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INVERSE STURM-LIOUVILLE PROBLEMS WITH A SPECTRAL PARAMETER IN THE BOUNDARY AND TRANSMISSION CONDITIONS

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Abstract. In this manuscript, we study the inverse problem for non self-adjoint Sturm–Liouville operator $-D^2+q$ with eigenparameter dependent boundary and discontinuity conditions inside a finite closed interval. By defining a new Hilbert space and using its spectral data of a kind, it is shown that the potential function can be uniquely determined by part of a set of values of eigenfunctions at some interior point and parts of two sets of eigenvalues.

1. Introduction

Let us consider the boundary value problem

$$\ell y := -y'' + qy = \lambda y, \tag{1.1}$$

$$U(y) := \lambda(y'(0) + h_1 y(0)) - h_2 y'(0) - h_3 y(0) = 0,$$
$$V(y) := \lambda(y'(\pi) + H_1 y(\pi)) - H_2 y'(\pi) - H_3 y(\pi) = 0, \tag{1.2}$$

with the jump conditions

$$U_1(y) := y(a+0) - a_1 y(a-0) = 0,$$
$$U_2(y) := y'(a+0) - a_2 y'(a-0) - a_3 y(a-0) = 0, \tag{1.3}$$

where $q(x)$ is a real function in $\in L^2(0,\pi)$, h_i, H_i, and a_i $(i = 1, 2, 3)$, are real with $a \in (0, \pi)$, $a_1 a_2 > 0$, $r_1 := h_3 - h_1 h_2 > 0$ and $r_2 := H_1 H_2 - H_3 > 0$.

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For simplicity we use the notation $L = L(q(x); h_1; h_2; h_3; H_1; H_2; H_3; a)$ for the problem (1.1)–(1.3). Here λ is the spectral parameter.

In the present paper, we are interested in the inverse problem, that is, the question of existence, uniqueness and reconstruction of the potential function q from given spectral data. These problems originated in the work of Ambarzumian(1929)[3], were continued by Borg(1945)[6], and have been gradually elucidated over the past seventy years. Here we want to look at the question of uniqueness for the above problem using two sets of spectra, or one spectrum plus a part of a set of value of eigenfunctions at some interior point. Such kind of problems have a long tradition and we refer the reader to [20]–[7], [34, 14, 25], and the references therein. In particular, the operator ℓ plays an important role as the one-dimensional Schrödinger operator in quantum mechanics and our transmission conditions include the case of point interactions (see e.g. the monographs [29] and [2]).

In section 2 we define a new Hilbert space for a non-self-adjoint Sturm–Liouville operator by using similar techniques as in [23, 1], to obtain the asymptotic form of solutions and eigenvalues. In section 3 we formulate a novel inverse spectral problem which is a generalization of [10, 28] and [22]–[11].

2. Asymptotic form of solutions and eigenvalues

In this section, we introduce a special inner product in the Hilbert space $(L_2(0,a) \oplus L_2(a,\pi)) \oplus \mathbb{C}^2$ and define a linear operator A in it such that the problem (1.1)-(1.3) can be interpreted as the eigenvalue problem of A. To this end we introduce the function

$$w(x) = \begin{cases} 1, & 0 \le x < a, \\ \frac{1}{a_1 a_2}, & a < x \le \pi. \end{cases} \tag{2.1}$$

Now we define a new Hilbert space inner product on

$$\mathcal{H} := (L_2(0,a) \oplus L_2(a,\pi)) \oplus \mathbb{C}^2,$$

by

$$\langle F, G \rangle_{\mathcal{H}} := \int_0^\pi f \bar{g} w + \frac{w(0)}{r_1} f_1 \bar{g}_1 + \frac{w(\pi)}{r_2} f_2 \bar{g}_2, \tag{2.2}$$

where $F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix}$ and $G = \begin{pmatrix} g(x) \\ g_1 \\ g_2 \end{pmatrix} \in \mathcal{H}$ and we let

$$R_1(u) := u'(0) + h_1 u(0), \qquad R'_1(u) := h_2 u'(0) + h_3 u(0),$$

$$R_2(u) := u'(\pi) + H_1 u(\pi), \qquad R'_2(u) := H_2 u'(\pi) + H_3 u(\pi).$$

In this Hilbert space we construct the operator

$$A : \mathcal{H} \to \mathcal{H}, \tag{2.3}$$

with domain

$$D(A) = \left\{ F = \begin{pmatrix} f(x) \\ f_1 \\ f_2 \end{pmatrix} \middle| \begin{array}{l} f(x), f'(x) \in AC[0,a) \cup (a,\pi] \text{ and, } f(a \pm 0), \\ f'(a \pm 0) \text{ is defined}, \ell f \in L_2(0,\pi), f_1 = R_1(f), \\ f_2 = R_2(f),\ U(f) = U_1(f) = U_2(f) = 0, \end{array} \right\} \tag{2.4}$$

and by action law

$$AF = \begin{pmatrix} \ell f \\ R_1'(f) \\ R_2'(f) \end{pmatrix} \qquad \text{with} \qquad F = \begin{pmatrix} f(x) \\ R_1(f) \\ R_2(f) \end{pmatrix} \in D(A).$$

So, we can change the boundary value problem (1.1)-(1.3) as following form

$$AY = \lambda Y \qquad Y := \begin{pmatrix} y(x) \\ R_1(y) \\ R_2(y) \end{pmatrix} \in D(A), \tag{2.5}$$

in the Hilbert space $\mathcal{H}$. It is easy to verify that the eigenvalues of the operator A coincide with those of the problem (1.1)-(1.3).

Theorem 2.1. *The operator A is self-adjoint.*

Proof. We omit the proof, since the arguments are the same as in [23]. □

Suppose that the functions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ are solutions of (1.1) under the initial conditions

$$\varphi(0,\lambda) = h_2 - \lambda, \qquad \varphi'(0,\lambda) = \lambda h_1 - h_3, \tag{2.6}$$

and

$$\psi(\pi,\lambda) = H_2 - \lambda, \qquad \psi'(\pi,\lambda) = \lambda H_1 - H_3. \tag{2.7}$$

By attaching a subscript 1 or 2 to the functions φ and ψ, we mean to refer to the first subinterval $[0,a)$ or to the second subinterval $(a,\pi]$. By virtue of [33], problem (1.1) under the initial conditions (2.6) or (2.7), has a unique solution $\varphi_1(x,\lambda)$ or $\psi_2(x,\lambda)$, which is an entire function of $\lambda \in \mathbb{C}$ for each fixed point $x \in [0,a)$ or $x \in (a,\pi]$. From the linear differential equations we obtain that the modified Wronskian

$$W(u,v) = w(x)\big(u(x)v'(x) - u'(x)v(x)\big), \tag{2.8}$$

is constant on $x \in [0,a) \cup (a,\pi]$ for two solutions $\ell u = \lambda u$, $\ell v = \lambda v$ satisfying the transmission conditions (1.3). Moreover, we set

$$\Delta(\lambda) := W(\varphi(\lambda), \psi(\lambda)). \tag{2.9}$$

Then $\Delta(\lambda)$ is an entire function whose roots coincide with the eigenvalues of problem L. Moreover, the eigenfunctions $\varphi_i(x,\lambda_n)$ and $\psi_i(x,\lambda_n)$

associated with a certain eigenvalue λ_n, satisfy the relation $\psi_i(x,\lambda_n) = \beta_n \varphi_i(x,\lambda_n)$, thus from (2.6),

$$\beta_n = \frac{\psi'(0,\lambda_n) + h_1 \psi(0,\lambda_n)}{r_1}. \tag{2.10}$$

We define the norming constant by

$$\gamma_n := \langle \Phi(x,\lambda_n), \Phi(x,\lambda_n) \rangle_{\mathcal{H}},$$

where

$$\Phi(x,\lambda) = (\varphi_1(x,\lambda), \varphi_2(x,\lambda), R_1(\varphi), R_2(\varphi))^T.$$

Lemma 2.2 ([27]). *The equality $\dot{\Delta}(\lambda_n) = -\gamma_n \beta_n$ holds for each eigenvalue λ_n.*

Lemma 2.3 ([27]). *The eigenvalues of the problem L are real and simple.*

Theorem 2.4. *Let $\lambda = \rho^2$ and $\tau := \mathrm{Im}\rho$. For equation (1.1) with boundary conditions (1.2) and jump conditions (1.3) as $|\lambda| \to \infty$, the following asymptotic formulas hold:*

$$\varphi(x;\lambda) = \begin{cases} \rho^2 \cos \rho x + \rho \left(-h_1 + \frac{1}{2}\int_0^x q(t)dt\right) \sin \rho x + O(\exp(|\tau| x)), & x < a, \\ \rho^2 (b_1 \cos \rho x + b_2 \cos \rho(2a - x)) + \rho \left(f_1(x) \sin \rho x \right. \\ \quad \left. + f_2(x) \sin \rho(2a - x)\right) + O(\exp(|\tau| x)), & x > a, \end{cases} \tag{2.11}$$

$$\varphi'(x;\lambda) = \begin{cases} -\rho^3 \sin \rho x + \rho^2 \left(-h_1 + \frac{1}{2}\int_0^x q(t)dt\right) \cos \rho x + O(\rho \exp(|\tau| x)) & x < a, \\ \rho^3 (-b_1 \sin \rho x + b_2 \sin \rho(2a - x)) + \rho^2 (f_1(x) \cos \rho x \\ \quad - f_2(x) \cos \rho(2a - x)) + O(\rho \exp(|\tau| x)), & x > a, \end{cases} \tag{2.12}$$

where

$$b_1 = (a_1 + a_2)/2, \qquad b_2 = (a_1 - a_2)/2, \tag{2.13}$$

and

$$f_1(x) = b_1 \left(-h_1 + \frac{1}{2}\int_0^x q(t)dt\right) + \frac{a_3}{2},$$
$$f_2(x) = b_2 \left(-h_1 - \frac{1}{2}\int_0^x q(t)dt + \int_0^a q(t)dt\right) + \frac{a_3}{2}.$$

Then the characteristic function is

$$\begin{aligned} \Delta(\lambda) = {} & \rho^5 (-b_1 \sin \rho\pi + b_2 \sin \rho(2a - \pi)) + \rho^4 \left[(f_1(\pi) + H_1 b_1) \cos \rho\pi \right. \\ & \left. + (-f_2(\pi) + H_1 b_2) \cos \rho(2a - \pi) \right] + O(\rho^3 \exp(|\tau| \pi)). \end{aligned} \tag{2.14}$$

Proof. We omit the proof, since the arguments are the same as in [27]. □

Define

$$\Delta_\circ(\rho) := \rho^5(-b_1 \sin\rho\pi + b_2 \sin\rho(2a-\pi)). \tag{2.15}$$

So we have the following Lemma:

Lemma 2.5. *The roots of characteristic function $\Delta_\circ(\rho)$ are*

$$\rho_n^\circ = n - \frac{5}{2} + \eta_n, \tag{2.16}$$

where $\eta_n \in (0,1)$ for $n \in \mathbb{N}$.

Proof. There exist a similar proof for obtaining $\eta_n \in (0,1)$ in Lemma 2.6 of [26]. □

Theorem 2.6. *The corresponding eigenvalues $\{\lambda_n\}$ of the boundary value problem L, admit the following asymptotic form as $n \to \infty$:*

$$\rho_n = n - \frac{5}{2} + \eta_n + O\left(\frac{1}{n}\right), \tag{2.17}$$

where $\eta_n \in (0,1)$.

Proof. From (2.14) and (2.15), we see that

$$\Delta(\rho) = \Delta_\circ(\rho) + O\left(\rho^4 \exp(|\tau|\pi)\right). \tag{2.18}$$

For sufficiently large value of ρ, the functions $\Delta(\rho)$ and $\Delta_\circ(\rho)$ have the same number of zeros counting multiplicities according to Rouche's theorem. So if ρ_n and ρ_n° are roots of $\Delta(\rho)$ and $\Delta_\circ(\rho)$ respectively, then we have $\rho_n = \rho_n^\circ + \varepsilon_n$ where $\lim_{n\to\infty} \varepsilon_n = 0$. Since numbers ρ_n are zeros of the characteristic function $\Delta(\rho)$, therefore by a simple calculation we get $\varepsilon_n = O\left(\frac{1}{n}\right)$. □

3. Main results

We consider the boundary value problem

$$\tilde{L} = L(\tilde{q}(x); \tilde{h}_1; \tilde{h}_2; \tilde{h}_3; H_1; H_2; H_3; a),$$

which is the same as L with diferent q, h_1, h_2 and h_3. Let $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ be the solutions of

$$\begin{gathered} -\varphi'' + q\varphi = \lambda\varphi, \\ \varphi(0,\lambda) = h_2 - \lambda,\ \varphi'(0,\lambda) = \lambda h_1 - h_3, \end{gathered} \tag{3.1}$$

and

$$\begin{gathered} -\tilde{\varphi}'' + \tilde{q}\tilde{\varphi} = \lambda\tilde{\varphi}, \\ \varphi(0,\lambda) = \tilde{h}_2 - \lambda,\ \varphi'(0,\lambda) = \lambda\tilde{h}_1 - \tilde{h}_3. \end{gathered} \tag{3.2}$$

Suppose b is an arbitrary and fixed point on $[0, \frac{\pi}{2}]$, then the following representation holds (see [19])

$$\begin{aligned}\varphi(x,\lambda) = (\lambda - h_2)\left[\cos(\rho x) + \int_0^x k_1(x,t)\cos(\rho t)dt\right] \\ + (h_3 - \lambda h_1)\left[\frac{\sin(\rho x)}{\rho} + \int_0^x k_2(x,t)\frac{\sin(\rho t)}{\rho}dt\right],\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\tilde{\varphi}(x,\lambda) = (\lambda - \tilde{h}_2)\left[\cos(\rho x) + \int_0^x \tilde{k}_1(x,t)\cos(\rho t)dt\right] \\ + (\tilde{h}_3 - \lambda\tilde{h}_1)\left[\frac{\sin(\rho x)}{\rho} + \int_0^x \tilde{k}_2(x,t)\frac{\sin(\rho t)}{\rho}dt\right],\end{aligned} \tag{3.4}$$

for $x \in (0, a)$, where $k_1(x,t)$, $k_2(x,t)$, $\tilde{k}_1(x,t)$, and $\tilde{k}_2(x,t)$ are continuous functions which does not depend on λ. Using (3.3) and (3.4) it is easy to verify that

$$\begin{aligned}\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) &= (\lambda - h_2)(\lambda - \tilde{h}_2)\left(\frac{1}{2}(1+\cos(2\rho x)) + \int_0^x v_1(x,t)\cos(2\rho t)dt\right) \\ &\quad + (h_3 - \lambda h_1)(\tilde{h}_3 - \lambda\tilde{h}_1)\left(\frac{1}{2}(1+\cos(2\rho x)) + \int_0^x v_2(x,t)\cos(2\rho t)dt\right) \\ &\quad + (\lambda - h_2)(\tilde{h}_3 - \lambda\tilde{h}_1)\left(\frac{1}{2\rho}(\sin(2\rho x) + \int_0^x v_3(x,t)\sin(2\rho t)dt)\right) \\ &\quad + (\lambda - \tilde{h}_2)(h_3 - \lambda h_1)\left(\frac{1}{2\rho}(\sin(2\rho x) + \int_0^x v_4(x,t)\sin(2\rho t)dt)\right) \\ &= \lambda^2\left(B_1(1+\cos(2\rho t)) + B_1\int_0^x R_1(x,t)\cos(2\rho t)dt\right) \\ &\quad + \lambda\left(B_2(1+\cos(2\rho t)) + B_2\int_0^x R_2(x,t)\cos(2\rho t)dt\right) \\ &\quad + B_3(1+\cos(2\rho t)) + B_3\int_0^x R_3(x,t)\cos(2\rho t)dt,\end{aligned} \tag{3.5}$$

for $x \in (0, a)$. Using a similar method of [34, 14] and a simple calculation we obtain

$$\begin{aligned}\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) = \lambda^2\left(B_1 A(x,\rho) + B_1\int_0^x R_1(x,t)\cos(2\rho t)dt\right) \\ + \lambda\left(B_2 A(x,\rho) + B_2\int_0^x R_2(x,t)\cos(2\rho t)dt\right) \\ + B_3 A(x,\rho) + B_3\int_0^x R_3(x,t)\cos(2\rho t)dt,\end{aligned} \tag{3.6}$$

for $a \le x \le b$; where

$$A(x,\rho) = 2A_1 + 2A_2\cos(2\rho x) + 2A_3\cos 2\rho(x-a) + 2A_4\cos 2\rho(x-2a).$$

Here

$$A_1 = \tfrac{1}{2a_1^2} + \tfrac{1}{2}(a_1^2 - a_2^2)\varphi(a-0,\lambda)\tilde{\varphi}(a-0,\lambda), \qquad A_2 = \tfrac{(a_1+a_2)^{-2}}{8},$$
$$A_3 = \tfrac{a_1^2-a_2^2}{4}, \qquad A_4 = \tfrac{(a_1-a_2)^2}{8},$$

and

$$B_1 = \frac{1+h_1\tilde{h}_1}{2}, \quad B_2 = -\frac{h_2+\tilde{h}_2+h_1\tilde{h}_3+\tilde{h}_1h_3}{2}, \quad B_3 = \frac{h_2\tilde{h}_2+h_3\tilde{h}_3}{2}.$$

The functions $R_i(x,t)$ $(i=1,2,3)$ are continuous which does not depend on λ.

Let $l(n)$ be a subsequence of natural numbers such that

$$l(n) = \frac{n}{\sigma_1}(1+\epsilon_{1n}), \quad 0 < \sigma_1 \le 1, \quad \epsilon_{1n} \to 0. \tag{3.7}$$

Suppose $\Delta_l(\lambda)$ is a restriction of $\Delta(\lambda)$ such that all roots of $\Delta_l(\lambda)$ is $\{\lambda_{l(n)}\}$.

Lemma 3.1. *Let $G_\eta = \{\rho : |\rho - \rho_{l(n)}| > \eta\}$ and fix $\eta > 0$, then there is a positive constant C_η such that*

$$|\Delta_l(\lambda)| \ge C_\eta |\rho|^5 \exp\left(\frac{|\tau|\pi}{\sigma_1}\right).$$

Proof. The proof of this lemma is similar to the proof of Theorem 1.1.3 [8, P. 11]. □

Theorem 3.2. *If $a \in (0,\pi)$ be a jump point, fix point $b \in [0,\frac{\pi}{2}]$, and $\sigma_1 > \frac{2b}{\pi}$*

$$\lambda_{l(n)} = \tilde{\lambda}_{l(n)}, \qquad \langle y_{l(n)}, \tilde{y}_{l(n)}\rangle_{(b)} = 0,$$

for any $n \in \mathbb{N}$, then $q(x) = \tilde{q}(x)$ almost everywhere(a.e.) on $[0,b]$ and $h_i = \tilde{h}_i$ for $i = 1,2,3$.

Proof. Define $Q(x) := \tilde{q}(x) - q(x)$. Multiplying $\tilde{\varphi}$ by (3.1), φ by (3.2), and subtracting the result and integrating on $[0,b] \cup [b,a) \cup (a,\pi]$, we obtain

$$G(\lambda) := \begin{cases} \int_0^b Q(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx + (\lambda - \tilde{h}_2)(\lambda h_1 - h_3) - (\lambda - h_2)(\lambda\tilde{h}_1 - \tilde{h}_3), & b \le a, \\ |a_1| \int_0^a Q(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx + \frac{1}{|a_2|}\int_a^b Q(x)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx \\ \quad +|a_1|(\lambda - \tilde{h}_2)(\lambda h_1 - h_3) - (\lambda - h_2)(\lambda\tilde{h}_1 - \tilde{h}_3), & a < b, \end{cases}$$

(3.8)

$$= \begin{cases} [\tilde{\varphi}'(x,\lambda)\varphi(x,\lambda) - \tilde{\varphi}(x,\lambda)\varphi'(x,\lambda)]_0^b + (\lambda - \tilde{h}_2)(\lambda h_1 - h_3) \\ \quad -(\lambda - h_2)(\lambda\tilde{h}_1 - \tilde{h}_3) & b \le a, \\ |a_1| [\tilde{\varphi}'(x,\lambda)\varphi(x,\lambda) - \tilde{\varphi}(x,\lambda)\varphi'(x,\lambda)]_0^a + \frac{1}{|a_2|}[\tilde{\varphi}'(x,\lambda)\varphi(x,\lambda) \\ \quad -\tilde{\varphi}(x,\lambda)\varphi'(x,\lambda)]_a^b + |a_1|(\lambda - \tilde{h}_2)(\lambda h_1 - h_3) - (\lambda - h_2)(\lambda\tilde{h}_1 - \tilde{h}_3), & a < b. \end{cases}$$

We now claim that $G(\lambda) = 0$ on the whole λ-plane. Using (3.8) we get $G(\lambda_{l(n)}) = 0$. From the inequality

$$|\cos(\rho x)| \le \exp(x|\tau|) \quad \text{and} \quad |\sin(\rho x)| \le \exp(x|\tau|), \tag{3.9}$$

we see that the entire function $G(\lambda)$ satisfies

(3.10)

$$|G(\lambda)| \le \begin{cases} C_1|\rho|^4 \exp(2b|\tau|), & b \le a, \\ |\rho|^4\Big(\frac{1}{2a_1^2} + C_2\exp(2a|\tau|) + A_2\exp(2b|\tau|) + A_3\exp(2(b-a)|\tau|) \\ \quad +A_4\exp(2|b-2a||\tau|) + C_3\exp(2b|\tau|)\Big), & b > a, \end{cases}$$

for some positive constants $C_1,\ C_2,\ C_3$ and $|\lambda|$ large enough. It is easy to see that $b = \max\{a, b, b-a, |b-2a|\}$ for the above case. From (3.8), (3.9), and (3.10) we see that the entire function $G(\lambda)$ satisfies

$$|G(\lambda)| \le M|\rho|^4 \exp(2|\tau|b), \tag{3.11}$$

for some positive constant M and $|\lambda|$ large enough. Define

$$\phi(\lambda) := \frac{G(\lambda)}{\Delta_l(\lambda)}.$$

The definition of $\Delta_l(\lambda)$ and $G(\lambda)$ implies $\phi(\lambda)$ is an entire function. It follows from (3.11) and Lemma 3.1 that

$$\phi(\lambda) = O\left(\frac{1}{|\rho|\exp(|\tau|(2b - \frac{\pi}{\sigma_1}))}\right),$$

for sufficiently large $|\lambda|$. From the condition of σ_1 the relation $2b - \frac{\pi}{\sigma_1} > 0$ holds. Hence Liouville's theorem implies

$$\phi(\lambda) = 0, \tag{3.12}$$

for all λ, therefore

$$G(\lambda) = 0, \quad \text{for all } \lambda. \tag{3.13}$$

Now we want prove that $q(x) = \tilde{q}(x)$ a.e. on $[0, b]$. We have two cases for obtaining this result.

Case (i): If $b \le a$, from (3.5), (3.8), and (3.13), we obtain

$$\begin{aligned} G(\lambda) = \lambda^2 &\left[B_1 \int_0^b Q(x)dx + B_1 \int_0^b \cos(2\rho t) \left(Q(t) + \int_t^b Q(t)R_1(x,t)dx \right) dt \right] \\ &+ \lambda \left[B_2 \int_0^b Q(x)dx + B_2 \int_0^b \cos(2\rho t) \left(Q(t) + \int_t^b Q(t)R_2(x,t)dx \right) dt \right] \\ &+ \left[B_3 \int_0^b Q(x)dx + B_3 \int_0^b \cos(2\rho t) \left(Q(t) + \int_t^b Q(t)R_3(x,t)dx \right) dt \right] \\ &+ \lambda^2(h_1 - \tilde{h}_1) + \lambda(h_3 - \tilde{h}_3 + \tilde{h}_1 h_2 - h_1 \tilde{h}_2) + h_3 \tilde{h}_2 - h_2 \tilde{h}_3 = 0, \end{aligned} \tag{3.14}$$

on the whole λ-plane. Letting $\lambda \to \infty$ for three section in (3.14) and real λ, we see that from the Riemann-Lebesgue lemma

$$\begin{aligned} &B_1 \int_0^b Q(x)dx + h_1 - \tilde{h}_1 = 0, \\ &B_2 \int_0^b Q(x)dx + h_3 - \tilde{h}_3 + \tilde{h}_1 h_2 - h_1 \tilde{h}_2 = 0, \\ &B_3 \int_0^b Q(x)dx + h_3 \tilde{h}_2 - h_2 \tilde{h}_3 = 0, \end{aligned}$$

and

$$B_i \int_0^b \cos(2\rho t) \left(Q(t) + \int_t^b Q(t)R_i(x,t)dx \right) dt = 0, \quad i = 1, 2, 3.$$

Note that there is at least one $B_i \neq 0$, $(i = 1, 2, 3)$. From the completeness of the function $\cos(2\rho t)$ on the interval $[0, b]$, we have

$$Q(t) + \int_t^b Q(x)R_i(x,t)dx = 0, \quad 0 < t < b.$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus $Q(x) = 0$ a.e. on $0 \le x \le b$, that is, $q(x) = \tilde{q}(x)$ a.e. on $[0, b]$. So, from the above equations we obtain $h_1 = \tilde{h}_1$, $h_2 = \tilde{h}_2$, and $h_3 = \tilde{h}_3$.

Case (ii): If $a < b \le \frac{\pi}{2}$, from (3.8), (3.13), and using the similar result of case (i) and changing the integral variables, we get

$$\begin{aligned}
G(\lambda) = B_1\lambda^2 \Bigg[&|a_1| \int_0^a Q(x)dx + \frac{2A_2}{|a_2|}\int_a^b Q(x)dx \\
&+ \int_0^b \cos(2\rho t)\left(F(t) + \int_t^b Q_1(t)R_1(x,t)dx\right)dt\Bigg] \\
+ B_2\lambda \Bigg[&|a_1| \int_0^a Q(x)dx + \frac{2A_2}{|a_2|}\int_a^b Q(x)dx \\
&+ \int_0^b \cos(2\rho t)\left(F(t) + \int_t^b Q_1(t)R_2(x,t)dx\right)dt\Bigg] \\
+ B_3 \Bigg[&|a_1| \int_0^a Q(x)dx + \frac{2A_2}{|a_2|}\int_a^b Q(x)dx \\
&+ \int_0^b \cos(2\rho t)\left(F(t) + \int_t^b Q_1(t)R_3(x,t)dx\right)dt\Bigg] \\
+ \lambda^2(h_1 - \tilde{h}_1) &+ \lambda(h_3 - \tilde{h}_3 + \tilde{h}_1 h_2 - h_1\tilde{h}_2) + h_3\tilde{h}_2 - h_2\tilde{h}_3 = 0,
\end{aligned} \tag{3.15}$$

for $i = 1, 2, 3$ and on the whole λ-plane. Letting $\rho \to \infty$ in (3.15), by the Riemann-Lebesgue lemma

$$B_1\Big[|a_1|\int_0^a Q(x)dx + \frac{2A_1}{|a_2|}\int_a^b Q(x)dx\Big] + h_1 - \tilde{h}_1 = 0,$$

$$B_2\Big[|a_1|\int_0^a Q(x)dx + \frac{2A_1}{|a_2|}\int_a^b Q(x)dx\Big] + h_3 - \tilde{h}_3 + \tilde{h}_1 h_2 - h_1\tilde{h}_2 = 0,$$

$$B_3\Big[|a_1|\int_0^a Q(x)dx + \frac{2A_1}{|a_2|}\int_a^b Q(x)dx\Big] + h_3\tilde{h}_2 - h_2\tilde{h}_3 = 0,$$

and

$$B_i \int_0^b \left(F(t) + \int_t^b R_i(x,t)Q_1(x)dx\right)\cos(2\rho t)dt = 0, \tag{3.16}$$

for all ρ and $i = 1, 2, 3$, where

$$Q_1(x) := \begin{cases} |a_1|Q(x), & x < a, \\ \frac{1}{|a_2|}Q(x), & a < x \le b. \end{cases} \tag{3.17}$$

Since the functions $\cos(2\rho t)$ is complete on $L_2[0, b]$, we have

$$F(t) + \int_t^b R_i(x,t)Q_1(x)dx = 0. \tag{3.18}$$

The form of $F(t)$ will help us to obtain that $Q(x) = 0$ a.e. in $[0, b]$. First, we consider the terms with $R_i(x, t)$ in (3.15). Since $R_i(x, t)$ is bounded on $(x,t) \in [0,\pi]\times[0,\pi]$ and $Q(x)$ is integrable on $[0,\pi]$, by the Fubini's Theorem,

$$\int_0^a Q_1(x)\int_0^x R_i(x,t)\cos(2\rho t)dtdx + \int_a^b Q_1(x)\int_0^x R_i(x,t)\cos(2\rho t)dtdx$$

$$= \int_0^b \int_t^b R_i(x,t)Q_1(x)dx\cos(2\rho t)dt. \tag{3.19}$$

Second, we consider the remainder terms in (3.15). Specifically, we have

$$\int_0^a Q(x)\cos(2\rho x)dx + \int_a^b A_2Q(x)\cos(2\rho x)dx = \int_0^b \widehat{Q}(x)\cos(2\rho x)dx, \tag{3.20}$$

where

$$\widehat{Q}(x) = \begin{cases} Q(t), & t\in[0,a], \\ 2A_2Q(t), & t\in[a,b], \end{cases}$$

$$\int_a^b A_3Q(x)\cos 2\rho(x-a)dx = \int_0^{b-a} A_3Q(x+a)\cos(2\rho x)dx, \tag{3.21}$$

and

$$\int_a^b A_4Q(x)\cos 2\rho(x-2a)dx = \begin{cases} \int_a^{2a-b} A_4Q(2a-x)\cos(2\rho x)dx, & b\le 2a, \\ \int_{-a}^{b-2a} A_4Q(x+2a)\cos(2\rho x)dx, & 2a<b. \end{cases} \tag{3.22}$$

The equations (3.20)-(3.22), imply that $F(t)$ in (3.19) has the following form. We have four cases for $F(t)$:

Case I: If $b < \frac{3}{2}a$

$$F(t) = \begin{cases} |a_1|Q(t) + \frac{2A_3}{|a_2|}Q(t+a), & t\in[0,b-a], \\ |a_1|Q(t), & t\in[b-a,2a-b], \\ |a_1|Q(t) + \frac{2A_4}{|a_2|}Q(2a-t)), & t\in[2a-b,a], \\ \frac{2A_2}{|a_2|}Q(t), & t\in[a,b]. \end{cases} \tag{3.23}$$

Case II: If $\frac{3}{2}a \le b < 2a$

$$F(t) = \begin{cases} |a_1|Q(t) + \frac{2A_3}{|a_2|}Q(t+a), & t\in[0,2a-b], \\ |a_1|Q(t) + \frac{2}{|a_2|}(A_3Q(t+a) + A_4Q(2a-t)), & t\in[2a-b,b-a], \\ |a_1|Q(t) + \frac{2A_4}{|a_2|}Q(2a-t), & t\in[b-a,a], \\ \frac{2A_2}{|a_2|}Q(t), & t\in[a,b]. \end{cases} \tag{3.24}$$

Case III: If $2a \le b < 3a$

(3.25)

$$F(t)=\begin{cases}\frac{2A_4}{|a_2|}Q(t+2a), & t\in[-a,0],\\ |a_1|Q(t)+\frac{2}{|a_2|}(A_3Q(t+a)+A_4Q(2a+t)), & t\in[0,b-2a],\\ |a_1|Q(t)+\frac{2A_3}{|a_2|}Q(t+a), & t\in[b-2a,a],\\ \frac{2}{|a_2|}(A_2Q(t)+A_3Q(t+a)), & t\in[a,b-a],\\ \frac{2A_2}{|a_2|}Q(t), & t\in[b-a,b].\end{cases}$$

Case IV: If $b \ge 3a$

(3.26)

$$F(t)=\begin{cases}\frac{2A_4}{|a_2|}Q(t+2a), & t\in[-a,0],\\ |a_1|Q(t)+\frac{2}{|a_2|}(A_3Q(t+a)+A_4Q(2a+t)), & t\in[0,a],\\ \frac{2}{|a_2|}(A_2Q(t)+A_3Q(t+a)+A_4Q(2a+t)), & t\in[a,b-2a],\\ \frac{2}{|a_2|}(A_2Q(t)+A_3Q(t+a)), & t\in[b-2a,b-a],\\ \frac{2A_2}{|a_2|}Q(t), & t\in[b-a,b].\end{cases}$$

To prove that $Q(x) = 0$ in $[0, b]$, we shall consider only case I. From (3.17), (3.18), and (3.24), we see that

$$\frac{1}{|a_2|}\left(2A_2Q(t)+\int_t^b Q(x)R_i(x,t)dx\right)=0, \quad t\in[a,b]. \tag{3.27}$$

Note that $\frac{1}{|a_2|} \neq 0$ and $A_2 \neq 0$. Since (3.27) is a homogenous Volterra integral equation, so $Q(t) = 0$ a.e. on [a,b]. From (3.14) and (3.15) we get $Q(x) = 0$ a.e. on $[0, b]$. So we get $q(x) = \tilde{q}(x)$ a.e. on $[0, b]$, $h_1 = \tilde{h}_1$, $h_2 = \tilde{h}_2$, and $h_3 = \tilde{h}_3$. By considering a similar way we can prove other cases. □

Suppose $m(n)$ and $r(n)$ are subsequences of natural numbers such that

$$m(n)=\frac{n}{\sigma_2}(1+\epsilon_{2n}), \quad 0<\sigma_2\le 1, \quad \epsilon_{2n}\to 0, \tag{3.28}$$

$$r(n)=\frac{n}{\sigma_3}(1+\epsilon_{3n}), \quad 0<\sigma_3\le 1, \quad \epsilon_{3n}\to 0. \tag{3.29}$$

Corollary 3.3. *Let $b \in (\frac{\pi}{2}, \pi)$, $a \in (0, \pi)$, and fix $\sigma_3 > 2 - \frac{2b}{\pi}$. If*

$$\lambda_{r(n)}=\tilde{\lambda}_{r(n)}, \quad \text{and} \quad \langle y_{r(n)},\tilde{y}_{r(n)}\rangle_{(b)}=0,$$

for each $n \in \mathbb{N}$, and $h_1 = \tilde{h}_1$, $h_2 = \tilde{h}_2$, and $h_3 = \tilde{h}_3$, then $q(x) = \tilde{q}(x)$ a.e. on $[b, \pi]$.

Proof. To prove that $q(x) = \tilde{q}(x)$ a.e. on $[b,\pi]$, we will consider the supplementary problem $\widehat{L}$ by changing x by $\pi - x$. Let $t = \pi - x$, then for problem (1.1) we have

$$-y'' + q(\pi - t)y = \lambda y.$$

We define $q_1(t) = q(\pi - t)$, then the above equation is the following form

$$\widehat{\ell}y := -y'' + q_1(t)y = \lambda y, \quad 0 < t < \pi$$

$$\lambda(y'(0) - H_1 y(0)) - H_2 y'(0) + H_3 y'(0) = 0,$$
$$\lambda(y'(\pi) - h_1 y(\pi)) - h_2 y'(\pi) + h_3 y(\pi) = 0,$$

with the discontinuous conditions

$$y(\pi - a + 0) = \frac{1}{a_1} y(\pi - a - 0),$$
$$y'(\pi - a + 0) = \frac{1}{a_2} y'(\pi - a - 0) + \frac{a_3}{a_1 a_2} y(\pi - a - 0). \tag{3.30}$$

By the similar proof of Theorem 3.2, we obtain $q_1(t) = \tilde{q}_1(t)$ a.e. on $[0, \pi - b]$. By replacement $x = \pi - t$ we have $q(x) = \tilde{q}(x)$ a.e. on $[b,\pi]$. □

Let μ_n be the eigenvalues of problems (3.1) and (3.31) and $\tilde{\mu}_n$ be the eigenvalues of problems (3.2) and (3.31) with the jump conditions (1.3) such that

$$\lambda(y'(\pi) + \mathfrak{H}_1 y(\pi)) - \mathfrak{H}_2 y'(\pi) - \mathfrak{H}_3 y'(\pi) = 0, \tag{3.31}$$

where $H_1 \neq \mathfrak{H}_1$ and $\mathfrak{H}_i, (i = 1,2,3)$ are real and $\mathfrak{H}_1\mathfrak{H}_3 - \mathfrak{H}_2 > 0$.

Theorem 3.4. *Let $b \in (\frac{\pi}{2}, \pi)$ and $a \in (0,\pi)$, fix $\sigma_2 > \frac{2b}{\pi} - 1$ and $\sigma_3 > 2 - \frac{2b}{\pi}$. If*

$$\lambda_n = \tilde{\lambda}_n, \qquad \mu_{m(n)} = \tilde{\mu}_{m(n)}, \qquad \langle y_{r(n)}, \tilde{y}_{r(n)} \rangle_{(b)} = 0,$$

for each $n \in \mathbb{N}$, then $q(x) = \tilde{q}(x)$ a.e. on $[b,\pi]$.

Proof. Let $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ be the eigenfunctions of L and $\tilde{L}$ corresponding to the eigenvalues λ_n, respectively. Since the eigenfunctions $y_n(x,\lambda_n)$ and $\tilde{y}_n(x,\lambda_n)$ have the same boundary condition at point π and from Corollary 3.3, $q(x) = \tilde{q}(x)$ a.e. on $[b,\pi]$, we obtain

$$\tilde{y}_n(x,\lambda_n) = a_n y_n(x,\lambda_n), \quad x \in (b,\pi], \ n \in N, \tag{3.32}$$

where a_n is constant. From (3.8) and assumptions we get

$$G(\lambda_n) = 0, \qquad G(\mu_{l(n)}) = 0. \tag{3.33}$$

We now show that $G(\lambda) = 0$, for all $\lambda \in \mathbb{C}$. By applying the similar method of the proof of Theorem 3.2 we obtain $q(x) = \tilde{q}(x)$ almost everywhere on $[0,b]$, $h_1 = \tilde{h}_1$, $h_2 = \tilde{h}_2$, and $h_3 = \tilde{h}_3$. Using Corollary

3.3 and the assumption of this theorem we obtain $q(x) = \tilde{q}(x)$ a.e. on $[b, \pi]$. □

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