ON MULTIPLICATIVE (STRONG) LINEAR PRESERVERS OF MAJORIZATIONS

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Abstract. In this paper, we study some kinds of majorizations on $\mathbb{M}_n$ and their linear or strong linear preservers. Also, we find the structure of linear or strong linear preservers which are multiplicative, i.e. linear or strong linear preservers like $\Phi$ with the property $\Phi(AB) = \Phi(A)\Phi(B)$ for every $A, B \in \mathbb{M}_n$.

1. Introduction

An $n \times n$ nonnegative real matrix $A$ is doubly stochastic if all of its row and column sums are one. For $X, Y \in \mathbb{M}_n$, it is said that $X$ is majorized by $Y$ (written as $X \prec Y$) if there exists a doubly stochastic matrix $D$ such that $X = DY$. If we omit the non-negativity condition, we reach to the set of all generalized $n \times n$ doubly stochastic matrices, denoted by $GD_n$. By considering some other kinds of stochastic matrices such as row stochastic, upper triangular row stochastic, tridiagonal row stochastic and even stochastic matrices, we reach to various kinds of majorizations. Let $T: \mathbb{M}_n \to \mathbb{M}_n$ be a linear operator. For a relation $\sim$ on $\mathbb{M}_n$, it is said that $T$ is a linear preserver of $\sim$, if $TX \sim TY$ whenever $X \sim Y$. Similarly, it is said that $T$ is a strong linear preserver of $\sim$, if

$$X \sim Y \iff TX \sim TY.$$ 

Also, it is said that $T$ is a multiplicative linear preserver of $\sim$, if in addition $T(XY) = T(X)T(Y)$, for all $X, Y \in \mathbb{M}_n$. In this paper, we characterize all multiplicative linear or strong linear preservers of some kinds of majorizations. It is well-known that nonzero linear and multiplicative operators on $\mathbb{M}_n$ are inner automorphisms. We use this fact in

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some proofs but we prefer to prove other theorems without using this. Assume $\mathcal{P}_n$ be the set of all $n \times n$ permutation matrices and $E_{ij}$ be an $n \times n$ matrix such that its $ij$th entry is 1, and all other entries are 0.

2. Multiplicative linear preservers of majorizations

In this section, we find multiplicative linear preservers of some kinds of majorizations. Let $X, Y \in M_n$. $X$ is said to be multivariately majorized by $Y$ (written as $X \prec_m Y$), if $X = DY$ for some $n \times n$ doubly stochastic matrix $D$ and called directionally majorized by $Y$ (written as $X \prec_d Y$), if for every $a \in \mathbb{R}^n$ there exists a doubly stochastic matrix $D_a$ such that $X_a = D_aY_a$. In [6] and [7], the authors found the linear preservers and strong linear preservers of multivariate and directional majorizations as follows.

In this paper, the letter $J$ indicates the matrix with all entries equal to 1, and $\text{tr}(x)$ uses for the summation of coordinates of a vector $x$.

Theorem 2.1 ([7], Theorem 1). Let $T : M_{n,m} \to M_{n,m}$ be a linear function. Then, $T$ preserves multivariate majorizations if and only if $T$ preserves directionally majorizations if and only if one of the following holds:

(a) There exist $A_1, \ldots, A_m \in M_{n,m}$ such that

$$TX = \sum_{i=1}^{m} (\text{tr}(x_i)) A_i,$$

where $x_i$ is the $i$’th column of $X$.

(b) There exist $P \in \mathcal{P}_n$ and $R, S \in M_m$ such that

$$TX = PXR + JXS.$$

Theorem 2.2 ([6], Theorem 2.4). Let $T : M_{n,m} \to M_{n,m}$ be a linear function. Then, $T$ strongly preserves multivariate majorizations if and only if $T$ strongly preserves directional majorizations if and only if there exist $P \in \mathcal{P}_n$ and $R, S \in M_m$ such that $TX = PXR + JXS$ and $R(R + nS)$ is invertible.

Let $T : M_n \to M_n$ has the form $TX = PXR + JXS$, for some $P \in \mathcal{P}_n$. Then, $T$ is a multiplicative linear preserver if and only if $QTQ^t$ is a multiplicative linear preserver for every $Q \in \mathcal{P}_n$. So, in the proof of the following theorem without loss of generality, we can assume that $P = I$.

Theorem 2.3. Let $T : M_n \to M_n$ be a linear function. Then, $T$ is a multiplicative linear preserver of a multivariate or a directional majorization if and only if $T$ has one of the following forms:

(i) $TX = 0$ for all $X \in M_n$. 

(ii) There exists a permutation $P \in \mathcal{P}_n$ such that
\[ TX = PXP^{-1}, \]
for all $X \in M_n$.

(iii) There exists a permutation $P \in \mathcal{P}_n$ and a scalar $\gamma \neq -n$ such that
\[ TX = (\gamma P + J)X(\gamma P + J)^{-1}, \]
for all $X \in M_n$.

**Proof.** It is easy to prove that each of the conditions (i), (ii) or (iii) are sufficient for $T$ to be a multiplicative linear preserver of a multivariate majorization. So, we only prove the necessity of the conditions. It is clear that $E_{ij}E_{kl} = 0$ if and only if $j \neq k$. Let $T$ be of the form (a) in Theorem 2.1. Then,
\[ TX = \sum_{i=1}^{n} (tr(x_i)) A_i, \]
where $A_1, \ldots, A_n \in M_n$. Let $1 \leq k \leq n$, and put $X = Y = E_{kk}$. Then
\[ A_kA_k = T(X)T(Y) = T(X) = A_k. \]
So, $A_k^2 = A_k$. On the other hand, set $X = E_{kk}$ and $Y = E_{jk}(j \neq k)$. Then,
\[ A_kA_k = T(X)T(Y) = T(XY) = T(0) = 0. \]
Therefore, $A_k = 0$ and hence $T = 0$.

Now, suppose that $T$ has the form (b) in Theorem 2.1. Then,
\[ T(E_{ij}) = E_{ij}R + JE_{ij}S \]
\[ = \begin{bmatrix} 0 & \cdots & 0 \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
r_{j1} & \cdots & r_{jn} \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} s_{j1} & \cdots & s_{jn} \\
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
s_{j1} & \cdots & s_{jn} \end{bmatrix}, \]
where \( r_j, \ldots, r_{jn} \) occur in the \( i \)'th row. Since \( k \neq j \),

\[
0 = T(0) = T(E_{ij}E_{kl}) = T(E_{ij})T(E_{kl})
\]

\[
= r_{jk} \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\end{bmatrix} + \left( \sum_{m=1}^{n} r_{jm} \right) \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\end{bmatrix} +
\]

\[
= s_{jk} \begin{bmatrix}
r_{11} & \cdots & r_{in} \\
\vdots & \ddots & \vdots \\
r_{11} & \cdots & r_{ln} \\
\end{bmatrix} + \left( \sum_{m=1}^{n} s_{jm} \right) \begin{bmatrix}
s_{11} & \cdots & s_{in} \\
\vdots & \ddots & \vdots \\
s_{11} & \cdots & s_{ln} \\
\end{bmatrix},
\]

where \( r_{11}, \ldots, r_{ln} \) and \( s_{11}, \ldots, s_{ln} \) in the first and second matrices occur in the \( i \)'th row. So,

\[
(2.1) \quad s_{jk} \begin{bmatrix}
r_{11} & \cdots & r_{in} \\
\vdots & \ddots & \vdots \\
r_{11} & \cdots & r_{ln} \\
\end{bmatrix} + \left( \sum_{m=1}^{n} s_{jm} \right) \begin{bmatrix}
s_{11} & \cdots & s_{in} \\
\vdots & \ddots & \vdots \\
s_{11} & \cdots & s_{ln} \\
\end{bmatrix} = 0,
\]

and

\[
(2.2) \quad r_{jk} \begin{bmatrix}
r_{11} & \cdots & r_{in} \\
\vdots & \ddots & \vdots \\
r_{11} & \cdots & r_{ln} \\
\end{bmatrix} + \left( \sum_{m=1}^{n} r_{jm} \right) \begin{bmatrix}
s_{11} & \cdots & s_{in} \\
\vdots & \ddots & \vdots \\
s_{11} & \cdots & s_{ln} \\
\end{bmatrix} = 0.
\]

We consider two cases.

Case 1: Let \( s_{jk} = 0 \) for all \( j, k (1 \leq j \neq k \leq n) \). Put \( l = j \) in (2.1). Then \( s_{jj} = 0 \) and hence \( S = 0 \). Again, put \( l = j \) in (2.2). Then \( r_{jk} = 0 \) for all \( j \neq k \) and hence

\[
R = \begin{bmatrix}
r_{11} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & r_{nn} \\
\end{bmatrix}.
\]

Since \( T \) is multiplicative, we have \( R = T(I) = T(I,I) = R^2 \), and hence

\[
r_{ii} = 0 \quad \text{or} \quad r_{ii} = 1.
\]

If \( R \neq 0 \) and \( R \neq I \), without loss of generality, we may assume that

\[
R = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & L \\
\end{bmatrix},
\]
where $L$ is a diagonal matrix whose diagonal entries are either 0 or 1. Set

$$X = \begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix} \in M_n.$$  

An easy calculation shows that $T(X^2) \neq T(X)^2$ which is a contradiction. So, $R = I$ or $R = 0$, and hence $T = I$ or $T = 0$.

Case 2: Let $s_{jk} \neq 0$ for some $j, k$ $(1 \leq j \neq k \leq n)$. Now, if

$$\sum_{m=1}^{n} s_{jm} = 0,$$

then, $R = 0$. If

$$\sum_{m=1}^{n} s_{jm} \neq 0,$$

then, $R = \gamma S$ and $T$ has the form $T(X) = (\gamma I + J)XS$. Since $k \neq j$ in (2.1) is arbitrary, we conclude that all $s_{jk}$ $(k \neq j)$ are the same for each $j$, say $\beta_j$. Also, we denote $s_{jj}$ by $\alpha_j$.

Now, let $j = k$. Then, $E_{ij}E_{kl} = E_{il}$, and $T(E_{ij})T(E_{kl}) = T(E_{il})$. If we write this in the matrix form, we have

$$\gamma^2 s_{jj} \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix} + \gamma \left( \sum_{m=1}^{n} s_{jm} \right) \begin{bmatrix}
0 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0
\end{bmatrix}$$

$$+ \gamma s_{jj} \begin{bmatrix}
s_{l1} & \cdots & s_{ln} \\
\vdots & \ddots & \vdots \\
s_{l1} & \cdots & s_{ln}
\end{bmatrix} + \left( \sum_{m=1}^{n} s_{jm} \right) \begin{bmatrix}
s_{l1} & \cdots & s_{ln} \\
\vdots & \ddots & \vdots \\
s_{l1} & \cdots & s_{ln}
\end{bmatrix}$$

$$= \gamma \begin{bmatrix}
s_{l1} & \cdots & s_{ln} \\
\vdots & \ddots & \vdots \\
s_{l1} & \cdots & s_{ln}
\end{bmatrix} + \begin{bmatrix}
s_{l1} & \cdots & s_{ln} \\
\vdots & \ddots & \vdots \\
s_{l1} & \cdots & s_{ln}
\end{bmatrix}.$$
where $s_{i1}, \ldots, s_{in}$ occurs in the $i'$th row. So,
\[
\begin{align*}
(\gamma \alpha_j + \sum_{m=1}^n s_{jm} - 1) [ \begin{array}{cccc}
s_{i1} & \cdots & s_{in} \\
\end{array} ] &= 0, \\
(\gamma \beta_j + \sum_{m=1}^n s_{jm}) [ \begin{array}{cccc}
s_{i1} & \cdots & s_{in} \\
\end{array} ] &= 0.
\end{align*}
\]
If $S = 0$, we are done. Assume $S \neq 0$. Then,
\[
\begin{align*}
\gamma \alpha_j + \alpha_j + (n - 1) \beta_j - 1 &= 0, \\
\gamma \beta_j + \alpha_j + (n - 1) \beta_j &= 0.
\end{align*}
\]
If $\gamma = -n$ or $\gamma = 0$, there is no solution. Otherwise, we have $\alpha_j = \frac{n+\gamma-1}{\gamma(n+\gamma)}$ and $\beta_j = \frac{-1}{\gamma(n+\gamma)}$, for all $j (1 \leq j \leq n)$. Now, we have
\[
\frac{1}{\gamma(n+\gamma)} \begin{bmatrix}
\gamma+1 & 1 & \cdots & 1 \\
1 & \ddots & 1 & \vdots \\
\vdots & 1 & \ddots & 1 \\
1 & \cdots & 1 & \gamma+1
\end{bmatrix} \begin{bmatrix}
n+\gamma-1 & -1 & \cdots & -1 \\
-1 & \ddots & \ddots & \vdots \\
\vdots & -1 & \ddots & -1 \\
-1 & \cdots & -1 & n+\gamma-1
\end{bmatrix} = I.
\]
This shows that $S^{-1} = \gamma I + J$ and the proof is complete. \qed

In the following section, we consider multiplicative strong linear pre-
servers of some kinds of majorizations.

3. MULTIPlicative STRONG LINEAR PRESERVERS OF MAJORIZATIONS

Similar to the linear preservers of multivariate and directional ma-
jorizations, some results hold for the strong linear preservers of multi-
variate and directional majorizations. Now, we state a theorem that is
useful in sequel.

Theorem 3.1 ([8], Theorem 1.1). Let $\mathbb{F}$ be an arbitrary field and $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ a bijective linear map satisfying
\[
\phi(AB) = \phi(A) \phi(B),
\]
where $A, B \in M_n(\mathbb{F})$. Then, there exists an invertible matrix $T \in M_n(\mathbb{F})$ such that
\[
\phi(A) = TAT^{-1},
\]
for every $A \in M_n(\mathbb{F})$.

A nonnegative matrix $A \in M_{n,m}$ is row stochastic if the sum of each row of it is 1. Denote the set of all row stochastic matrices by $\mathcal{R}_n$. For $A, B \in M_{n,m}$, it is said that $A$ is a matrix majorized from left (resp. from right) by $B$ if $A = RB$ (resp. $A = BR$) for some rowstochastic matrices $R \in M_n$ (resp. $R \in M_m$), and denoted by $\prec_l$ (resp. $\prec_r$). In [8], the authors found the structure of strong linear preservers of $\prec_l$ as follows.
Theorem 3.2 ([3], Theorem 5.2). A linear operator $T : M_n \rightarrow M_n$ strongly preserves the matrix majorization $\prec_l$ if and only if there exist a permutation matrix $P$ and an invertible matrix $L$ in $M_n$ such that $TX = PXL$ for all $X \in M_n$. Moreover, if $TI = I$, then $L = P^t$.

The structure of multiplicative strong linear preservers of a matrix majorization from left is as follows. The case of a matrix majorization from right is similar.

Theorem 3.3. A linear operator $T : M_n \rightarrow M_n$ is a multiplicative strong linear preserver of the matrix majorization $\prec_l$ if and only if there exists a permutation matrix $P$ such that $TX = PXP^t$ for all $X \in M_n$.

Proof. Since $I^2 = I$, $T(I)^2 = T(I)$. Hence $T(I)(T(I) - I) = 0$. But $T(I) = PL$ is invertible. So, $T(I) = I$, and hence by Theorem 5.2, $TX = PXP^t$ for all $X \in M_n$, which shows $T$ is multiplicative as well. $\square$

Now, we find the structure of multiplicative strong linear preservers of lw-column majorizations. In fact, we show that the structure of multiplicative strong linear preservers of a majorization and multiplicative strong linear preservers of rw-row majorization on $M_n$ are the same.

For $x, y \in M_{n,1}$, it is said that $x$ is lw-majorized by $y$, if $x = Ry$ for some $R \in \mathcal{R}_n$. Let $A, B \in M_{n,m}$. It is said that that $A$ is lw-column majorized by $B$ if every column of $A$ is lw-majorized by the corresponding column of $B$.

Theorem 3.4 ([2], Theorem 3.10). Let $T : M_{n,m} \rightarrow M_{n,m}$ be a linear function. Then, $T$ strongly preserves lw-column majorization if and only if there exist $b_1, \ldots, b_m \in \bigcup_{i=1}^m \text{span} \{e_i\}$, $P_1, \ldots, P_m \in \mathcal{P}_n$ such that $B := [b_1 | \cdots | b_m]$ is invertible and $TX = [P_1Xb_1 | \cdots | P_mXb_m]$.

The following theorem characterizes all multiplicative linear preservers of lw-column majorization on $M_n$.

Theorem 3.5. Let $T : M_n \rightarrow M_n$ be a linear operator. Then, $T$ is a multiplicative strong linear preserver of lw-column majorization if and only if $TX = PXP^t$ for some permutations $P$.

Proof. By Theorem 6.4, there exist permutation matrices $P_1, \ldots, P_n \in \mathcal{P}_n$ and vectors $b_1, \ldots, b_n \in \bigcup_{i=1}^n \text{span} \{e_i\}$ such that $B := [b_1 | \cdots | b_n]$ is invertible and

$$TX = [P_1Xb_1 | \cdots | P_nXb_n].$$

By Theorem 6.4, $T$ has also the form $TX = CXC^{-1}$. Now, suppose

$$[P_1Xb_1 | \cdots | P_nXb_n] = [CXD_1 | \cdots | CXD_n],$$

where $D_i$ is the $i$th column of $C^{-1}$. So, $P_kXb_k = CXD_k$. Since $B$ is invertible, it may be assumed that $(b_k)_{ik} \neq 0$, where $(b_k)_j$ stands for the $j$th coordinate of $b_k$. Setting $X = E_{jik}$, we have
Since $P_k$ and $C$ are invertible, then $(b_k)_i = 0$ if and only if $(d_k)_i = 0$. Now, we have

$$
(b_k)_{ik} \begin{bmatrix} (P_k)_{1j} \\ \vdots \\ (P_k)_{nj} \end{bmatrix} = (d_k)_{ik} \begin{bmatrix} (C)_{1j} \\ \vdots \\ (C)_{nj} \end{bmatrix}.
$$

Since $j$ and $k$ variate independently, we have $P_k = \gamma C$ and $D_k = \gamma b_k$. Put $P := \gamma C$. Then,

$$
TX = [P_1 X b_1 | \cdots | P_n X b_n] = PX P^t.
$$

4. **Multiplicative linear preservers of generalized majorizations**

For $x, y \in \mathbb{R}^n$, it is said that $x$ is gs-majorized by $y$ (denoted by $y \succ_{gs} x$), if there exists a generalized doubly stochastic matrix $D$ such that $x = Dy$, see [3]. For $A, B \in M_{n,m}$, it is said that $B$ is gd-majorized by $A$ (written as $A \succ_{gd} B$), if $Ax \succ_{gs} Bx$, for all $x \in \mathbb{R}^m$. Actually, $A \succ_{gd} B$ if and only if, for every $x \in \mathbb{R}^m$, there exists a g-doubly stochastic matrix $D_x$ such that $Bx = D_x(Ax)$. The following theorem gives the possible structures of all linear functions from $M_{n,m}$ to $M_{n,k}$ that preserve $\succ_{gd}$.

**Theorem 4.1** ([3], Theorem 1.3). Let $T : M_{n,m} \rightarrow M_{n,k}$ be a linear function that preserves $\succ_{gd}$. Then, one of the following holds.
(i) There exist \(A_1, \ldots, A_m \in M_{n,k}\) such that
\[
T(X) = \sum_{j=1}^{m} \text{tr}(x_j)A_j,
\]
where \(x_j\) is the \(j\)th column of \(X\).

(ii) There exist \(R, S \in M_{n,k}\) and an invertible matrix \(D \in GD_n\) such that
\[
T(X) = DXR + JXS.
\]

(iii) There exist \(S \in M_{m,k}, a \in \mathbb{R}^m, r_1, \ldots, r_k \in \mathbb{R}\) and invertible matrices \(D_1, \ldots, D_k \in GD_n\) such that
\[
T(X) = [r_1D_1Xa] \cdots [r_kD_kXa] + JXS.
\]

Now, we characterize the multiplicative linear preservers of \(gd\)-majorization on \(M_n\).

Lemma 4.2. Let \(S \in M_n\). The linear function \(T : M_n \to M_n\) defined by \(TX = JXS\) is multiplicative if and only if \(T = 0\).

Proof. It is obvious that for all \(i, j, k \in \{1, \ldots, n\}\),
\[
T(E_{ij}) = \begin{bmatrix}
    s_{j1} & \cdots & s_{jn} \\
    \vdots & \ddots & \vdots \\
    s_{j1} & \cdots & s_{jn}
\end{bmatrix}
= T(E_{kj}).
\]
Then, for \(k \neq j\),
\[
0 = T(0) = T(E_{ij}E_{kl}) = T(E_{ij})T(E_{kl}) = T(E_{ij})T(E_{jl}) = T(E_{ij}E_{jl}) = T(E_{il}).
\]
So, \(T = 0\). \(\square\)

Theorem 4.3. Let \(T : M_n \to M_n\) be a linear function. Then, \(T\) is a multiplicative linear preserver of \(gd\)-majorization if and only if \(T\) has one of the following forms:

(i) \(TX = 0\) for all \(X \in M_n\).

(ii) There exists an invertible matrix \(D \in GD_n\) such that
\[
TX = DXD^{-1},
\]
for all \(X \in M_n\).
(iii) There exist an invertible matrix $D \in GD_n$ and a scalar $\gamma \neq -n$ such that

$$TX = (\gamma D + J)X(\gamma D + J)^{-1},$$

for all $X \in M_n$.

Proof. Let $T : M_n \rightarrow M_n$ be a linear function that preserves $\succeq_{gd}$. If $T$ is in the form of (i) or (ii) in Theorem 4.1, the proof is similar to that of Theorem 2.3. If $T$ is in the form of (iii), we show that $T = 0$.

First, suppose $n = 2$. If $r_1 = 0$, we can choose $D_1 = D_2$ and $T$ is in the form of (ii) in Theorem 4.1. So, without loss of generality, we can assume that

$$T(X) = [Xa|rDXa] + JXS.$$ 

Now, let

$$a = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, \quad D = \begin{bmatrix} d & 1 - d \\ 1 - d & d \end{bmatrix},$$

and for convenience, $t = r(2d - 1)$.

Suppose $X = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ and $Y = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. So,

$$T(X) = \alpha \begin{bmatrix} 1 & t \\ -1 & -t \end{bmatrix},$$

and

$$T(Y) = \beta \begin{bmatrix} 1 & t \\ -1 & -t \end{bmatrix}.$$ 

Since $X^2 = X$, $T(X)^2 = T(X)$ and hence $\alpha^2 (1 - t) = \alpha$. So, $\alpha = 0$ or $\alpha = \frac{1}{1 - t}$. Note that if $t = 1$ then $\alpha = 0$. On the other hand, since $Y^2 = -Y$, $\beta = 0$ or $\beta = -\frac{1}{1 - t}$, and if $t = 1$, then $\beta = 0$ and $T = 0$, by Lemma 4.2. So, assume that $t \neq 1$. Since $XY = Y$, $\alpha \beta (1 - t) = \beta$, and if $\alpha = 0$, we conclude that $\beta = 0$. Similarly, since $YX = -X$, we conclude that if $\beta = 0$, then $\alpha = 0$. So, we have $\alpha = \beta = 0$ or $\alpha = -\beta = \frac{1}{1 - t}$.

An easy calculation shows that

$$T(E_{11}) = \begin{bmatrix} \alpha + s_{11} & r \alpha d + s_{12} \\ s_{11} & r \alpha (1 - d) + s_{12} \end{bmatrix},$$

$$T(E_{12}) = \begin{bmatrix} -\alpha + s_{21} & -r \alpha d + s_{22} \\ s_{21} & -r \alpha (1 - d) + s_{22} \end{bmatrix},$$

$$T(E_{22}) = \begin{bmatrix} s_{21} & -r \alpha (1 - d) + s_{22} \\ -\alpha + s_{21} & -r \alpha d + s_{22} \end{bmatrix}.$$
The solution of the equations

\[
\begin{align*}
T(E_{12}) T(E_{11}) &= 0, \\
T(E_{11}) T(E_{22}) &= 0,
\end{align*}
\]

is

\[
\begin{align*}
s_{11} &= -\frac{\alpha}{1+t}, \\
s_{12} &= -\frac{\alpha(\text{tr} + rd(1-t))}{1+t}, \\
s_{21} &= \frac{\alpha t}{1+t}, \\
s_{22} &= \frac{\alpha r(1-d+dt)}{1+t}.
\end{align*}
\]

Note that \( \alpha = -s_{11} (1 + t) \). Assume that \( t \neq -1 \). (If \( t = -1, \alpha = 0 \), and we are done.)

Calculating \( T(E_{11}) T(E_{21}) \), shows that

\[
(\alpha + s_{11}) (\text{rad} + s_{12} + s_{11}) = 0.
\]

Now, if \( \alpha + s_{11} = 0 \), and \( \alpha \neq 0 \), we have \( t = 0 \). Since \( D \) is invertible, \( d \neq \frac{1}{2} \). So, \( r = 0 \) and we are done. If \( \text{rad} + s_{12} + s_{11} = 0 \), then \( t = 1 \) or \( t = -1 \), which is a contradiction.

Now, suppose that \( n \geq 3 \). Let

\[
T(X) = [r_1 D_1 X a] \ldots [r_n D_n X a] + JXS,
\]

for some \( S \in M_n, a \in \mathbb{R}^n, r_1, \ldots, r_n \in \mathbb{R} \) and invertible matrices \( D_1, \ldots, D_n \in GD_n \). Without loss of generality, assume that \( r_1, a_1 \neq 0 \), where \( a_1 \) is the first coordinate of \( a \). Set

\[
X = \begin{bmatrix}
1 & 0 & \cdots & 0 \\
-1 & 0 & \cdots \\
0 & 0 & \cdots & \vdots \\
\vdots & \vdots & \ddots \\
0 & 0 & \cdots & 0
\end{bmatrix}.
\]

Then, \( X^2 = X \) and hence \( T(X)^2 = T(X) \). So,
\[
\begin{bmatrix}
    a_1^2 \\
    a_1^2 \\
    \vdots \\
    a_1^2 \\
\end{bmatrix}
= a_1 
\begin{bmatrix}
    r_1 ((D_1)_{1,1} - (D_1)_{1,2}) & \cdots & r_n ((D_n)_{1,1} - (D_n)_{1,2}) \\
    \vdots & \ddots & \vdots \\
    r_1 ((D_1)_{n,1} - (D_1)_{n,2}) & \cdots & r_n ((D_n)_{n,1} - (D_n)_{n,2}) \\
\end{bmatrix}
\]

The first column of the matrix in the right hand side of the above equation cannot be equal to zero. Since otherwise, the first and the second columns of \(D_1\) are equal and the matrix cannot be invertible. Let \(\beta = (D_1)_{i,1} - (D_1)_{i,2}\), for some \(i\) such that \((D_1)_{i,1} - (D_1)_{i,2} \neq 0\) and \(\alpha\) be the corresponding entry in the right hand side matrix in the above equation. It is clear that \(\alpha \neq 0\). So, \(a_1^2 r_1 \alpha = a_1 r_1 \beta\).

If we put

\[
X = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
0 & 0 & 0 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 \\
\end{bmatrix}
\]

we get to the equation \(X^2 = -X\) and hence \(T(X)^2 = -T(X)\), which gives the equation \(a_1^2 r_1 \alpha = -a_2 r_1 \beta\). Again, without loss of generality, we may assume that \(a_2 \neq 0\). If we shift the column vector

\[
\begin{bmatrix}
1 \\
-1 \\
0 \\
\vdots \\
0 \\
\end{bmatrix}
\]

to the other columns, we have \(X^2 = 0\), and hence \(T(X)^2 = 0\), which gives \(a_k \alpha = 0 (k \geq 3)\), where \(a_k\) is the \(k\)th coordinate of \(a\) and \(\alpha \neq 0\).
So, $a$ must have the form
\[
\begin{bmatrix}
    a_1 \\
    -a_1 \\
    0 \\
    \vdots \\
    0
\end{bmatrix}.
\]

Now, set
\[
X = \begin{bmatrix}
    0 & 0 & 0 & \cdots & 0 \\
    0 & 1 & 0 & \cdots \\
    0 & -1 & 0 & \cdots & \vdots \\
    0 & 0 & 0 & \cdots \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0
\end{bmatrix}.
\]

Then, $T(X) \neq 0$, and we can obtain the corresponding $\alpha$ and $\beta$. Using these $\alpha$ and $\beta$, and setting
\[
X = \begin{bmatrix}
    0 & 0 & \cdots & 0 \\
    1 & 0 & \cdots \\
    -1 & 0 & \cdots & 0 \\
    0 & 0 & \cdots \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0
\end{bmatrix},
\]
we get to $X^2 = 0$ and $T(X)^2 = 0$. This implies that $a_1$ or $r_1 = 0$ and we reach to a contradiction. So, $T(X)$ has the form $T(X) = JXS$, and hence by Lemma \[4.2\], we have $T = 0$. □

5. **Multiplicative strong linear preservers of generalized majorizations**

The definitions, theorems and proofs of multiplicative strong linear preservers of gs and gd-majorization are similar to multivariate and directional majorizations, see \[3\].

Now, we consider gut-majorizations.

An $n \times m$ matrix $R$ is called g-row stochastic if the sum of all entries in each row of $R$ is 1. Denote the set of all $n \times n$ upper triangular g-row stochastic matrices by $R_n^{gut}$. By $E$, we mean a matrix whose entries in the last column are 1 and the other entries are 0. For $A, B \in M_{n,m}$, it is said that $A$ is gut-majorized by $B$, and written as $A \prec_{gut} B$, if there exists $R \in R_n^{gut}$ such that $A = RB$. In \[3\] the authors found the structure of all strong linear preservers of gut-majorizations as follows.
Theorem 5.1 ([3], Theorem 1.3). Let $T : M_{n,m} \to M_{n,m}$ be a linear function. Then, $T$ strongly preserves $\prec_{\text{gut}}$ if and only if $TX = AXR + EXS$ for some $R, S \in M_m$ and invertible matrix $A \in R_{n}^{\text{gut}}$ such that $R(R + S)$ is invertible.

Let $T : M_n \to M_n$ be a strong linear preserver of gut-majorization. It is obvious that $T$ is multiplicative if and only if $QTQ^{-1}$ is multiplicative for every $Q \in R_{n}^{\text{gut}}$.

Theorem 5.2. Let $T : M_n \to M_n$ be a multiplicative strong linear preserver of gut-majorization. Then, there exists $Q \in R_{n}^{\text{gut}}$ such that $TX = QXQ^{-1}$ for every $X \in M_n$.

Proof. Without loss of generality, we can suppose that $TX = XR + EXS$. By Theorem 3.1, there exists an invertible matrix $D \in M_n$ such that

$$TX = XR + EXS = D XD^{-1}.$$

If $X$ is an arbitrary $n \times n$ matrix with $n$th row equal to zero, we have $XR = D XD^{-1}$. Suppose that

$$D^{-1} = \begin{bmatrix} A_1 & a_{1n} \\ \vdots & \vdots \\ a_{n1} \cdots a_{n,n-1} & a_{nn} \end{bmatrix},$$

$$RD = \begin{bmatrix} B_1 & b_{1n} \\ \vdots & \vdots \\ b_{n1} \cdots b_{n,n-1} & b_{nn} \end{bmatrix},$$

and

$$X = \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix}.$$

We have $A_1 X_1 B_1 = X_1$. Since $X_1 \in M_{n-1}$ is arbitrary, $A_1 = B_1 = I_{n-1}$. If we take $X_1 = I_{n-1}$, then,

$$\begin{bmatrix} I_{n-1} & b_{1n} \\ \vdots & \vdots \\ a_{1n} \cdots a_{n-1} & b_{n-1,n} \end{bmatrix} = \begin{bmatrix} I_{n-1} & 0 \\ 0 & 0 \end{bmatrix},$$
and hence \(a_{n1} = \cdots = a_{nn-1} = b_{1n} = \cdots = b_{n-1n} = 0\). So,
\[
D^{-1} = \begin{bmatrix}
I_{n-1} & a_{1n} \\
\vdots & \vdots \\
0 & a_{n-1n}
\end{bmatrix},
\]
\[
RD = \begin{bmatrix}
I_{n-1} & 0 \\
\vdots & \vdots \\
b_{1n} & b_{nn}
\end{bmatrix},
\]
and
\[
D = \begin{bmatrix}
I_{n-1} & -\frac{a_{1n}}{a_{nn}} \\
\vdots & \vdots \\
0 & -\frac{a_{n-1n}}{a_{nn}}
\end{bmatrix}.
\]
Put \(X = E_{1n}\) in the equation \(XRD = DX\). Since \(b_{n1} = \cdots = b_{nn-1} = 0\) and \(b_{nn} = 1\), then \(R = D^{-1}\).

Now,
\[
XR + EXS = DXD^{-1},
\]
and
\[
XRD + EXSD = DX.
\]
Putting \(X = E_{n1}, \ldots, E_{nn}\) in this equation, we get
\[
-\frac{a_{1n}}{a_{nn}} = \cdots = -\frac{a_{n-1n}}{a_{nn}} = \frac{1 - a_{nn}}{a_{nn}},
\]
\[
(SD)_{11} = \cdots = (SD)_{nn},
\]
and
\[
(SD)_{ij} = 0, \quad i \neq j.
\]
So,
\[
SD = -\frac{a_{1n}}{a_{nn}} I \quad \Rightarrow \quad S = -\frac{a_{1n}}{a_{nn}} D^{-1},
\]
\[
D^{-1} = \begin{bmatrix}
I_{n-1} & \alpha \\
\vdots & \vdots \\
0 & 1 + \alpha
\end{bmatrix},
\]
and
\[
D = \begin{bmatrix}
I_{n-1} & -\frac{\alpha}{1+\alpha} \\
\vdots & \vdots \\
0 & -\frac{\alpha}{1+\alpha}
\end{bmatrix}.
\]
where $\alpha = a_{1n} = \cdots = a_{n-1n}$. Therefore,

$$T(X) = XD^{-1} - \frac{\alpha}{1 + \alpha} EXD^{-1}$$

$$= \left( I - \frac{\alpha}{1 + \alpha} E \right) XD^{-1}.$$

Hence $TX = DXD^{-1}$, and the proof is complete. $\square$

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**References**