

## THE APPROXIMATE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS ON CANTOR SETS WITHIN LOCAL FRACTIONAL OPERATORS

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ABSTRACT. In this paper, we apply the local fractional Adomian decomposition and variational iteration methods to obtain the analytic approximate solutions of Fredholm integral equations of the second kind within local fractional derivative operators. The iteration procedure is based on local fractional derivative. The obtained results reveal that the proposed methods are very efficient and simple tools for solving local fractional integral equations.

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### 1. INTRODUCTION

Integral equation is encountered in a variety of applications in many fields including continuum mechanics, potential theory, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in physics and biology, renewal theory, quantum mechanics, radiation, optimization, optimal control systems, communication theory, mathematical economics, population genetics, queuing theory, medicine, mathematical problems of radioactive equilibrium, the particle transport problems of astrophysics and reactor theory, acoustics, fluid mechanics, steady state heat conduction, and radioactive heat transfer problems. Fredholm integral equation is one of the most important integral equations [2, 5]. Many initial and boundary value problems associated with ordinary differential equations and partial differential equation can be transformed into problems of solving some approximate integral equations [10]. The most standard form of Fredholm integral equations of the second

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kind within local fractional derivative operator is given by the form

$$(1.1) \quad u(x) = f(x) + \frac{1}{\Gamma(1 + \alpha)} \int_a^b K(x, t)u(t)(dt)^\alpha, \quad 0 < \alpha \leq 1$$

or equivalently

$$(1.2) \quad u(x) = f(x) + \frac{1}{\Gamma(1 + \alpha)} g(x) \int_a^b h(t)u(t)(dt)^\alpha, \quad 0 < \alpha \leq 1$$

where  $K(x, t)$  is the kernel of the local fractional integral equation, and  $f(x)$  a local fractional continuous function. The limits of integration  $a$  and  $b$  are constants and the unknown function  $u(x)$  appears linearly under the integral sign.

In this paper our aim is to investigate the application of the local fractional Adomian decomposition method and local fractional variational iteration method for solving the local fractional integral equations in the sense of local fractional derivative operators. To illustrate the validity and advantages of the methods, we will apply them to the local fractional Fredholm integral equations of the second kind. We will mostly use degenerate or separable kernels. A degenerate or a separable kernel is a function that can be expressed as the sum of the product of two functions which depends only on one variable. Such a kernel can be expressed in the form  $K(x, t) = g(x)h(t)$ . This paper is organized as follows: In Section 2, the basic mathematical tools are reviewed. Section 3 gives the analysis of the methods used in this paper. An illustrative example is shown in Section 4. Conclusions are in Section 5.

## 2. PRELIMINARIES

In this section, we present some basic definitions and notations of the local fractional operators (see [1, 3, 4, 6, 7, 8, 9, 11, 12]).

**Definition 2.1.** The local fractional derivative of  $\psi(x)$  of order  $\alpha$  at  $x = x_0$  is given by

$$(2.1) \quad \begin{aligned} \psi^{(\alpha)}(x_0) &= \frac{d^\alpha}{dx^\alpha} \psi(x)|_{x=x_0} \\ &= \lim_{x \rightarrow x_0} \frac{\Delta^\alpha(\psi(x) - \psi(x_0))}{(x - x_0)^\alpha}, \end{aligned}$$

where  $\Delta^\alpha(\psi(x) - \psi(x_0)) \cong \Gamma(\alpha + 1)(\psi(x) - \psi(x_0))$ .

The formulas of local fractional derivatives of special functions used in this paper are as follows:

$$(2.2) \quad D_x^{(\alpha)} a\psi(x) = aD_x^{(\alpha)}\psi(x),$$

$$(2.3) \quad \frac{d^\alpha}{dx^\alpha} \left( \frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \frac{x^{(n-1)\alpha}}{\Gamma(1+(n-1)\alpha)}$$

**Definition 2.2.** The local fractional integral of  $\psi(x)$  in the interval  $[a, b]$  is given by

$$(2.4) \quad \begin{aligned} {}_a I_b^{(\alpha)} \psi(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b \psi(t)(dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \rightarrow 0} \sum_{j=0}^{N-1} \psi(t_j)(\Delta t_j)^\alpha, \end{aligned}$$

where the partition of the interval  $[a, b]$  is denoted as  $(t_j, t_{j+1}), j = 0, \dots, N - 1, t_0 = a$  and  $t_N = b$  with  $\Delta t_j = t_{j+1} - t_j$  and  $\Delta t = \max \{ \Delta t_0, \Delta t_1, \dots \}$ .

The formulas of local fractional integrals of special functions used in current paper are as follows:

$$(2.5) \quad {}_0 I_x^{(\alpha)} a\psi(t) = a{}_0 I_x^{(\alpha)} \psi(t),$$

$$(2.6) \quad {}_0 I_x^{(\alpha)} \left( \frac{t^{n\alpha}}{\Gamma(1+n\alpha)} \right) = \frac{x^{(n+1)\alpha}}{\Gamma(1+(n+1)\alpha)}.$$

**Definition 2.3.** The Mittag Leffler function, sine function and cosine function are defined as

$$(2.7) \quad E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)}, \quad 0 < \alpha \leq 1,$$

$$(2.8) \quad \sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{(2k+1)\alpha}}{\Gamma(1+(2k+1)\alpha)}, \quad 0 < \alpha \leq 1$$

$$(2.9) \quad \cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k\alpha}}{\Gamma(1+2k\alpha)}, \quad 0 < \alpha \leq 1.$$

### 3. ANALYTICAL METHODS

**3.1. Local Fractional Adomian Decomposition Method.** The local fractional Adomian decomposition method consists of decomposing the unknown function  $u(x)$  of any equation into a sum of an infinite number of components defined by the decomposition series

$$(3.1) \quad u(x) = \sum_{n=0}^{\infty} u_n(x),$$

where the components  $u_n(x)$ ,  $n \geq 0$  will be determined recurrently. The local fractional Adomian decomposition method concerns itself with finding the components  $u_0, u_1, u_2, \dots$  individually. As we have seen before, the determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated. To establish the recurrence relation, we substitute (3.1) into (1.1) to obtain

$$(3.2) \quad \sum_{n=0}^{\infty} u_n(x) = f(x) + \frac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t) \left( \sum_{n=0}^{\infty} u_n(t) \right) (dt)^\alpha,$$

or equivalently

$$(3.3) \quad u_0(x) + u_1(x) + \dots = f(x) + \frac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t) (u_0(t) + u_1(t) + \dots) (dt)^\alpha.$$

The zeroth component  $u_0(x)$  is identified by all terms that are not included under the integral sign. This means that the components  $u_n(x)$ ,  $n \geq 0$  of the unknown function  $u(x)$  are completely determined by setting the recurrence relation

$$(3.4) \quad \begin{aligned} u_0(x) &= f(x), \\ u_{n+1}(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b K(x,t) (u_n(t)) (dt)^\alpha, \quad n \geq 0. \end{aligned}$$

**3.2. Local Fractional Variational Iteration Method.** The local fractional variational iteration method works effectively if the kernel  $K(x,t)$  is separable and can be written in the form  $K(x,t) = g(x)h(t)$ . Differentiating both sides of (1.2) with respect to  $x$  gives

$$(3.5) \quad u^{(\alpha)}(x) = f^{(\alpha)}(x) + \frac{1}{\Gamma(1+\alpha)} g^{(\alpha)}(x) \int_a^b h(t) u(t) (dt)^\alpha.$$

According to the rule of local fractional variational iteration method, the correction local fractional functional for (3.5) is given by

$$(3.6) \quad \begin{aligned} u_{n+1}(x) &= u_n(x) + {}_0 I_x^{(\alpha)} \left( \frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} \left[ u_n^{(\alpha)}(\xi) - f^{(\alpha)}(\xi) \right. \right. \\ &\quad \left. \left. - \frac{1}{\Gamma(1+\alpha)} g^{(\alpha)}(\xi) \int_a^b h(r) u_n^\sim(r) (dr)^\alpha \right] \right), \end{aligned}$$

where  $\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}$  is a general fractal Lagrange multiplier. The local fractional variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier  $\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)}$  that can be identified optimally via integration by parts and by using a

restricted variation. However,  $\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} = -1$  for local fractional integro-differential equation (3.5) of order  $\alpha$ . Having determined the Lagrange multiplier, an iteration formula, without restricted variation, given by

$$(3.7) \quad u_{n+1}(x) = u_n(x) -_0 I_x^{(\alpha)} \left( u_n^{(\alpha)}(\xi) - f^{(\alpha)}(\xi) - \frac{1}{\Gamma(1+\alpha)} g^{(\alpha)}(\xi) \int_a^b h(r) u_n(r) (dr)^\alpha \right), \quad n \geq 0,$$

is used for determination of the successive approximations of the solution  $u_{n+1}(x), n \geq 0$ . The zeroth approximation  $u_0(x)$  can be any selective function. However, the given initial value  $u(0)$  is preferably used for the selective zeroth approximation  $u_0(x)$  as will be seen later. Consequently, the solution is given by

$$(3.8) \quad u(x) = \lim_{n \rightarrow \infty} u_n(x).$$

#### 4. AN ILLUSTRATIVE PARADIGM

Let us consider the following Fredholm integral equation involving local fractional operator in the form:

$$(4.1) \quad u(x) = E_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u(t) (dt)^\alpha,$$

(I) Local Fractional Adomian Decomposition Method (LFADM).

Suppose that the solution

$$(4.2) \quad u(x) = \sum_{n=0}^{\infty} u_n(x).$$

Substituting (4.2) into both sides of (4.1) gives

$$(4.3) \quad \sum_{n=0}^{\infty} u_n(x) = E_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} \left( \sum_{n=0}^{\infty} u_n(t) \right) (dt)^\alpha.$$

From (4.3), we obtain the following recurrence relation

$$(4.4) \quad u_0(x) = E_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)},$$

$$u_{n+1}(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u_n(t) (dt)^\alpha, \quad n \geq 0.$$

Therefore, we have

$$u_0(x) = E_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)},$$

$$\begin{aligned} u_1(x) &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u_0(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^1 \frac{t^\alpha}{\Gamma(1+\alpha)} \left( E_\alpha(t^\alpha) - \frac{t^\alpha}{\Gamma(1+\alpha)} \right) (dt)^\alpha \\ &= \frac{2}{3} \frac{x^\alpha}{\Gamma(1+\alpha)}, \end{aligned}$$

$$\begin{aligned} u_2(x) &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u_1(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^1 \frac{t^\alpha}{\Gamma(1+\alpha)} \left( \frac{2}{3} \frac{t^\alpha}{\Gamma(1+\alpha)} \right) (dt)^\alpha \\ &= \frac{2}{9} \frac{x^\alpha}{\Gamma(1+\alpha)}, \end{aligned}$$

$$\begin{aligned} u_3(x) &= \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{t^\alpha}{\Gamma(1+\alpha)} u_2(t) (dt)^\alpha \\ &= \frac{1}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^1 \frac{t^\alpha}{\Gamma(1+\alpha)} \left( \frac{2}{9} \frac{t^\alpha}{\Gamma(1+\alpha)} \right) (dt)^\alpha \\ &= \frac{2}{27} \frac{x^\alpha}{\Gamma(1+\alpha)}, \end{aligned}$$

$$u_4(x) = \frac{2}{81} \frac{x^\alpha}{\Gamma(1+\alpha)},$$

and so on. Using (4.2) gives the series solution

$$(4.5) \quad u(x) = E_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{2}{3} \frac{x^\alpha}{\Gamma(1+\alpha)} \left[ 1 + \frac{1}{3} + \frac{1}{9} + \dots \right].$$

The series solution (4.5) converges to the closed form solution

$$(4.6) \quad u(x) = E_\alpha(x^\alpha).$$

(II) Local Fractional Variational Iteration Method (LFVIM).

We can write (4.1) in the form

$$(4.7) \quad u(x) = E_\alpha(x^\alpha) - \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \frac{x^\alpha}{\Gamma(1+\alpha)} \int_0^1 \frac{t^\alpha}{\Gamma(1+\alpha)} u(t)(dt)^\alpha.$$

In view of (3.7) and (4.7) the local fractional iteration algorithm can be written as follows:

$$(4.8) \quad u_{n+1}(x) = u_n(x) - {}_0I_x^{(\alpha)} \left( u_n^{(\alpha)}(\xi) - E_\alpha(\xi^\alpha) + 1 - \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{r^\alpha}{\Gamma(1+\alpha)} u_n(r)(dr)^\alpha \right), \quad n \geq 0,$$

where we used  $\frac{\lambda(\xi)^\alpha}{\Gamma(1+\alpha)} = -1$ . Notice that the initial condition  $u(0) = 1$  is obtained by substituting  $x = 0$  into (4.8). Therefore, we have

$$\begin{aligned} u_0(x) &= 1, \\ u_1(x) &= u_0(x) - {}_0I_x^{(\alpha)} \left( u_0^{(\alpha)}(\xi) - E_\alpha(\xi^\alpha) + 1 - \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{r^\alpha}{\Gamma(1+\alpha)} u_0(r)(dr)^\alpha \right) \\ &= E_\alpha(x^\alpha) - \frac{1}{2} \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ u_2(x) &= u_1(x) - {}_0I_x^{(\alpha)} \left( u_1^{(\alpha)}(\xi) - E_\alpha(\xi^\alpha) + 1 - \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{r^\alpha}{\Gamma(1+\alpha)} u_1(r)(dr)^\alpha \right) \\ &= E_\alpha(x^\alpha) - \frac{1}{6} \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ u_3(x) &= u_2(x) - {}_0I_x^{(\alpha)} \left( u_2^{(\alpha)}(\xi) - E_\alpha(\xi^\alpha) + 1 - \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{r^\alpha}{\Gamma(1+\alpha)} u_2(r)(dr)^\alpha \right) \\ &= E_\alpha(x^\alpha) - \frac{1}{18} \frac{x^\alpha}{\Gamma(1+\alpha)}, \\ u_n(x) &= E_\alpha(x^\alpha) - \frac{1}{2 \times 3^{n-1}} \frac{x^\alpha}{\Gamma(1+\alpha)}, \quad n \geq 1. \end{aligned}$$

Finally, the solution is

$$(4.9) \quad \begin{aligned} u(x) &= \lim_{n \rightarrow \infty} u_n(x) \\ &= \lim_{n \rightarrow \infty} \left[ E_\alpha(x^\alpha) - \frac{1}{2 \times 3^{n-1}} \frac{x^\alpha}{\Gamma(1+\alpha)} \right] \\ &= E_\alpha(x^\alpha). \end{aligned}$$

## 5. CONCLUSIONS

In this work, the analytical approximate solutions for the Fredholm integral equations of the second kind involving local fractional derivative operators are investigated by using the local fractional Adomian decomposition method and local fractional variational iteration method. The obtained results demonstrate the reliability of the methodology and its

wider applicability to local fractional integral equations and hence can be extended to other problems of diversified nonlinear nature.

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