Sahand Communications in Mathematical Analysis (SCMA) Vol. 3 No. 2 (2016), 91-106
http://scma.maragheh.ac.ir

ON MULTIPLICATIVE (STRONG) LINEAR PRESERVERS OF MAJORIZATIONS

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Abstract. In this paper, we study some kinds of majorizations on $\mathbf{M}_n$ and their linear or strong linear preservers. Also, we find the structure of linear or strong linear preservers which are multiplicative, i.e. linear or strong linear preservers like Φ with the property $\Phi(AB) = \Phi(A)\Phi(B)$ for every $A, B \in \mathbf{M}_n$.

1. Introduction

An $n \times n$ nonnegative real matrix A is doubly stochastic if all of its row and column sums are one. For $X, Y \in \mathbf{M}_n$, it is said that X is majorized by Y (written as $X \prec Y$) if there exists a doubly stochastic matrix D such that $X = DY$. If we omit the non-negativity condition, we reach to the set of all generalized $n \times n$ doubly stochastic matrices, denoted by GD_n. By considering some other kinds of stochastic matrices such as row stochastic, upper triangular row stochastic, tridiagonal row stochastic and even stochastic matrices, we reach to various kinds of majorizations. Let $T : \mathbf{M}_n \to \mathbf{M}_n$ be a linear operator. For a relation $\sim$ on $\mathbf{M}_n$, it is said that T is a linear preserver of $\sim$, if $TX \sim TY$ whenever $X \sim Y$. Similarly, it is said that T is a strong linear preserver of $\sim$, if

$$X \sim Y \quad \Leftrightarrow \quad TX \sim TY.$$

Also, it is said that T is a multiplicative linear preserver of $\sim$, if in addition $T(XY) = T(X)T(Y)$, for all $X, Y \in \mathbf{M}_n$. In this paper, we characterize all multiplicative linear or strong linear preservers of some kinds of majorizations. It is well-known that nonzero linear and multiplicative operators on M_n are inner automorphisms. We use this fact in

2010 *Mathematics Subject Classification.* 15A04, 15A21.
Key words and phrases. Doubly stochastic matrix, Linear preserver, Multiplicative map .
Received: 24 December 2015, Accepted: 27 January 2016.
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some proofs but we prefer to prove other theorems without using this. Assume $\mathcal{P}_n$ be the set of all $n \times n$ permutation matrices and E_{ij} be an $n \times n$ matrix such that its ijth entry is 1, and all other entries are 0.

2. Multiplicative linear preservers of majorizations

In this section, we find multiplicative linear preservers of some kinds of majorizations. Let $X, Y \in \mathbf{M}_n$. X is said to be multivariately majorized by Y (written as $X \prec_m Y$), if $X = DY$ for some $n \times n$ doubly stochastic matrix D and called directionally majorized by Y (written as $X \prec_d Y$), if for every $a \in \mathbb{R}^n$ there exists a doubly stochastic matrix D_a such that $Xa = D_a Ya$. In [6] and [7], the authors found the linear preservers and strong linear preservers of multivariate and directional majorizations as follows.

In this paper, the letter J indicates the matrix with all entries equal to 1, and $tr(x)$ uses for the summation of coordinates of a vector x.

Theorem 2.1 ([7], Theorem 1)**.** *Let $T : M_{n,m} \to M_{n,m}$ be a linear function. Then, T preserves multivariate majorizations if and only if T preserves directionally majorizations if and only if one of the following holds:*

(a) *There exist $A_1, \ldots, A_m \in M_{n,m}$ such that*

$$TX = \sum_{i=1}^{m} (tr(x_i))\, A_i,$$

where x_i is the i'th column of X.

(b) *There exist $P \in \mathcal{P}_n$ and $R, S \in M_m$ such that*

$$TX = PXR + JXS.$$

Theorem 2.2 ([6], Theorem 2.4)**.** *Let $T : M_{n,m} \to M_{n,m}$ be a linear function. Then, T strongly preserves multivariate majorizations if and only if T strongly preserves directional majorizations if and only if there exist $P \in \mathcal{P}_n$ and $R, S \in M_m$ such that $TX = PXR + JXS$ and $R(R + nS)$ is invertible.*

Let $T : \mathbf{M}_n \to \mathbf{M}_n$ has the form $TX = PXR + JXS$, for some $P \in \mathcal{P}_n$. Then, T is a multiplicative linear preserver if and only if QTQ^t is a multiplicative linear preserver for every $Q \in \mathcal{P}_n$. So, in the proof of the following theorem without loss of generality, we can assume that $P = I$.

Theorem 2.3. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a linear function. Then, T is a multiplicative linear preserver of a multivariate or a directional majorization if and only if T has one of the following forms:*

(i) *$TX = 0$ for all $X \in \boldsymbol{M}_n$.*

(ii) *There exists a permutation* $P \in \mathcal{P}_n$ *such that*

$$TX = PXP^{-1},$$

for all $X \in \boldsymbol{M}_n$.

(iii) *There exists a permutation* $P \in \mathcal{P}_n$ *and a scalar* $\gamma \neq -n$ *such that*

$$TX = (\gamma P + J)X(\gamma P + J)^{-1},$$

for all $X \in \boldsymbol{M}_n$.

Proof. It is easy to prove that each of the conditions (i), (ii) or (iii) are sufficient for T to be a multiplicative linear preserver of a multivariate majorization. So, we only prove the necessity of the conditions. It is clear that $E_{ij}E_{kl} = 0$ if and only if $j \neq k$. Let T be of the form (a) in Theorem 2.1. Then,

$$TX = \sum_{i=1}^{n} (tr(x_i))\, A_i,$$

where $A_1, \ldots, A_n \in M_n$. Let $1 \leq k \leq n$, and put $X = Y = E_{kk}$. Then

$$\begin{aligned} A_k A_k &= T(X)T(Y) \\ &= T(X) \\ &= A_k. \end{aligned}$$

So, $A_k^2 = A_k$. On the other hand, set $X = E_{kk}$ and $Y = E_{jk} (j \neq k)$. Then,

$$\begin{aligned} A_k A_k &= T(X)T(Y) \\ &= T(XY) \\ &= T(0) \\ &= 0. \end{aligned}$$

Therefore, $A_k = 0$ and hence $T = 0$.

Now, suppose that T has the form (b) in Theorem 2.1. Then,

$$\begin{aligned} T(E_{ij}) &= E_{ij}R + JE_{ij}S \\ &= \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ r_{j1} & \cdots & r_{jn} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} s_{j1} & \cdots & s_{jn} \\ \vdots & \vdots & \vdots \\ s_{j1} & \cdots & s_{jn} \end{bmatrix}, \end{aligned}$$

where $r_{j1}, \ldots, r_{jn}$ occur in the i'th row. Since $k \neq j$,

$$\begin{aligned}
0 &= T(0) \\
&= T(E_{ij}E_{kl}) \\
&= T(E_{ij})T(E_{kl}) \\
&= r_{jk}\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ r_{l1} & \cdots & r_{ln} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} + \left(\sum_{m=1}^{n} r_{jm}\right)\begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ s_{l1} & \cdots & s_{ln} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \\
&\quad + s_{jk}\begin{bmatrix} r_{l1} & \cdots & r_{ln} \\ \vdots & \cdots & \vdots \\ r_{l1} & \cdots & r_{ln} \end{bmatrix} + \left(\sum_{m=1}^{n} s_{jm}\right)\begin{bmatrix} s_{l1} & \cdots & s_{ln} \\ \vdots & \cdots & \vdots \\ s_{l1} & \cdots & s_{ln} \end{bmatrix},
\end{aligned}$$

where $r_{l1}, \ldots, r_{ln}$ and $s_{l1}, \ldots, s_{ln}$ in the first and second matrices occur in the i'th row. So,

$$s_{jk}\begin{bmatrix} r_{l1} & \ldots & r_{ln} \end{bmatrix} + \left(\sum_{m=1}^{n} s_{jm}\right)\begin{bmatrix} s_{l1} & \ldots & s_{ln} \end{bmatrix} = 0, \tag{2.1}$$

and

$$r_{jk}\begin{bmatrix} r_{l1} & \ldots & r_{ln} \end{bmatrix} + \left(\sum_{m=1}^{n} r_{jm}\right)\begin{bmatrix} s_{l1} & \ldots & s_{ln} \end{bmatrix} = 0. \tag{2.2}$$

We consider two cases.

Case 1: Let $s_{jk} = 0$ for all $j, k\,(1 \leq j \neq k \leq n)$. Put $l = j$ in (2.1). Then $s_{jj} = 0$ and hence $S = 0$. Again, put $l = j$ in (2.2). Then $r_{jk} = 0$ for all $j \neq k$ and hence

$$R = \begin{bmatrix} r_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & r_{nn} \end{bmatrix}.$$

Since T is multiplicative, we have $R = T(I) = T(I.I) = R^2$, and hence

$$r_{ii} = 0 \quad \text{or} \quad r_{ii} = 1.$$

If $R \neq 0$ and $R \neq I$, without loss of generality, we may assume that

$$R = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & L \end{bmatrix},$$

where L is a diagonal matrix whose diagonal entries are either 0 or 1. Set

$$X = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \end{array}\right] \in M_n.$$

An easy calculation shows that $T\left(X^2\right) \neq T(X)^2$ which is a contradiction. So, $R = I$ or $R = 0$, and hence $T = I$ or $T = 0$.

Case 2: Let $s_{jk} \neq 0$ for some j, k $(1 \leq j \neq k \leq n)$. Now, if

$$\sum_{m=1}^{n} s_{jm} = 0,$$

then, $R = 0$. If

$$\sum_{m=1}^{n} s_{jm} \neq 0,$$

then, $R = \gamma S$ and T has the form $T(X) = (\gamma I + J)XS$. Since $k \neq j$ in (2.1) is arbitrary, we conclude that all s_{jk} $(k \neq j)$ are the same for each j, say β_j. Also, we denote s_{jj} by α_j.
Now, let $j = k$. Then, $E_{ij}E_{kl} = E_{il}$, and $T\left(E_{ij}\right)T\left(E_{kl}\right) = T\left(E_{il}\right)$. If we write this in the matrix form, we have

$$\gamma^2 s_{jj} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ s_{l1} & \cdots & s_{ln} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} + \gamma \left(\sum_{m=1}^{n} s_{jm}\right) \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ s_{l1} & \cdots & s_{ln} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$
$$+ \gamma s_{jj} \begin{bmatrix} s_{l1} & \cdots & s_{ln} \\ \vdots & \cdots & \vdots \\ s_{l1} & \cdots & s_{ln} \end{bmatrix} + \left(\sum_{m=1}^{n} s_{jm}\right) \begin{bmatrix} s_{l1} & \cdots & s_{ln} \\ \vdots & \cdots & \vdots \\ s_{l1} & \cdots & s_{ln} \end{bmatrix}$$
$$= \gamma \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \\ s_{l1} & \cdots & s_{ln} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} s_{l1} & \cdots & s_{ln} \\ \vdots & \cdots & \vdots \\ s_{l1} & \cdots & s_{ln} \end{bmatrix},$$

where $s_{l1},\ldots,s_{ln}$ occurs in the i'th row. So,

$$\begin{cases} \left(\gamma\alpha_j+\sum_{m=1}^{n}s_{jm}-1\right)\begin{bmatrix} s_{l1} & \cdots & s_{ln}\end{bmatrix}=0,\\ \left(\gamma\beta_j+\sum_{m=1}^{n}s_{jm}\right)\begin{bmatrix} s_{l1} & \cdots & s_{ln}\end{bmatrix}=0.\end{cases}$$

If $S=0$, we are done. Assume $S\neq 0$. Then,

$$\begin{cases} \gamma\alpha_j+\alpha_j+(n-1)\,\beta_j-1=0,\\ \gamma\beta_j+\alpha_j+(n-1)\,\beta_j=0.\end{cases}$$

If $\gamma=-n$ or $\gamma=0$, there is no solution. Otherwise, we have $\alpha_j=\frac{n+\gamma-1}{\gamma(n+\gamma)}$ and $\beta_j=\frac{-1}{\gamma(n+\gamma)}$, for all $j\,(1\le j\le n)$. Now, we have

$$\frac{1}{\gamma\,(n+\gamma)}\begin{bmatrix} \gamma+1 & 1 & \cdots & 1\\ 1 & \ddots & 1 & \vdots\\ \vdots & 1 & \ddots & 1\\ 1 & \cdots & 1 & \gamma+1\end{bmatrix}\begin{bmatrix} n+\gamma-1 & -1 & \cdots & -1\\ -1 & \ddots & -1 & \vdots\\ \vdots & -1 & \ddots & -1\\ -1 & \cdots & -1 & n+\gamma-1\end{bmatrix}=I.$$

This shows that $S^{-1}=\gamma I+J$ and the proof is complete. □

In the following section, we consider multiplicative strong linear preservers of some kinds of majorizations.

3. Multiplicative strong linear preservers of majorizations

Similar to the linear preservers of multivariate and directional majorizations, some results hold for the strong linear preservers of multivariate and directional majorizations. Now, we state a theorem that is useful in sequel.

Theorem 3.1 ([8], Theorem 1.1)**.** *Let $\mathbb{F}$ be an arbitrary field and $\phi: M_n(\mathbb{F})\to M_n(\mathbb{F})$ a bijective linear map satisfying*

$$\phi\,(AB)=\phi\,(A)\,\phi\,(B)\,,$$

where $A,B\in M_n(\mathbb{F})$. Then, there exists an invertible matrix $T\in M_n(\mathbb{F})$ such that

$$\phi\,(A)=TAT^{-1},$$

for every $A\in M_n(\mathbb{F})$.

A nonnegative matrix $A\in M_{n,m}$ is row stochastic if the sum of each row of it is 1. Denote the set of all row stochastic matrices by $\mathcal{R}_n$. For $A,B\in M_{n,m}$, it is said that A is a matrix majorized from left(resp. from right) by B if $A=RB$(resp. $A=BR$) for some rowstochastic matrices $R\in M_n$(resp. $R\in M_m$), and denoted by $\prec_l$(resp. $\prec_r$). In [6], the authors found the structure of strong linear preservers of $\prec_l$ as follows.

Theorem 3.2 ([6], Theorem 5.2)**.** *A linear operator $T : M_n \to M_n$ strongly preserves the matrix majorization $\prec_l$ if and only if there exist a permutation matrix P and an invertible matrix L in M_n such that $TX = PXL$ for all $X \in M_n$. Moreover, if $TI = I$, then $L = P^t$.*

The structure of multiplicative strong linear preservers of a matrix majorization from left is as follows. The case of a matrix majorization from right is similar.

Theorem 3.3. *A linear operator $T : M_n \to M_n$ is a multiplicative strong linear preserver of the matrix majorization $\prec_l$ if and only if there exists a permutation matrix P such that $TX = PXP^t$ for all $X \in M_n$.*

Proof. Since $I^2 = I$, $T(I)^2 = T(I)$. Hence $T(I)(T(I) - I) = 0$. But $T(I) = PL$ is invertible. So, $T(I) = I$, and hence by Theorem 3.2, $TX = PXP^t$ for all $X \in M_n$, wich shows T is multiplicative as well. □

Now, we find the structure of multiplicative strong linear preservers of lw-column majorizations. In fact, we show that the structure of multiplicative strong linear preservers of a majorization and multiplicative strong linear preservers of rw-row majorization on M_n are the same.

For $x, y \in M_{n,1}$, it is said that x is lw-majorized by y, if $x = Ry$ for some $R \in \mathcal{R}_n$. Let $A, B \in M_{n,m}$. It is said that that A is lw-column majorized by B if every column of A is lw-majorizwd by the corresponding column of B.

Theorem 3.4 ([2], Theorem 3.10)**.** *Let $T : M_{n,m} \to M_{n,m}$ be a linear function. Then, T strongly preserves lw-column majorization if and only if there exist $b_1, \dots, b_m \in \bigcup_{i=1}^m Span\{e_i\}$, $P_1, \dots P_m \in \mathcal{P}_n$ such that $B := [b_1|\cdots|b_m]$ is invertible and $TX = [P_1Xb_1|\dots|P_mXb_m]$.*

The following theorem characterizes all multiplicative linear preservers of lw-column majorization on M_n.

Theorem 3.5. *Let $T : M_n \to M_n$ be a linear operator. Then, T is a multiplicative strong linear preserver of lw-column majorization if and only if $TX = PXP^t$ for some permutations P.*

Proof. By Theorem 3.4, there exist permutation matrices $P_1, \dots, P_n \in \mathcal{P}_n$ and vectors $b_1, \dots, b_n \in \bigcup_{i=1}^n span\{e_i\}$ such that $B := [b_1|\cdots|b_n]$ is invertible and

$$TX = [P_1Xb_1|\cdots|P_nXb_n].$$

By Theorem 3.1, T has also the form $TX = CXC^{-1}$. Now, suppose

$$[P_1Xb_1|\cdots|P_nXb_n] = [CXD_1|\cdots|CXD_n],$$

where D_i is the ith column of C^{-1}. So, $P_kXb_k = CXD_k$. Since B is invertible, it may be assumed that $(b_k)_{i_k} \neq 0$, where $(b_k)_j$ stands for the jth coordinate of b_k. Setting $X = E_{ji_k}$, we have

$$\begin{bmatrix} (P_k)_{11} & \cdots & (P_k)_{1n} \\ \vdots & \ddots & \vdots \\ (P_k)_{n1} & \cdots & (P_k)_{nn} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} (b_k)_1 \\ \vdots \\ (b_k)_n \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & \cdots & C_{1n} \\ \vdots & \ddots & \vdots \\ C_{n1} & \cdots & C_{nn} \end{bmatrix} \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} (d_k)_1 \\ \vdots \\ (d_k)_n \end{bmatrix}.$$

Since P_k and C are invertible, then $(b_k)_i = 0$ if and only if $(d_k)_i = 0$. Now, we have

$$(b_k)_{i_k} \begin{bmatrix} (P_k)_{1j} \\ \vdots \\ (P_k)_{nj} \end{bmatrix} = (d_k)_{i_k} \begin{bmatrix} (C)_{1j} \\ \vdots \\ (C)_{nj} \end{bmatrix}.$$

Since j and k variate independently, we have $P_k = \gamma C$ and $D_k = \gamma b_k$. Put $P := \gamma C$. Then,

$$TX = [P_1 X b_1 | \cdots | P_n X b_n] = PXP^t.$$

□

4. Multiplicative linear preservers of generalized majorizations

For $x, y \in \mathbb{R}^n$, it is said that x is gs-majorized by y (denoted by $y \succ_{gs} x$), if there exists a generalized doubly stochastic matrix D such that $x = Dy$, see [3]. For $A, B \in M_{n,m}$, it is said that B is gd-majorized by A (written as $A \succ_{gd} B$), if $Ax \succ_{gs} Bx$, for all $x \in \mathbb{R}^m$. Actually, $A \succ_{gd} B$ if and only if, for every $x \in \mathbb{R}^m$, there exists a g-doubly stochastic matrix D_x such that $Bx = D_x(Ax)$. The following theorem gives the possible structures of all linear functions from $M_{n,m}$ to $M_{n,k}$ that preserve $\succ_{gd}$.

Theorem 4.1 ([3], Theorem 1.3). *Let $T : M_{n,m} \to M_{n,k}$ be a linear function that preserves $\succ_{gd}$. Then, one of the following holds.*

(i) There exist $A_1, \ldots, A_m \in M_{n,k}$ *such that*

$$T(X) = \sum_{j=1}^{m} tr(x_j) A_j,$$

where x_j *is the* j*th column of* X*.*

(ii) There exist $R, S \in M_{m,k}$ *and an invertible matrix* $D \in GD_n$ *such that*

$$T(X) = DXR + JXS.$$

(iii) There exist $S \in M_{m,k}$, $a \in \mathbb{R}^m$, $r_1, \ldots, r_k \in \mathbb{R}$ *and invertible matrices* $D_1, \ldots, D_k \in GD_n$ *such that*

$$T(X) = [r_1 D_1 X a| \cdots |r_k D_k X a] + JXS.$$

Now, we characterize the multiplicative linear preservers of gd-majorization on M_n.

Lemma 4.2. *Let* $S \in M_n$*. The linear function* $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ *defined by* $TX = JXS$ *is multiplicative if and only if* $T = 0$*.*

Proof. It is obvious that for all $i, j, k \in \{1, \ldots, n\}$,

$$\begin{aligned} T\left(E_{ij}\right) &= \begin{bmatrix} s_{j1} & \cdots & s_{jn} \\ \vdots & & \vdots \\ s_{j1} & \cdots & s_{jn} \end{bmatrix} \\ &= T\left(E_{kj}\right). \end{aligned}$$

Then, for $k \neq j$,

$$\begin{aligned} 0 &= T(0) \\ &= T\left(E_{ij} E_{kl}\right) \\ &= T\left(E_{ij}\right) T\left(E_{kl}\right) \\ &= T\left(E_{ij}\right) T\left(E_{jl}\right) \\ &= T\left(E_{ij} E_{jl}\right) \\ &= T\left(E_{il}\right). \end{aligned}$$

So, $T = 0$. □

Theorem 4.3. *Let* $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ *be a linear function. Then,* T *is a multiplicative linear preserver of gd-majorization if and only if* T *has one of the following forms:*

(i) $TX = 0$ *for all* $X \in \boldsymbol{M}_n$*.*

(ii) *There exists an invertible matrix* $D \in GD_n$ *such that*

$$TX = DXD^{-1},$$

for all $X \in \boldsymbol{M}_n$*.*

(iii) *There exist an invertible matrix $D \in GD_n$ and a scalar $\gamma \neq -n$ such that*

$$TX = (\gamma D + J)X(\gamma D + J)^{-1},$$

for all $X \in \boldsymbol{M}_n$.

Proof. Let $T : M_n \to M_n$ be a linear function that preserves $\succ_{gd}$. If T is in the form of (i) or (ii) in Theorem 4.1, the proof is similar to that of Theorem 2.3. If T is in the form of (iii), we show that $T = 0$.

First, suppose $n = 2$. If $r_1 = 0$, we can choose $D_1 = D_2$ and T is in the form of (ii) in Theorem 4.1. So, without loss of generality, we can assume that

$$T(X) = [Xa|rDXa] + JXS.$$

Now, let

$$a = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \qquad D = \begin{bmatrix} d & 1-d \\ 1-d & d \end{bmatrix},$$

and for convenience, $t = r(2d-1)$.

Suppose $X = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. So,

$$T(X) = \alpha \begin{bmatrix} 1 & t \\ -1 & -t \end{bmatrix},$$

and

$$T(Y) = \beta \begin{bmatrix} 1 & t \\ -1 & -t \end{bmatrix}.$$

Since $X^2 = X$, $T(X)^2 = T(X)$ and hence $\alpha^2(1-t) = \alpha$. So, $\alpha = 0$ or $\alpha = \frac{1}{1-t}$. Note that if $t = 1$ then $\alpha = 0$. On the other hand, since $Y^2 = -Y$, $\beta = 0$ or $\beta = -\frac{1}{1-t}$, and if $t = 1$, then $\beta = 0$ and $T = 0$, by Lemma 4.2. So, assume that $t \neq 1$. Since $XY = Y$, $\alpha\beta(1-t) = \beta$, and if $\alpha = 0$, we conclude that $\beta = 0$. Similarly, since $YX = -X$, we conclude that if $\beta = 0$, then $\alpha = 0$. So, we have $\alpha = \beta = 0$ or $\alpha = -\beta = \frac{1}{1-t}$.

An easy calculation shows that

$$T(E_{11}) = \begin{bmatrix} \alpha + s_{11} & r\alpha d + s_{12} \\ s_{11} & r\alpha(1-d) + s_{12} \end{bmatrix},$$

$$T(E_{12}) = \begin{bmatrix} -\alpha + s_{21} & -r\alpha d + s_{22} \\ s_{21} & -r\alpha(1-d) + s_{22} \end{bmatrix},$$

$$T(E_{22}) = \begin{bmatrix} s_{21} & -r\alpha(1-d) + s_{22} \\ -\alpha + s_{21} & -r\alpha d + s_{22} \end{bmatrix}.$$

The solution of the equations

$$\begin{cases} T(E_{12})\,T(E_{11}) = 0, \\ T(E_{11})\,T(E_{22}) = 0, \end{cases}$$

is

$$\begin{cases} s_{11} = -\frac{\alpha}{1+t}, \\ s_{12} = -\frac{\alpha(tr+rd(1-t))}{1+t}, \\ s_{21} = \frac{\alpha t}{1+t}, \\ s_{22} = \frac{\alpha r(1-d+dt)}{1+t}. \end{cases}$$

Note that $\alpha = -s_{11}(1+t)$. Assume that $t \neq -1$. (If $t=-1$, $\alpha = 0$, and we are done.)

Calculating $T(E_{11})\,T(E_{21})$, shows that

$$(\alpha + s_{11})(r\alpha d + s_{12} + s_{11}) = 0.$$

Now, if $\alpha + s_{11} = 0$, and $\alpha \neq 0$, we have $t = 0$. Since D is invertible, $d \neq \frac{1}{2}$. So, $r = 0$ and we are done. If $r\alpha d + s_{12} + s_{11} = 0$, then $t = 1$ or $t = -1$, which is a contradiction.

Now, suppose that $n \geq 3$. Let

$$T(X) = [r_1 D_1 X a| \dots |r_n D_n X a] + JXS,$$

for some $S \in M_n$, $a \in \mathbb{R}^n$, $r_1, \dots, r_n \in \mathbb{R}$ and invertible matrices $D_1, \dots, D_n \in GD_n$. Without loss of generality, assume that $r_1, a_1 \neq 0$, where a_1 is the first coordinate of a. Set

$$X = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 0 & \cdots & \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Then, $X^2 = X$ and hence $T(X)^2 = T(X)$. So,

$$a_1^2\begin{bmatrix} r_1\sum_{m=1}^{n} r_m\Big((D_m)_{1,1}-(D_m)_{1,2}\Big)\Big((D_1)_{m,1}-(D_1)_{m,2}\Big) & \cdots & r_n\sum_{m=1}^{n} r_m\Big((D_m)_{1,1}-(D_m)_{1,2}\Big)\Big((D_n)_{m,1}-(D_n)_{m,2}\Big) \\ \vdots & \cdots & \vdots \\ r_1\sum_{m=1}^{n} r_m\Big((D_m)_{n,1}-(D_m)_{n,2}\Big)\Big((D_1)_{m,1}-(D_1)_{m,2}\Big) & \cdots & r_n\sum_{m=1}^{n} r_m\Big((D_m)_{n,1}-(D_m)_{n,2}\Big)\Big((D_n)_{m,1}-(D_n)_{m,2}\Big) \end{bmatrix}$$

$$= a_1\begin{bmatrix} r_1\Big((D_1)_{1,1}-(D_1)_{1,2}\Big) & \cdots & r_n\Big((D_n)_{1,1}-(D_n)_{1,2}\Big) \\ \vdots & & \vdots \\ r_1\Big((D_1)_{n,1}-(D_1)_{n,2}\Big) & \cdots & r_n\Big((D_n)_{n,1}-(D_n)_{n,2}\Big) \end{bmatrix}.$$

The first column of the matrix in the right hand side of the above equation cannot be equal to zero. Since otherwise, the first and the second columns of D_1 are equal and the matrix cannot be invertible. Let $\beta = (D_1)_{i,1} - (D_1)_{i,2}$, for some i such that $(D_1)_{i,1} - (D_1)_{i,2} \neq 0$ and α be the corresponding entry in the right hand side matrix in the above equation. It is clear that $\alpha \neq 0$. So, $a_1^2 r_1 \alpha = a_1 r_1 \beta$.

If we put

$$X = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 0 & & \\ 0 & 0 & 0 & & \vdots \\ \vdots & & & \ddots & \\ 0 & & \cdots & & 0 \end{bmatrix},$$

we get to the equation $X^2 = -X$ and hence $T(X)^2 = -T(X)$, which gives the equation $a_2^2 r_1 \alpha = -a_2 r_1 \beta$. Again, without loss of generality, we may assume that $a_2 \neq 0$. If we shift the column vector

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

to the other columns, we have $X^2 = 0$, and hence $T(X)^2 = 0$, which gives $a_k \alpha = 0 (k \geq 3)$, where a_k is the kth coordinate of a and $\alpha \neq 0$.

So, a must have the form

$$\begin{bmatrix} a_1 \\ -a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Now, set

$$X = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \\ 0 & -1 & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \\ \vdots & \vdots & \vdots & \ddots & \\ 0 & & \cdots & & 0 \end{bmatrix}.$$

Then, $T(X) \neq 0$, and we can obtain the corresponding α and β. Using these α and β, and setting

$$X = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & & \vdots \\ -1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

we get to $X^2 = 0$ and $T(X)^2 = 0$. This implies that a_1 or $r_1 = 0$ and we reach to a contradiction. So, $T(X)$ has the form $T(X) = JXS$, and hence by Lemma 4.2, we have $T = 0$. □

5. Multiplicative strong linear preservers of generalized majorizations

The definitions, theorems and proofs of multiplicative strong linear preservers of gs and gd-majorization are similar to multivariate and directional majorizations, see [3].

Now, we consider gut-majorizations.

An $n \times m$ matrix R is called g-row stochastic if the sum of all entries in each row of R is 1. Denote the set of all $n \times n$ upper triangular g-row stochastic matrices by $\mathcal{R}_n^{gut}$. By E, we mean a matrix whose entries in the last column are 1 and the other entries are 0. For $A, B \in M_{n,m}$, it is said that A is gut-majorized by B, and written as $A \prec_{gut} B$, if there exists $R \in \mathcal{R}_n^{gut}$ such that $A = RB$. In [4] the authors found the structure of all strong linear preservers of gut-majorizations as follows.

Theorem 5.1 ([4], Theorem 1.3)**.** *Let $T : M_{n,m} \to M_{n,m}$ be a linear function. Then, T strongly preserves $\prec_{gut}$ if and only if $TX = AXR + EXS$ for some $R, S \in M_m$ and invertible matrix $A \in \mathcal{R}_n^{gut}$ such that $R(R+S)$ is invertible.*

Let $T : M_n \to M_n$ be a strong linear preserver of gut-majorization. It is obvious that T is multiplicative if and only if QTQ^{-1} is multiplicative for every $Q \in \mathcal{R}_n^{gut}$.

Theorem 5.2. *Let $T : M_n \to M_n$ be a multiplicative strong linear preserver of gut-majorization. Then, there exists $Q \in \mathcal{R}_n^{gut}$ such that $TX = QXQ^{-1}$ for every $X \in M_n$.*

Proof. Without loss of generality, we can suppose that $TX = XR + EXS$. By Theorem 3.1, there exists an invertible matrix $D \in M_n$ such that

$$\begin{aligned} TX &= XR + EXS \\ &= DXD^{-1}. \end{aligned}$$

If X is an arbitrary $n \times n$ matrix with nth row equal to zero, we have $XR = DXD^{-1}$. Suppose that

$$D^{-1} = \left[\begin{array}{ccc|c} & & & a_{1n} \\ & A_1 & & \vdots \\ & & & a_{n-1n} \\ \hline a_{n1} & \cdots & a_{nn-1} & a_{nn} \end{array}\right],$$

$$RD = \left[\begin{array}{ccc|c} & & & b_{1n} \\ & B_1 & & \vdots \\ & & & b_{n-1n} \\ \hline b_{n1} & \cdots & b_{nn-1} & b_{nn} \end{array}\right],$$

and

$$X = \left[\begin{array}{c|c} X_1 & 0 \\ \hline 0 & 0 \end{array}\right].$$

We have $A_1X_1B_1 = X_1$. Since $X_1 \in M_{n-1}$ is arbitrary, $A_1 = B_1 = I_{n-1}$. If we take $X_1 = I_{n-1}$, then,

$$\left[\begin{array}{ccc|c} & & & b_{1n} \\ & I_{n-1} & & \vdots \\ & & & b_{n-1n} \\ \hline a_{n1} & \cdots & a_{nn-1} & * \end{array}\right] = \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & 0 \end{array}\right],$$

and hence $a_{n1} = \cdots = a_{nn-1} = b_{1n} = \cdots = b_{n-1n} = 0$. So,

$$D^{-1} = \left[\begin{array}{c|c} I_{n-1} & \begin{matrix} a_{1n} \\ \vdots \\ a_{n-1n} \end{matrix} \\ \hline 0 & a_{nn} \end{array}\right],$$

$$RD = \left[\begin{array}{ccc|c} & I_{n-1} & & 0 \\ \hline b_{n1} & \cdots & b_{nn-1} & b_{nn} \end{array}\right],$$

and

$$D = \left[\begin{array}{c|c} I_{n-1} & \begin{matrix} -\frac{a_{1n}}{a_{nn}} \\ \vdots \\ -\frac{a_{n-1n}}{a_{nn}} \end{matrix} \\ \hline 0 & \frac{1}{a_{nn}} \end{array}\right].$$

Put $X = E_{1n}$ in the equation $XRD = DX$. Since $b_{n1} = \cdots = b_{nn-1} = 0$ and $b_{nn} = 1$, then $R = D^{-1}$.

Now,

$$XR + EXS = DXD^{-1},$$

and

$$XRD + EXSD = DX.$$

Putting $X = E_{n1}, \ldots, E_{nn}$ in this equation, we get

$$-\frac{a_{1n}}{a_{nn}} = \cdots = -\frac{a_{n-1n}}{a_{nn}} = \frac{1 - a_{nn}}{a_{nn}},$$

$$(SD)_{11} = \cdots = (SD)_{nn},$$

and

$$(SD)_{ij} = 0, \quad i \neq j.$$

So,

$$SD = -\frac{a_{1n}}{a_{nn}} I \quad \Rightarrow \quad S = -\frac{a_{1n}}{a_{nn}} D^{-1},$$

$$D^{-1} = \left[\begin{array}{c|c} I_{n-1} & \begin{matrix} \alpha \\ \vdots \\ \alpha \end{matrix} \\ \hline 0 & 1 + \alpha \end{array}\right],$$

and

$$D = \left[\begin{array}{c|c} I_{n-1} & \begin{matrix} -\frac{\alpha}{1+\alpha} \\ \vdots \\ -\frac{\alpha}{1+\alpha} \end{matrix} \\ \hline 0 & \frac{1}{1+\alpha} \end{array}\right],$$

where $\alpha = a_{1n} = \cdots = a_{n-1n}$. Therefore,

$$T(X) = XD^{-1} - \frac{\alpha}{1+\alpha}EXD^{-1}$$
$$= \left(I - \frac{\alpha}{1+\alpha}E\right)XD^{-1}.$$

Hence $TX = DXD^{-1}$, and the proof is complete. □

Acknowledgments. The authors would like to thank the referee(s) for useful comments and remarks.

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