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ON STRONGLY JORDAN ZERO-PRODUCT PRESERVING MAPS

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Abstract. In this paper, we give a characterization of strongly Jordan zero-product preserving maps on normed algebras as a generalization of Jordan zero-product preserving maps. In this direction, we give some illustrative examples to show that the notions of strongly zero-product preserving maps and strongly Jordan zero-product preserving maps are completely different. Also, we prove that the direct product and the composition of two strongly Jordan zero-product preserving maps are again strongly Jordan zero-product preserving maps. But this fact is not the case for tensor product of them in general. Finally, we prove that every $*-$preserving linear map from a normed $*-$algebra into a C^*-algebra that strongly preserves Jordan zero-products is necessarily continuous.

1. Introduction and preliminaries

The author of the current study recently has introduced and investigated the notions of strongly zero-product and strongly Jordan zero-product preserving maps on a class of normed algebras [4]. These notions are generalization of the concepts " zero-product and Jordan zero-product preserving maps " respectively.

In this direction, the basic properties of strongly zero-product preserving maps were investigated in [5] on general normed algebras.
A linear map $\varphi : A \longrightarrow B$ between two algebras A and B, over a field F is said to be a zero-product preserving map if, $\varphi(a)\varphi(c) = 0$ whenever

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$ac = 0$, $(a, c \in A)$. Also, φ is said to be a Jordan zero-product preserving map if, $\varphi(a) \circ \varphi(c) = 0$ whenever $a \circ c = 0$, where " $\circ$ " denotes the Jordan product $a \circ c = ac + ca$. The notions of zero-product and Jordan zero-product preserving maps are different notions and one does not imply the other in general (see Example 3.1 in this paper). Some useful references in the field of zero-product and Jordan zero-product preserving maps are [1, 2].

Let A and B be two normed algebras over a field F. We say that a linear map $\varphi : A \longrightarrow B$ is a strongly zero-product preserving map if, for any two sequences $\{a_n\}_n, \{c_n\}_n$ in A, $\varphi(a_n)\varphi(c_n) \longrightarrow 0$ whenever $a_n c_n \longrightarrow 0$.

Also, we say that φ is a strongly Jordan zero-product preserving map if, for any two sequences $\{a_n\}_n, \{c_n\}_n$ in A, $\varphi(a_n) \circ \varphi(c_n) \longrightarrow 0$ whenever $a_n \circ c_n \longrightarrow 0$.

The following Remark similar to the [5, Remark 3.4] holds.

Remark 1.1.
(i) Let A and B be normed algebras and let B be unital. Then every surjective strongly Jordan zero-product preserving map is continuous.
(ii) Let A and B be two unital normed algebras with the units 1_A and 1_B, respectively. Also let $\varphi : A \longrightarrow B$ be a strongly Jordan zero-product preserving map such that $\varphi(1_A) = 1_B$. Then φ is continuous.

For the normed algebras A and B over a field F, we will denote by $A \otimes B$ the algebraic tensor product of A and B. It is well known that $A \otimes B$ is a normed algebra with the following projective cross norm given by

$$\|u\| = \inf \left\{ \sum_{k=1}^{k=n} \|a_k\| \|b_k\|, u = \sum_{k=1}^{k=n} a_k \otimes b_k, a_k \in A, b_k \in B, n \in \mathbb{N} \right\},$$

for all $u \in A \otimes B$.

2. Examples

Example 2.1.
(i) Every continuous homomorphism between normed algebras is strongly zero-product and also strongly Jordan zero-product preserving map.
(ii) Let $\mathcal{V}$ be an infinite dimensional normed vector space with the basis $\beta = \{e_1, e_2, e_3, \ldots\}$ such that $\|e_n\| = 1$ for all $n \geq 1$. Also, let $f \in \mathcal{V}^*$ be a continuous linear functional satisfying $f(e_1) = 1$ and $f(e_n) = 0$ for all $n \geq 2$. So $\ker f = \overline{\text{span}}\{e_2, e_3, e_4, \ldots\}$. For all $a, b \in \mathcal{V}$ define $a \cdot b = f(a)b$. Clearly, $(\mathcal{V}, \cdot)$ is an associative

normed algebra (for the basic properties of this algebra see [3]). We denote it by $\mathcal{V}_f$. Define $\varphi : \mathcal{V}_f \longrightarrow \mathcal{V}_f$ such that $\varphi(e_1) = 0$ and $\varphi(e_n) = 2^n e_2$ for all $n \geq 2$. Since $f \circ \varphi \equiv 0$; it is obvious that φ is a strongly zero-product and also strongly Jordan zero-product preserving map. We show that φ is neither a continuous map nor a homomorphism on $\mathcal{V}_f$. To this end let $a_n = \frac{e_n}{n}$. So,

$$\|a_n\| = \frac{1}{n}\|e_n\| = \frac{1}{n} \longrightarrow 0.$$

But

$$\lim_{n\to\infty} \|\varphi(a_n)\| = \lim_{n\to\infty} \frac{2^n}{n}\|e_2\| = \lim_{n\to\infty} \frac{2^n}{n} = \infty.$$

This shows that φ is not a continuous map. Also, $4e_2 = \varphi(e_2) = \varphi(e_1 e_2) \neq \varphi(e_1)\varphi(e_2) = 0$. So, φ is not a homomorphism.

The part (ii) of Example 2.1 shows that strongly zero-product preserving maps and strongly Jordan zero-product preserving maps are not continuous maps or homomorphisms in general.

Remark 2.2. It is obvious that every strongly zero-product (strongly Jordan zero-product) preserving map is a zero-product (Jordan zero-product) preserving map. But the converse is not the case in general. The following example shows this fact.

Example 2.3. Let $\mathcal{V}$ be an infinite dimensional normed vector space with the basis $\beta = \{e_1, e_2, e_3, \ldots\}$ such that $\|e_n\| = 1$ for all $n \geq 1$. Also let $f \in \mathcal{V}^*$ be a continuous linear functional satisfying $f(e_1) = 1$ and $f(e_n) = 0$ for all $n \geq 2$. So, $\ker f = \overline{\operatorname{span}}\{e_2, e_3, e_4, \ldots\}$. Define $\varphi : \mathcal{V}_f \longrightarrow \mathcal{V}_f$ such that $\varphi(a) = f(a)e_1 + \theta(a)$, where $\theta : \mathcal{V}_f \longrightarrow \ker f$ is a linear map such that $\theta(e_1) = 0$ and $\theta(e_n) = 2^n e_2$ for all $n \geq 2$. It is obvious that $\varphi(\ker f) \subseteq \ker f$. So, by [4, Theorems 2.1 and 2.2], φ is a zero-product (Jordan zero-product) preserving map. But we show that φ is not a strongly Jordan zero-product (strongly zero-product) preserving map. To this end, let $a_n = \frac{e_1}{n}$ and $c_n = e_{n+1}$. Clearly, $\|a_n \circ c_n\| \longrightarrow 0$ but $\lim_{n\to\infty} \|\varphi(a_n) \circ \varphi(c_n)\| = \lim_{n\to\infty} \frac{2^{n+1}}{n}\|e_2\| = \lim_{n\to\infty} \frac{2^{n+1}}{n} = \infty$. This shows that φ is not a strongly Jordan zero-product preserving map. A similar argument can be applied to show that $\|a_n c_n\| \longrightarrow 0$ but $\|\varphi(a_n)\varphi(c_n)\| \longrightarrow \infty$. So φ is not a strongly zero-product preserving map.

3. The notions of strongly zero-product and strongly Jordan zero-product preserving maps are different

In this section, we give some illustrative examples to show that the notions of strongly zero-product and strongly Jordan zero-product preserving maps are different and one does not imply the other.

Example 3.1. (i) Let $A = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C} \right\}$ and $B = M_{2\times 2}(\mathbb{C})$. It is obvious that A and B with the usual addition and multiplication and with the norm

$$\left\| \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \right\| = 2 \max \left\{ |\alpha|,\ |\beta|,\ |\gamma|,\ |\delta| \right\},$$

are Banach algebras. Define $\varphi : A \longrightarrow B$ such that

$$\varphi\left(\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \alpha & \alpha \\ \beta & 0 \end{bmatrix}.$$

The linearity of φ is obvious. We show that φ is a strongly Jordan zero-product preserving map. For each $a = \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix}$ and $c = \begin{bmatrix} \lambda & \mu \\ 0 & 0 \end{bmatrix} \in A$,

$$\begin{aligned} \|\varphi(a) \circ \varphi(c)\| &= \left\| \begin{bmatrix} 2\alpha\lambda + \alpha\mu + \lambda\beta & 2\alpha\lambda \\ \beta\lambda + \alpha\mu & \beta\lambda + \alpha\mu \end{bmatrix} \right\| \\ &\leq 2\|a \circ c\|. \end{aligned}$$

It follows that φ is a strongly Jordan zero-product preserving map. But φ is not a strongly zero-product preserving map. Indeed, let $a_n = \begin{bmatrix} 0 & n \\ 0 & 0 \end{bmatrix}$ and $c_n = \begin{bmatrix} n & 0 \\ 0 & 0 \end{bmatrix}$. Clearly, $a_n c_n \longrightarrow 0$ but

$$\begin{aligned} \|\varphi(a_n)\varphi(c_n)\| &= \left\| \begin{bmatrix} 0 & 0 \\ n^2 & n^2 \end{bmatrix} \right\| \\ &= 2n^2 \longrightarrow \infty. \end{aligned}$$

(Also note that φ is not a zero-product preserving map).

(ii) Let $A = \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C} \right\}$ and $B = M_{2\times 2}(\mathbb{C})$. It is obvious that A and B are normed algebras over the real field $F = \mathbb{R}$, with the mentioned norm in part (i). Also every non-zero element of A is invertible. Define $\varphi : A \longrightarrow B$ such that

$$\varphi\left(\begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \right) = \begin{bmatrix} 0 & 0 \\ \alpha & \beta \end{bmatrix}.$$

Clearly, φ is an $\mathbb{R}-$linear map. We show that φ is a strongly zero-product preserving map.
Let $a = \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix}$ and $c = \begin{bmatrix} \lambda & \mu \\ -\bar{\mu} & \bar{\lambda} \end{bmatrix}$ be two elements of A. It is obvious that

$$\begin{aligned} ac &= \begin{bmatrix} \alpha\lambda - \beta\bar{\mu} & \alpha\mu + \beta\bar{\lambda} \\ -\bar{\beta}\lambda - \bar{\alpha}\bar{\mu} & -\bar{\beta}\mu + \bar{\alpha}\bar{\lambda} \end{bmatrix} \\ &= \begin{bmatrix} s & t \\ -\bar{t} & \bar{s} \end{bmatrix} \end{aligned}$$

where,

$$\begin{cases} \alpha\lambda - \beta\bar{\mu} = s \\ \alpha\mu + \beta\bar{\lambda} = t, \end{cases} \tag{3.1}$$

Let $c \neq 0$. So, $|\lambda|^2 + |\mu|^2 \neq 0$. From (3.1) we have

$$\beta = \frac{-\mu}{|\lambda|^2 + |\mu|^2} s + \frac{\lambda}{|\lambda|^2 + |\mu|^2} t.$$

So,

$$\beta\lambda = \frac{-\lambda\mu}{|\lambda|^2 + |\mu|^2} s + \frac{\lambda^2}{|\lambda|^2 + |\mu|^2} t$$

and

$$\beta\mu = \frac{-\mu^2}{|\lambda|^2 + |\mu|^2} s + \frac{\lambda\mu}{|\lambda|^2 + |\mu|^2} t.$$

It follows that

$$\begin{aligned} |\beta\lambda| &\leq \frac{|-\lambda\mu|}{|\lambda|^2 + |\mu|^2} |s| + \frac{|\lambda^2|}{|\lambda|^2 + |\mu|^2} |t| \\ &\leq \frac{1}{2} |s| + |t| \\ &\leq \|ac\|. \end{aligned}$$

Similarly, $|\beta\mu| \leq \|ac\|$. So,

$$\begin{aligned} \|\varphi(a)\varphi(c)\| &= \left\| \begin{bmatrix} 0 & 0 \\ \beta\lambda & \beta\mu \end{bmatrix} \right\| \\ &\leq 2\|ac\|. \end{aligned}$$

This shows that φ is a strongly zero-product preserving map. We shall show that φ is not a strongly Jordan zero-product preserving map. Indeed, let $a_n = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $c_n = \begin{bmatrix} ni & 0 \\ 0 & -ni \end{bmatrix}$.

Clearly, $a_n \circ c_n = 0$ so $\lim_{n\longrightarrow\infty} a_n \circ c_n = 0$. But

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|\varphi(a_n) \circ \varphi(c_n)\| &= \lim_{n\longrightarrow\infty} \left\| \begin{bmatrix} 0 & 0 \\ ni & 0 \end{bmatrix} \right\| \\ &= \lim_{n\longrightarrow\infty} 2|ni| \\ &= \infty.\end{aligned}$$

This shows that φ is not a strongly Jordan zero-product preserving map. (Also note that φ is not a Jordan zero-product preserving map).

Remark 3.2. Example 3.1 shows that the notions of zero-product and Jordan zero-product preserving maps are completely different and one does not imply the other.

4. Main Results

In this section, we give a characterization of strongly Jordan zero-product preserving maps on normed algebras. Also, we prove that every $*-$preserving linear map from a normed $*-$algebra into a C^*-algebra that strongly preserves Jordan zero-products is necessarily continuous.

Theorem 4.1. *Let A and B be normed algebras. Then a linear map $\varphi : A \longrightarrow B$ is a strongly Jordan zero-product preserving map if and only if there exists $M > 0$ such that*

$$\|\varphi(a) \circ \varphi(c)\| \leq M\|a \circ c\|, (a, c \in A).$$

Proof. Similar to the proof of [5, Theorem 3.1], by contradiction, suppose that φ is a strongly Jordan zero-product preserving map and the desired inequality is not true for all $M > 0$. So, for $M = 1$ there exist $a_1,\ c_1 \in A$ such that,

$$\|\varphi(a_1) \circ \varphi(c_1)\| > \|a_1 \circ c_1\|.$$

For $M = \frac{2}{\|\varphi(a_1)\circ\varphi(c_1)\|}$ there exist $a_2,\ c_2 \in A$ such that

$$\|\varphi(a_2) \circ \varphi(c_2)\| > \frac{2}{\|\varphi(a_1) \circ \varphi(c_1)\|}\|a_2 \circ c_2\|.$$

It follows that

$$\left\| \frac{a_2}{\|\varphi(a_2) \circ \varphi(c_2)\|} \circ c_2 \right\| < \frac{\|\varphi(a_1) \circ \varphi(c_1)\|}{2}.$$

A similar argument can be applied to show that, for

$$M = \frac{n}{\|\varphi(a_1) \circ \varphi(c_1)\|},$$

there exist $a_n,\ c_n \in A$ such that

$$\left\| \frac{a_n}{\|\varphi(a_n) \circ \varphi(c_n)\|} \circ c_n \right\| < \frac{\|\varphi(a_1) \circ \varphi(c_1)\|}{n}.$$

Let $a'_n = \frac{a_n}{\|\varphi(a_n)\circ\varphi(c_n)\|}$ and $c'_n = c_n$. As $a'_n \circ c'_n \longrightarrow 0$, it follows that

$$\varphi(a'_n) \circ \varphi(c'_n) \longrightarrow 0.$$

That is a contradiction. The converse is obvious. □

Definition 4.2. Let A and B be two $*-$algebras. We say that a linear map $\varphi : A \longrightarrow B$ is $*-$preserving if, $\varphi(a^*) = \varphi(a)^*, \quad (a \in A)$.

Proposition 4.3. *Let A be a normed $*-$ algebra and B be a C^*-algebra. Also let $\varphi : A \longrightarrow B$ be a $*-$preserving linear map that strongly preserves Jordan zero-products. Then, φ is continuous.*

Proof. Let $\varphi : A \longrightarrow B$ be a $*-$preserving linear map that strongly preserves Jordan zero-products. Also, Let $\{a_n\}_n$ be a sequence in A such that $a_n \longrightarrow 0$. So, $a_n = b_n + ic_n$, where $b_n = \frac{a_n+a_n^*}{2}$ and $c_n = \frac{a_n-a_n^*}{2i}$. Obviously, b_n and c_n are self adjoint elements such that

$$\lim_{n\longrightarrow\infty} b_n = \lim_{n\longrightarrow\infty} c_n = 0.$$

It follows that $b_n \circ b_n \longrightarrow 0$ and $c_n \circ c_n \longrightarrow 0$. So, $\varphi(b_n) \circ \varphi(b_n) \longrightarrow 0$ and $\varphi(c_n) \circ \varphi(c_n) \longrightarrow 0$. So

$$\begin{aligned} \|\varphi(b_n)\|^2 &= \|\varphi(b_n)\varphi(b_n)^*\| \\ &= \|\varphi(b_n)\varphi(b_n^*)\| \\ &= \|\varphi(b_n)\varphi(b_n)\| \\ &= \frac{1}{2}\|\varphi(b_n) \circ \varphi(b_n)\| \longrightarrow 0. \end{aligned}$$

This shows that $\varphi(b_n) \longrightarrow 0$. Similarly $\varphi(c_n) \longrightarrow 0$. So, $\varphi(a_n) \longrightarrow 0$ and equivalently φ is continuous. □

5. Hereditary Properties

In this section, similar to [5], we present the hereditary properties of strongly Jordan zero-product preserving maps. We prove that the direct product and the composition of two strongly Jordan zero-product preserving maps are again strongly Jordan zero-product preserving maps. But this fact is not the case for the tensor product of them in general. The proofs of the following results are similar to [5, Proposition 4.1 and proposition 4.2].

Proposition 5.1. *Let A, B, C, D be normed algebras and let $\varphi : A \longrightarrow B$ and $\psi : C \longrightarrow D$ be two strongly Jordan zero-product preserving maps. Then, $\varphi \oplus \psi : A \oplus C \longrightarrow B \oplus D$ is a strongly Jordan zero-product preserving map.*

Proof. As φ and ψ are strongly Jordan zero-product preserving maps, there exist $M,\ N > 0$ such that $\|\varphi(a) \circ \varphi(a')\| \leq M\|a \circ a'\|$ and

$$\|\psi(c) \circ \psi(c')\| \leq N\|c \circ c'\|,\ (a,\ a' \in A,\ c,\ c' \in C).$$

So,

$$\begin{aligned}
\|(\varphi \oplus \psi)(a, c) \circ (\varphi \oplus \psi)(a', c')\| &= \|(\varphi(a), \psi(c)) \circ (\varphi(a'), \psi(c'))\| \\
&= \|(\varphi(a) \circ \varphi(a'), \psi(c) \circ \psi(c'))\| \\
&= \|\varphi(a) \circ \varphi(a')\| + \|\psi(c) \circ \psi(c')\| \\
&\leq M\|a \circ a'\| + N\|c \circ c'\| \\
&\leq M(\|a \circ a'\| + \|c \circ c'\|) \\
&\quad + N(\|a \circ a'\| + \|c \circ c'\|) \\
&= (M + N)(\|a \circ a'\| + \|c \circ c'\|) \\
&= (M + N)\|(a \circ a', c \circ c')\| \\
&= (M + N)\|(a, c) \circ (a', c')\|.
\end{aligned}$$

Applying Theorem 4.1 shows that $\varphi \oplus \psi$ is a strongly Jordan zero-product preserving map. □

Proposition 5.2. *Let A, B and C be normed algebras and let $\varphi : A \longrightarrow B$ and $\psi : B \longrightarrow C$ be two strongly Jordan zero-product preserving maps. Then, $\psi \circ \varphi : A \longrightarrow C$ is a strongly Jordan zero-product preserving map.*

Proof. As φ and ψ are strongly Jordan zero-product preserving maps, there exist $M, N > 0$ such that $\|\varphi(a) \circ \varphi(a')\| \leq M\|a \circ a'\|$ and

$$\|\psi(b) \circ \psi(b')\| \leq N\|b \circ b'\|, \quad (a,\ a' \in A,\ b,\ b' \in B).$$

So,

$$\begin{aligned}
\|(\psi \circ \varphi)(a) \circ (\psi \circ \varphi)(a')\| &= \|\psi(\varphi(a)) \circ \psi(\varphi(a'))\| \\
&\leq N\|\varphi(a) \circ \varphi(a')\| \\
&\leq MN\|a \circ a'\|,\ (a,\ a' \in A).
\end{aligned}$$

This shows that $\psi \circ \varphi$ is a strongly Jordan zero-product preserving map. □

The following example shows that the tensor product of two strongly Jordan zero-product preserving maps need not be a strongly Jordan zero-product preserving map in general.

Example 5.3. Let

$$A = \left\{ \begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \,\middle|\, \alpha, \beta \in \mathbb{C} \right\}$$

and $B = M_{2\times 2}(\mathbb{C})$. Define $\varphi : A \longrightarrow B$ such that

$$\varphi\left(\begin{bmatrix} \alpha & \beta \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} \alpha & \alpha \\ \beta & 0 \end{bmatrix}.$$

By Example 3.1 φ is a strongly Jordan zero-product preserving map. We shall show that $\varphi \otimes \varphi : A \otimes A \longrightarrow B \otimes B$ is not a strongly Jordan zero-product preserving map. To this end, let $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $y = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. One can simply verify that $x \circ y = 0$. But

$$(\varphi \otimes \varphi)(x) \circ (\varphi \otimes \varphi)(y) = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \neq 0.$$

This shows that $\varphi \otimes \varphi$ is not a Jordan zero-product preserving map which implies that $\varphi \otimes \varphi$ is not a strongly Jordan zero-product preserving map.

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