PARABOLIC STARLIKE Mappings of the Unit Ball $B^n$

SAMIRA RAHROVI

Abstract. Let $f$ be a locally univalent function on the unit disk $U$. We consider the normalized extensions of $f$ to the Euclidean unit ball $B^n \subset \mathbb{C}^n$ given by

$$
\Phi_{n,\gamma}(f)(z) = \left( f(z_1), \left( f'(z_1) \right)^\gamma \hat{z} \right),
$$

where $\gamma \in [0, 1/2]$, $z = (z_1, \hat{z}) \in B^n$ and

$$
\Psi_{n,\beta}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^\beta \hat{z} \right),
$$

in which $\beta \in [0, 1]$, $f(z_1) \neq 0$ and $z = (z_1, \hat{z}) \in B^n$. In the case $\gamma = 1/2$, the function $\Phi_{n,\gamma}(f)$ reduces to the well known Roper-Suffridge extension operator. By using different methods, we prove that if $f$ is parabolic starlike mapping on $U$ then $\Phi_{n,\gamma}(f)$ and $\Psi_{n,\beta}(f)$ are parabolic starlike mappings on $B^n$.

1. Introduction

Let $\mathbb{C}^n$ be the vector space of $n$-complex variables $z = (z_1, \ldots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{k=1}^{n} z_k \bar{w}_k$ and Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$. The open ball $\{z \in \mathbb{C}^n : \|z\| < r\}$ is denoted by $B^n_r$ and the unit ball $B^n_1$ by $B^n$. In the case of one complex variable, $B^1$ is denoted by $U$. It is convenient, if $n \geq 2$ to write a vector $z \in \mathbb{C}^n$ as $z = (z_1, \hat{z})$, where $z_1 \in \mathbb{C}$ and $\hat{z} = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$.

Let $H(B^n, \mathbb{C}^n)$ denotes the topological vector space of all holomorphic mappings $F : B^n \rightarrow \mathbb{C}^n$. Let $F \in H(B^n)$, we say that $F$ is normalized if $F(0) = 0$ and $DF(0) = I$, where $DF$ is the Fréchet differential of $F$ and $I$ is the identity operator on $\mathbb{C}^n$. Let $S(B^n)$ be the set of normalized holomorphic functions.

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biholomorphic mappings on $B^n$, and $S_1 = S$ is the classical family of univalent mappings of $U$.

A map $f \in S(B^n)$ is said to be a convex if its image is convex domain in $\mathbb{C}^n$, and starlike if its image is a starlike domain with respect to 0. We denote the classes of normalized convex and starlike mappings on $B^n$ respectively by $K(B^n)$ and $S^*(B^n)$.

In 1995, Roper and Suffridge [8] introduced an extension operator which gives a way of extending a (locally) univalent function on the unit disk $U$ to a (locally) univalent mapping of $B^n$ into $\mathbb{C}^n$. This operator is defined for a normalized locally biholomorphic function $f$ in the unit disk $U$ in $\mathbb{C}$ by (see [3] and [8])

$$[\Phi_n(f)](z) = \left( f(z_1), \sqrt{f'(z_1)} \hat{z} \right),$$

where $z = (z_1, \hat{z}) \in B^n$ and we choose the branch of the square root such that $\sqrt{f'(z_1)}|_{z_1=0} = 1$.

The following results illustrate the importance and usefulness of the Roper-Suffridge extension operator

$$\Phi_n(K) \subseteq K(B^n), \quad \Phi_n(S^*) \subseteq S^*(B^n).$$

The first was proved by Roper and Suffridge when they introduced their operator [8], while the second result was given by Graham and Kohr [11]. Till now, it has been difficult to make constant the concrete convex mappings and starlike mappings on $B^n$. By making use of the Roper-Suffridge extension operator, we may easily give many concrete examples about these mappings. This is one important reason why people are interested in this extension operator. A good treatment of further applications of the Roper-Suffridge extension operator can be found in the recent book by Graham and Kohr [3].

The authors [3] considered the following operator

$$\Phi_{n,\gamma}(f)(z) = (f(z_1), (f'(z_1))^\gamma \hat{z}), \quad z = (z_1, \hat{z}) \in B^n,$$

where $\gamma \in [0, 1/2]$ and $f$ is a locally univalent function in $U$, normalized by $f(0) = f'(0) - 1 = 0$. We choose the branch of the power function such that $(f'(z_1))^\gamma|_{z_1=0} = 1$. Of course when $\gamma = 1/2$, we obtain the Roper-Suffridge extension operator. In [11], a number of extension results were obtained related to the operator $\Phi_{n,\gamma}$, $\gamma \in [0, 1/2]$; if $f \in S$, then $\Phi_{n,\gamma}(f)$ can be embedded in a Loewner chain and moreover $\Phi_{n,\gamma}(f) \in S^0(B^n)$. In particular, if $f \in S^*$, then $\Phi_{n,\gamma}(f) \in S^*(B^n)$. It was also proved that convexity is preserved only if $\gamma = 1/2$. 
In [2], Graham and Kohr introduced another extension operator for the locally biholomorphic function $f$ on $U$ by

$$
\Psi_{n,\beta}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^\beta \hat{z} \right), \quad z = (z_1, \hat{z}) \in B^n,
$$

where $\beta \in [0,1]$ and $f(z_1) \neq 0$, when $z_1 \in U \setminus \{0\}$, and we choose the branch of the power function such that $\left( \frac{f(z_1)}{z_1} \right)^\beta |_{z_1=0} = 1$. They proved that the operator $\Psi_{n,\beta}(f)$ maps the normalized starlike function on $U$ to a normalized starlike mapping on $B^n$. When $\beta = 1$, it was proved and discussed by Pfaltzgraff and Suffridge [7].

**Remark 1.1.** Let $g : U \to \mathbb{C}$ be a holomorphic univalent function such that $g(0) = 1$, $g(\bar{\eta}) = \overline{g(\eta)}$ for $\eta \in U$ (so, $g$ has real coefficients in its power series expansion), $\text{Reg}(\eta) > 0$ on $U$ and assume $g$ satisfies the following

$$
\begin{aligned}
\min_{|\eta|=r} \text{Reg}(\eta) &= \min \{g(r), g(-r)\}, \\
\max_{|\eta|=r} \text{Reg}(\eta) &= \max \{g(r), g(-r)\},
\end{aligned}
$$

for $r \in (0,1)$.

For example, the condition (1.1) is satisfied by all functions which are convex in the direction of the imaginary axis and symmetric about the real axis (see [3]).

Let

$$
\mathcal{M}_g = \left\{ h \in H(B^n) : h(0) = 0, \quad Dh(0) = I_n, \quad \left( h(z), \frac{z}{\|z\|^2} \right) \in g(U), \quad z \in B^n \setminus \{0\} \right\}.
$$

For $g(\xi) = \frac{1+i\xi}{1+i\bar{\xi}}$, $\xi \in U$, we obtain the well known set $\mathcal{M}_g = \mathcal{M}$ of mapping with a positive real part on $B^n$, i.e.

$$
\mathcal{M}_g = \left\{ h \in H(B^n) : h(0) = 0, \quad Dh(0) = I_n, \quad \text{Re} \left( h(z), \frac{z}{\|z\|^2} \right) > 0, \quad z \in B^n \setminus \{0\} \right\}.
$$

Now, we give the definition of parabolic starlike mappings on $B^n$ (see [3]). Let

$$
q(\eta) = 1 + \frac{4}{\pi^2} \left( \log \frac{1 + \sqrt{\eta}}{1 - \sqrt{\eta}} \right)^2.
$$
Then $q$ is a biholomorphic mapping from $U$ onto domain $\Omega$, where

$$\Omega = \{ w = u + iv : v^2 < 4u \}$$

$$= \{ w : |w - 1| < 1 + \Re w \}.$$

We note that $\Omega$ is a parabolic region in the right half-plane.

**Definition 1.2.** Let $f$ be a normalized locally biholomorphic mapping on $B^n$, we say that $f$ is a parabolic starlike mapping if

$$\left\langle \left[ Df(z) \right]^{-1} f(z), \frac{z}{\|z\|^2} \right\rangle \in g(U), \quad z \in B^n \setminus \{0\},$$

where $g = \frac{1}{q}$.

Let $f$ be a parabolic starlike mapping on $B^n$. Since

$$\Re \left\langle \left[ Df(z) \right]^{-1} f(z), z \right\rangle > 0,$$

parabolic starlike mappings are starlike mappings by Suffridge [9].

In order to prove the main results, we need the following lemma:

**Lemma 1.3.** [5]. Let $g = \frac{1}{q}$. Then, $g(U)$ is starlike with respect to 1.

## 2. Main Results

**Theorem 2.1.** Let $f : U \to C$ be a normalized locally univalent function, which satisfies the condition

$$|z_1 f'(z_1) - 1| < 1, \quad z_1 \in U.$$  

(2.1)

Also, let $F = \Phi_{n, \gamma}(f)$, then

$$\left| \frac{\|z\|^2}{\langle DF^{-1}(z) F(z), z \rangle} - 1 \right| < 1, \quad z \in B^n \setminus \{0\},$$

and hence $F$ is a parabolic starlike mapping on $B^n$.

**Proof.** Without loss of generality, we may assume that $f$ is holomorphic on the closed unit disk $\overline{U}$, since otherwise we use the function $f_r(z_1) = f(r z_1)/r$ for $r \in (0, 1)$, which is holomorphic on $\overline{U}$. Taking into account the minimum principle for harmonic functions, we have to prove that

$$\left| \frac{1}{\langle DF^{-1}(z) F(z), z \rangle} - 1 \right| < 1, \quad z = (z_1, \hat{z}) \in B^n, \quad \|z\| = 1,$$

i.e.

$$\Re \langle DF^{-1}(z) F(z), z \rangle > \frac{1}{2}, \quad z \in \partial B^n.$$

We know that the inequality (2.1) is equivalent to

$$\Re \left\{ \frac{f(z_1)}{z_1 f'(z_1)} \right\} > \frac{1}{2}, \quad z_1 \in U.$$
Since \( f \) is parabolic starlike, \( F = \Phi_{n,\gamma}(f) \) is starlike on \( B^n \), and hence biholomorphic. A short computation yields that
\[
DF(z) = \begin{bmatrix}
  f'(z_1) & 0 \\
  \gamma (f'(z_1))^{-1} f''(z_1)\hat{z} & (f'(z_1))^\gamma
\end{bmatrix},
\]
therefore,
\[
DF^{-1}(z)F(z) = \begin{bmatrix}
  f(z_1) & \gamma (f'(z_1))^{-1} f''(z_1)\hat{z} \\
  -(f'(z_1))^\gamma \hat{z} + \hat{z}
\end{bmatrix}.
\]
Then we have
\[
\langle DF(z)^{-1}F(z), \hat{z} \rangle = \frac{f(z_1)}{f'(z_1)} \hat{z}_1 + \|\hat{z}\|^2 \left( 1 - \gamma \frac{f(z_1)f''(z_1)}{(f'(z_1))^2} \right) = |z_1|^2 \frac{f(z_1)}{z_1 f'(z_1)} + \|\hat{z}\|^2 - \|\hat{z}\|^2 \gamma \frac{f(z_1)f''(z_1)}{(f'(z_1))^2},
\]
for \( z = (z_1, \hat{z}) \in B^n \). We must show that
\[
(2.2) \quad \text{Re} \langle DF(z)^{-1}F(z), \hat{z} \rangle > \frac{1}{2}, \quad z \in \partial B^n.
\]
Therefore, by making use of the equality \( |z_1|^2 + \sum_{j=2}^n |z_j|^2 = 1 \), we must show that
\[
|z_1|^2 \text{Re} \left\{ \frac{f(z_1)}{z_1 f'(z_1)} \right\} + (1 - |z_1|^2) - \gamma (1 - |z_1|^2) \text{Re} \left\{ \frac{f(z_1)f''(z_1)}{(f'(z_1))^2} \right\} > \frac{1}{2}.
\]
If \( z = (z_1, 0) \in B^n \), then
\[
\text{Re} \langle DF(z)^{-1}F(z), \hat{z} \rangle > |z_1|^2 \text{Re} \left\{ \frac{f(z_1)}{z_1 f'(z_1)} \right\} > \frac{\|\hat{z}\|^2}{2},
\]
and hence we may assume \( \hat{z} \neq 0 \). Then \( F \) is holomorphic in a neighborhood of each \( z \in B^n \) for each \( \hat{z} \neq 0 \). In view of the minimum principle for harmonic functions, it suffices to prove that
\[
\text{Re} \langle DF(z)^{-1}F(z), \hat{z} \rangle \geq \frac{1}{2}, \quad \|\hat{z}\| = 1.
\]
Let \( p(z_1) = \frac{f(z_1)}{z_1 f'(z_1)} \) for \( |z_1| < 1 \). Since \( f \) is parabolic starlike on \( U \), we obtain that \( \text{Re} p(z_1) > \frac{1}{2} \). Also let \( q(z_1) = 2p(z_1) - 1 \). Then \( \text{Re} q(z_1) > 0 \) for \( |z_1| < 1 \), and thus (see e.g. [x], Theorem 2.1.3)
\[
\text{Re} \left\{ z_1 q'(z_1) \right\} \geq \frac{-2|z_1|}{1 - |z_1|^2} \text{Re} q(z_1), \quad |z_1| < 1.
\]
Therefore, we deduce that

\[(2.3) \quad \Re\left\{z_1f'(z_1)\right\} \geq \frac{-2|z_1|}{1 - |z_1|^2}\Re(z_1) + \frac{|z_1|}{1 - |z_1|^2}.\]

On the other hand, since

\[
\frac{f''(z_1)zf(z_1)}{(f'(z_1))^2} = 1 - z_1f'(z_1) - p(z_1),
\]

we deduce from (2.3) that

\[
|z_1|^2\Re\left\{\frac{f(z_1)}{z_1f'(z_1)}\right\} + (1 - |z_1|^2) - \gamma (1 - |z_1|^2) \Re\left\{\frac{f(z_1)f''(z_1)}{(f'(z_1))^2}\right\}
= |z_1|^2\Re(z_1) + (1 - |z_1|^2) - \gamma (1 - |z_1|^2) \Re\left\{1 - z_1f'(z_1) - p(z_1)\right\}
= (|z_1|^2 + \gamma (1 - |z_1|^2)) \Re(z_1) + (1 - |z_1|^2) (1 - \gamma)
+ \gamma (1 - |z_1|^2) \left(\frac{-2|z_1|}{1 - |z_1|^2}\Re(z_1) + \frac{|z_1|}{1 - |z_1|^2}\right)
= (|z_1|^2 + \gamma (1 - |z_1|^2)) \Re(z_1) - 2\gamma |z_1|\Re(z_1) + \gamma |z_1|
+ (1 - |z_1|^2) (1 - \gamma)
\geq \frac{1}{2} (\gamma + (1 - \gamma)|z_1|^2) + (1 - |z_1|^2) (1 - \gamma)
= \frac{1}{2} \gamma + \frac{1 - \gamma}{2}|z_1|^2 + (1 - |z_1|^2) (1 - \gamma)
= \frac{1}{2} \gamma + (1 - \gamma) \left(1 - \frac{1}{2}|z_1|^2\right)
\geq \frac{1}{2} \gamma + \frac{1 - \gamma}{2} = \frac{1}{2}.
\]

Hence the relation (2.2) holds, as desired. This completes the proof. \(\square\)

In view of Theorem 2.1, we obtain the following particular cases. This result, was obtained in [1], in the case \(\gamma = \frac{1}{2}\).

**Corollary 2.2.** Let \(f : U \to C\) be a normalized locally univalent function, which satisfies the condition

\[
\left|\frac{z_1f'(z_1)}{f(z_1)} - 1\right| < 1, \quad z_1 \in U.
\]

Also, let \(F = \Phi_n(f)\) where

\[
\Phi_n(f)(z) = (f(z_1), \sqrt{f''(z_1)}\hat{z}), \quad z = (z_1, \hat{z}) \in B^n.
\]
then
\[ \left| \frac{\|z\|^2}{\langle DF^{-1}(z)F(z), z \rangle} - 1 \right| < 1, \quad z \in B^n \setminus \{0\}, \]
and hence \( F \) is a parabolic starlike mapping on \( B^n \).

In the next Theorem, through a different method, we show that \( \Psi_{n,\beta}(f) \) is also a parabolic starlike mapping on \( B^n \).

**Theorem 2.3.** Assume that \( f \) be a parabolic starlike function on \( U \). For any \( \beta \in [0, 1] \), let \( F = \Psi_{n,\beta}(f) \). Then \( \Psi_{n,\beta}(f) \) is a parabolic starlike mapping on \( B^n \).

**Proof.** Let \( f \in S \) be a parabolic starlike mapping on \( U \), since parabolic starlike mappings are starlike mappings with respect to 1 (Lemma 1.3) then we must show that
\[
F(z) = \Phi_{n,\beta}(f)(z) = \left[ \begin{array}{c} f(z_1) \\ (f(z_1)/z_1)^\beta \hat{z} \end{array} \right],
\]
is a parabolic starlike map with respect to 1 on \( B^n \). For this purpose, through simple calculations we have
\[
DF(z) = \left[ \begin{array}{cc} \beta \left( f(z_1)/z_1 \right)^{\beta-1} \left( f'(z_1) - f(z_1)/z_1^2 \right) \hat{z} & 0 \\ \hat{z} & \hat{z} \end{array} \right],
\]
and
\[
DF^{-1}(z)F(z) = \left[ \begin{array}{c} \left( f(z_1)/z_1 f'(z_1) \right) \hat{z} + \hat{z} \end{array} \right],
\]
therefore
\[
\langle DF(z)^{-1}F(z), z \rangle = \frac{f(z_1)}{f'(z_1)} \hat{z}_1 + \beta \|\hat{z}\|^2 \left( \frac{f(z_1)}{z_1 f'(z_1)} - 1 \right) + \|\hat{z}\|^2.
\]
Now we will show that
\[
\left| \frac{1}{\|z\|^2} \langle DF(z)^{-1}F(z), z \rangle - 1 \right| < 1.
\]
For this purpose we have
\[
\left| \frac{1}{\|z\|^2} \left\{ \frac{|z_1|^2 f(z_1)}{z_1 f'(z_1)} + \beta \|\hat{z}\|^2 \left( \frac{f(z_1)}{z_1 f'(z_1)} - 1 \right) + \|\hat{z}\|^2 \right\} - 1 \right|.
\]
where $0 \leq \beta \leq 1$. This completes the proof. \hfill \Box

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References


\footnote{DEPARTMENT OF MATHEMATICS, FACULTY OF BASIC SCIENCE, UNIVERSITY OF BONAB, P.O. Box 5551-761167, BONAB, IRAN. 
E-mail address: sarahrovi@gmail.com}