Sahand Communications in Mathematical Analysis (SCMA) Vol. 3 No. 1 (2016), 29-35
http://journal.maragheh.ac.ir

SOME STUDY ON THE GROWTH PROPERTIES OF ENTIRE FUNCTIONS REPRESENTED BY VECTOR VALUED DIRICHLET SERIES IN THE LIGHT OF RELATIVE RITT ORDERS

SANJIB KUMAR DATTA[1*], TANMAY BISWAS[2], AND PRANAB DAS[3]

ABSTRACT. For entire functions, the notions of their growth indicators such as Ritt order are classical in complex analysis. But the concepts of relative Ritt order of entire functions and as well as their technical advantages of not comparing with the growths of $\exp\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of entire functions in the light of their relative Ritt order are the prime concern of this paper. Actually in this paper we establish some newly developed results related to the growth rates of entire functions on the basis of their relative Ritt order (respectively, relative Ritt lower order).

1. Introduction

Let $f(s)$ be an entire function of the complex variable $s = \sigma + it$ (σ and t are real variables) defined by everywhere absolutely convergent vector valued Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{s\lambda_n}, \tag{1.1}$$

where a_n's belong to a Banach space $(E, \|.\|)$ and λ_n's are non-negative real numbers such that $0 < \lambda_n < \lambda_{n+1}\,(n \geq 1)$, $\lambda_n \to \infty$ as $n \to \infty$ and satisfy the conditions

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty,$$

2010 *Mathematics Subject Classification.* 30B50, 30D15, 30D99.
Key words and phrases. Vector valued, Dirichlet series (VVDS), Relative Ritt order, Relative Ritt lower order, Growth.
Received: 26 July 2015, Accepted: 21 September 2015.
* Corresponding author.

and

$$\limsup_{n\to\infty}\frac{\log\|a_n\|}{\lambda_n}=-\infty.$$

If σ_a and σ_c denote respectively the abscissa of convergence and absolute convergence of (1.1), then in this case clearly $\sigma_a=\sigma_c=\infty$.

The function $M_f(\sigma)$ known as maximum modulus function corresponding to an entire function $f(s)$ defined by (1.1), is written as follows

$$M_f(\sigma)=\underset{-\infty<t<\infty}{l.u.b.}|f(\sigma+it)|.$$

However, the Ritt order [1] of $f(s)$, denoted by ρ_f which is generally used in computational is defined in terms of the growth of $f(s)$ with respect to the $\exp\exp z$ function as

$$\begin{aligned}\rho_f&=\limsup_{\sigma\to\infty}\frac{\log\log M_f(\sigma)}{\log\log M_{\exp\exp z}(\sigma)}\\&=\limsup_{\sigma\to\infty}\frac{\log\log M_f(\sigma)}{\sigma}.\end{aligned}$$

During the past decades, several authors {cf. [1, 2, 3, 5, 7],} made closed investigations on the properties of entire Dirichlet series related to Ritt order. Further, B.L. Srivastava [6] defined different growth parameters such as order and lower order of entire functions represented by vector valued Dirichlet series. He also obtained the results for coefficient characterization of order.

G.S. Srivastava [4] introduced the relative Ritt order between two entire functions represented by vector valued Dirichlet series to avoid comparing growth just with $\exp\exp z$ which is as follows:

$$\begin{aligned}\rho_g(f)&=\inf\{\mu>0:M_f(\sigma)<M_g(\sigma\mu)\text{ for all }\sigma>\sigma_0(\mu)\}\\&=\limsup_{\sigma\to\infty}\frac{M_g^{-1}M_f(\sigma)}{\sigma}.\end{aligned}$$

Similarly, one can define the relative Ritt lower order of $f(s)$ with respect to $g(s)$, denoted by $\lambda_g(f)$ in the following manner:

$$\lambda_g(f)=\liminf_{\sigma\to\infty}\frac{M_g^{-1}M_f(\sigma)}{\sigma}.$$

For entire functions, the notions of their growth indicators such as Ritt order, is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in different directions using the classical growth indicators. But at that time, the concepts of relative Ritt order of entire functions and as well as their technical advantages of not comparing with the growths of $\exp\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of entire functions in the light of their relative Ritt order are the prime concern of this paper. Actually in this paper we establish some newly developed results related to the growth rates of entire functions on the basis of their relative Ritt order (respectively, relative Ritt lower order).

2. Theorems

In this section, we present the main results of the paper.

Theorem 2.1. *If f, g, h and k be any four entire functions represented by vector valued Dirichlet series such that $0 < \lambda_h(f) \leq \rho_h(f) < \infty$ and $0 < \lambda_k(g) \leq \rho_k(g) < \infty$, then*

$$\begin{aligned}\frac{\lambda_h(f)}{\rho_k(g)} &\leq \liminf_{\sigma\to\infty} \frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \\ &\leq \frac{\lambda_h(f)}{\lambda_k(g)} \\ &\leq \limsup_{\sigma\to\infty} \frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \\ &\leq \frac{\rho_h(f)}{\lambda_k(g)}.\end{aligned}$$

Proof. From the definition of $\rho_k(g)$ and $\lambda_h(f)$, we have for arbitrary positive ε and for all sufficiently large values of σ, that

$$M_h^{-1}M_f(\sigma) \geq (\lambda_h(f) - \varepsilon)\,\sigma, \tag{2.1}$$

and

$$M_k^{-1}M_g(\sigma) \leq (\rho_k(g) + \varepsilon)\,\sigma. \tag{2.2}$$

Now from (2.1) and (2.2), it follows for all sufficiently large values of σ, that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{(\lambda_h(f) - \varepsilon)\,\sigma}{(\rho_k(g) + \varepsilon)\,\sigma}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain

$$\liminf_{\sigma\to\infty} \frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{\lambda_h(f)}{\rho_k(g)}. \tag{2.3}$$

Again for a sequence of values of σ tending to infinity,

$$M_h^{-1}M_f(\sigma) \leq (\lambda_h(f) + \varepsilon)\,\sigma, \tag{2.4}$$

and for all large values of σ,

$$M_k^{-1}M_g(\sigma) \geq (\lambda_k(g) - \varepsilon)\,\sigma. \tag{2.5}$$

Combining (2.4) and (2.5), we get for a sequence of values of σ tending to infinity, that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{(\lambda_h(f)+\varepsilon)\,\sigma}{(\lambda_k(g)-\varepsilon)\,\sigma}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\lambda_h(f)}{\lambda_k(g)}. \tag{2.6}$$

Also for a sequence of values of σ tending to infinity, it follows that

$$M_k^{-1}M_g(\sigma) \leq (\lambda_k(g)+\varepsilon)\,\sigma. \tag{2.7}$$

Now from (2.1) and (2.7), we obtain for a sequence of values of σ tending to infinity, that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{(\lambda_h(f)-\varepsilon)\,\sigma}{(\lambda_k(g)+\varepsilon)\,\sigma}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{\lambda_h(f)}{\lambda_k(g)}. \tag{2.8}$$

Also for all sufficiently large values of σ,

$$M_h^{-1}M_f(\sigma) \leq (\rho_h(f)+\varepsilon)\,\sigma. \tag{2.9}$$

Now it follows from (2.5) and (2.9), for all sufficiently large values of σ, that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{(\rho_h(f)+\varepsilon)\,\sigma}{(\lambda_k(g)-\varepsilon)\,\sigma}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain

$$\limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\rho_h(f)}{\lambda_k(g)}. \tag{2.10}$$

Thus the theorem follows from (2.3), (2.6), (2.8) and (2.10). □

Theorem 2.2. *If f, g, h and k be any four entire functions represented by vector valued Dirichlet series such that $0<\rho_h(f)<\infty$ and $0<\rho_k(g)<\infty$, then*

$$\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\rho_h(f)}{\rho_k(g)} \leq \limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)}.$$

Proof. From the definition of $\rho_k(g)$, we get for a sequence of values of σ tending to infinity, that

$$M_k^{-1}M_g(\sigma) \geq (\rho_k(g)-\varepsilon)\,\sigma. \tag{2.11}$$

Now from (2.9) and (2.11), it follows for a sequence of values of σ tending to infinity, that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{(\rho_h(f)+\varepsilon)\,\sigma}{(\rho_k(g)-\varepsilon)\,\sigma}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain

$$\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \leq \frac{\rho_h(f)}{\rho_k(g)}. \tag{2.12}$$

Again for a sequence of values of σ tending to infinity, we have

$$M_h^{-1}M_f(\sigma) \geq (\rho_h(f)-\varepsilon)\,\sigma. \tag{2.13}$$

So, combining (2.2) and (2.13), we get for a sequence of values of σ tending to infinity, that

$$\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{(\rho_h(f)-\varepsilon)\,\sigma}{(\rho_k(g)+\varepsilon)\,\sigma}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} \geq \frac{\rho_h(f)}{\rho_k(g)}. \tag{2.14}$$

Thus the theorem follows from (2.12) and (2.14). □

The following theorem is a natural consequence of Theorem 2.1 and Theorem 2.2.

Theorem 2.3. *If f, g, h and k are any four entire functions represented by vector valued Dirichlet series such that $0<\lambda_h(f)\leq\rho_h(f)<\infty$ and $0<\lambda_k(g)\leq\rho_k(g)<\infty$ then*

$$\begin{aligned}\liminf_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)} &\leq \min\left\{\frac{\lambda_h(f)}{\lambda_k(g)},\frac{\rho_h(f)}{\rho_k(g)}\right\}\\ &\leq \max\left\{\frac{\lambda_h(f)}{\lambda_k(g)},\frac{\rho_h(f)}{\rho_k(g)}\right\}\\ &\leq \limsup_{\sigma\to\infty}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)}.\end{aligned}$$

Theorem 2.4. *Let f, g, h and k be any four entire functions represented by vector valued Dirichlet series such that $\rho_k(g)<\infty$. If $\lambda_h(f)=\infty$ then*

$$\lim_{r\to\sigma}\frac{M_h^{-1}M_f(\sigma)}{M_k^{-1}M_g(\sigma)}=\infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of σ tending to infinity

$$M_h^{-1} M_f(\sigma) \leq \beta M_k^{-1} M_g(\sigma)\,. \tag{2.15}$$

Again from the definition of $\rho_k(g)$, it follows that for all sufficiently large values of σ, that

$$M_k^{-1} M_g(\sigma) \leq (\rho_k(g) + \varepsilon)\,\sigma. \tag{2.16}$$

Thus from (2.15) and (2.16), we have for a sequence of values of σ tending to infinity, that

$$M_h^{-1} M_f(\sigma) \leq \beta\,(\rho_k(g) + \varepsilon)\,\sigma,$$

$$\text{i.e.,} \qquad \frac{M_h^{-1} M_f(\sigma)}{\sigma} \leq \frac{\beta(\rho_k(g)+\varepsilon)\sigma}{\sigma},$$

$$\text{i.e.,} \qquad \liminf_{r\to\sigma} \frac{M_h^{-1} M_f(\sigma)}{\sigma} = \lambda_h(f) < \infty.$$

This is a contradiction. This proves the theorem. □

Remark 2.5. Theorem 2.4 is also valid with "limit superior" instead of "limit" if $\lambda_h(f) = \infty$ is replaced by $\rho_h(f) = \infty$ and the other conditions remaining the same.

References

1. Q.I. Rahaman, *The Ritt order of the derivative of an entire function*, Annales Polonici Mathematici., 17 (1965) 137–140.
2. C.T. Rajagopal and A.R. Reddy, *A note on entire functions represented by Dirichlet series*, Annales Polonici Mathematici., 17 (1965) 199-208.
3. J.F. Ritt, *On certain points in the theory of Dirichlet series*, Amer. Jour. Math., 50 (1928) 73–86.
4. G.S. Srivastava, *A note on relative type of entire functions represented by vector valued dirichlet series*, Journal of Classicial Analysis, 2(1) (2013) 61-72.
5. G.S. Srivastava and A. Sharma, *On generalized order and generalized type of vector valued Dirichlet series of slow growth*, Int. J. Math. Archive, 2(12) (2011) 2652-2659.
6. B.L. Srivastava, *A study of spaces of certain classes of vector valued Dirichlet series*, Thesis, I. I. T., Kanpur, 1983.
7. R.P. Srivastav and R.K. Ghosh, *On entire functions represented by Dirichlet series*, Annales Polonici Mathematici., 13 (1963) 93–100.

[1] Department of Mathematics, University of Kalyani, P.O.-Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India.

E-mail address: sanjib_kr_datta@yahoo.co.in

[2] Rajbari, Rabindrapalli, R. N. Tagore Road, P.O.-Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India.

E-mail address: tanmaybiswas_math@rediffmail.com

[3] Department of Mathematics, University of Kalyani, P.O.-Kalyani, Dist-Nadia, PIN- 741235, West Bengal, India.

E-mail address: pranabdas90@gmail.com