disk $|z|<1$ satisfying the condition

$$\mathfrak{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{1}{2}-\mu,\qquad |z|<1, \tag{1.3}$$

for some $-1/2<\mu\leq 1$. The class $\mathcal{F}(1)$ was studied by Ponnusamy et al. [12]. Also, $\mathcal{F}(1/2)=\mathcal{K}$. Clearly, $\mathcal{F}(\mu)\subset\mathcal{K}\subset\mathcal{S}^*$ for all $\mu\in(-1/2,1/2)$. Functions $\mathcal{F}(1)$ are known to be univalent and close-to-convex in $\mathbb{D}$, i.e., complement of can be written as the union of non-intersecting half-lines (see [4, 5, 6, 7]).

Generally, the class $\mathcal{F}(\mu)$ plays a crucial role in the discussion on certain extremal problems for the class of complex-valued and sense-preserving harmonic convex functions and some other related problems in determining univalence criteria for sense-preserving harmonic mappings (see [3, 8]).

Discussions will continue with a example.

Example 1.2. Consider $\beta>0$ and let $f_\beta:\mathbb{D}\to\mathbb{C}$ be defined by $f_\beta(z)=\frac{1}{\beta}\left(1-e^{-\beta z}\right)$. It is easily seen that the function f belongs to the class $\mathcal{A}$ and it is convex function. Since

$$\mathfrak{Re}\left(1+\frac{zf_\beta''(z)}{f_\beta'(z)}\right)=\mathfrak{Re}\,(1-\beta z)>1-\beta,\qquad z\in\mathbb{D},$$

it follows that $f\in\mathcal{F}\,(\beta-1/2)$. In particular, the function $f_{3/2}(z)\in\mathcal{F}(1)$. As a result, the function $f_{3/2}(z)$ is a univalent and close-to-convex function (see [11]). However, the function $f_\beta(z)$ belongs to the class $\mathcal{F}(\mu)$ iff $0<\beta\leq 3/2$.

Recently D. Breaz et al. [2], for analytic functions f_j $(j=1,2,\ldots,n)$, given by (1.1) introduced the general integral operator $G_\beta(z)$ by

$$G_\beta(z)=\int_0^z\prod_{j=1}^{n}\left(\frac{f_j(t)}{t}\right)^{\beta_j}dt, \tag{1.4}$$

where $\beta_j > 0$ $(j = 1, 2, \ldots, n)$, and $\sum_{j=1}^{n} \beta_j = \beta$. A simple computation shows that

$$G'_\beta(z) = \prod_{j=1}^{n} \left(\frac{f_j(z)}{z} \right)^{\beta_j}, \tag{1.5}$$

and

$$zG''_\beta(z) = \prod_{j=1}^{n} \left(\frac{f_j(z)}{z} \right)^{\beta_j} \sum_{j=1}^{n} \beta_j \left(\frac{zf'_j(z)}{f_j(z)} - 1 \right). \tag{1.6}$$

2. Main Results

One of our main results is contained in the following theorem:

Theorem 2.1. *Assume that $\alpha \in \mathbb{R}$, $\beta_j > 0$ and $\sum_{j=1}^{n} \beta_j = \beta$. If $2\beta + 1 < \alpha < 2$ and $f_j \in \mathcal{A}$ satisfy the inequality*

$$\left| \frac{zf'_j(z)}{f_j(z)} - 1 \right| < 1, \qquad (j = 1, 2, \ldots, n), \tag{2.1}$$

then

$$G_\beta(z) = \int_0^z \prod_{j=1}^{n} \left(\frac{f_j(t)}{t} \right)^{\beta_j} dt \in \mathcal{G}(\alpha).$$

Proof. It is easy to see that

$$f \in \mathcal{G}(\alpha) \quad \Leftrightarrow \quad \mathfrak{Re} \left(\frac{\alpha f'(z) - 2zf''(z)}{2f'(z)} \right) > 0, \qquad z \in \mathbb{D}.$$

By using

$$|w - 1| < 1 \quad \Leftrightarrow \quad \mathfrak{Re} \frac{1}{w} > \frac{1}{2}, \tag{2.2}$$

(see [1]), we let

$$\frac{1}{w} = \frac{\alpha G'_\beta(z) - 2zG''_\beta(z)}{2G'_\beta(z)}, \qquad z \in \mathbb{D},$$

where $G_\beta(z)$ defined by (1.4). Then

$$|w - 1| = \left| \frac{(2 - \alpha)G'_\beta(z) + 2zG''_\beta(z)}{\alpha G'_\beta(z) - 2zG''_\beta(z)} \right|, \qquad z \in \mathbb{D}.$$

From (2.1), (1.5) and (1.6) and make a simple computation to get

$$\begin{aligned}|w-1| &= \left|\frac{2-\alpha+\sum_{j=1}^{n} 2\beta_j\left(\frac{zf_j'(z)}{f_j(z)}-1\right)}{\alpha-\sum_{j=1}^{n} 2\beta_j\left(\frac{zf_j'(z)}{f_j(z)}-1\right)}\right| \\ &< \frac{2-\alpha+2\beta}{\alpha-2\beta} \\ &= \frac{2}{\alpha-2\beta}-1 \\ &< 1.\end{aligned}$$

In view of conclusion (2.2), we have $G_\beta(z) \in \mathcal{G}(\alpha)$. This completes the proof. □

Theorem 2.2. *Let $\beta_j > 0$, $\sum_{j=1}^{n} \beta_j = \beta$ and $\alpha \in \mathbb{R}$. If $2 < \alpha < 3$ and $f_j \in \mathcal{A}$ satisfy the inequality*

$$\left|\frac{zf_j'(z)}{f_j(z)}-1\right| < \frac{1}{2\beta}, \qquad (j=1,2,\ldots,n),$$

then

$$G_\beta(z) = \int_0^z \prod_{j=1}^{n}\left(\frac{f_j(t)}{t}\right)^{\beta_j} dt \in \mathcal{G}(\alpha).$$

Proof. The proof is similar of Theorem 2.1 and we omit the details. □

Theorem 2.3. *Let $\mu \in (-1/2, 1]$ and $f \in \mathcal{A}$. If $G_\beta(z) \in \mathcal{F}(\mu)$ then*

$$\left|\frac{zf_j'(z)}{f_j(z)}-1\right| \geq 1, \qquad (j=1,2,\ldots,n),$$

where $G_\beta(z)$ defined by (1.4).

Proof. From definition of $\mathcal{F}(\mu)$ we have

$$f \in \mathcal{F}(\mu) \quad \Leftrightarrow \quad \mathfrak{Re}\left(\frac{(1+2\mu)f'(z)+2zf''(z)}{2f'(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

Assume that

$$\frac{1}{w} = \frac{(1+2\mu)G_\beta'(z)+2zG_\beta''(z)}{2G_\beta'(z)}, \qquad z \in \mathbb{D},$$

then we have

$$\begin{aligned}|w-1| &= \left|\frac{(1-2\mu)G_\beta'(z)-2zG_\beta''(z)}{(1+2\mu)G_\beta'(z)+2zG_\beta''(z)}\right| \\ &\leq \frac{2\mu-1+2\beta}{1-2\mu-2\beta}.\end{aligned}$$

Since $\frac{2\mu-1+2\beta}{1-2\mu-2\beta}$ is negative, we conclude that $\mathfrak{Re}\{1/w\} \not> 0$. This completes the proof. □

Theorem 2.4. *If $\gamma \in [0,1)$, $0 < \beta + \gamma < 1/2$ and $f \in \mathcal{A}$ satisfy the inequality*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| < 1, \qquad (j = 1, 2, \ldots, n),$$

then $G_\beta(z) \in \mathcal{K}(\gamma)$, where $G_\beta(z)$ defined by (1.4).

Proof. We se that

$$f \in \mathcal{K}(\gamma) \quad \Leftrightarrow \quad \mathfrak{Re}\left(\frac{(1-\gamma)f'(z) + zf''(z)}{f'(z)}\right) > 0, \qquad z \in \mathbb{D}.$$

If we take

$$\frac{1}{w} = \frac{(1-\gamma)G_\beta'(z) + zG_\beta''(z)}{G_\beta'(z)}, \qquad z \in \mathbb{D},$$

then we have $|w - 1| < \frac{\beta+\gamma}{1-(\gamma+\beta)}$. This shows that $G_\beta(z)$ is convex operator.

□

Corollary 2.5. *Assume that $0 < \beta < 1/2$ and f satisfy the inequality*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| < 1, \qquad (j = 1, 2, \ldots, n),$$

then $G_\beta(z)$ is a convex function.

Corollary 2.6. *Assume that $0 < \beta+\gamma < 1/2$ and f satisfy the inequality*

$$\left|\frac{zf_j'(z)}{f_j(z)} - 1\right| < 1, \qquad (j = 1, 2, \ldots, n),$$

then

$$z\prod_{j=1}^{n}\left(\frac{f_j(z)}{z}\right)^{\beta_j} \in \mathcal{S}^*(\gamma).$$

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