Sahand Communications in Mathematical Analysis (SCMA) Vol. 3 No. 1 (2016), 21-27
http://scma.maragheh.ac.ir

SOME PROPERTIES OF FUZZY REAL NUMBERS

BAYAZ DARABY[1] AND JAVAD JAFARI[2*]

Abstract. In the mathematical analysis, there are some theorems and definitions that established for both real and fuzzy numbers. In this study, we try to prove Bernoulli's inequality in fuzzy real numbers with some of its applications. Also, we prove two other theorems in fuzzy real numbers which are proved before, for real numbers.

1. Introduction

According to [5] if η is a non-negative fuzzy real number, then for any real number $p \neq 0$, η^p is a fuzzy real number, wich some of their properties were proved before. (This paper is trying to study some properties wich are proved in [5] for two real fuzzy numbers η^p and δ^p). Also in [4], Bernoulli's inequality states that, if α is a real number whenever $1+\alpha \geq 0$, then we have $(1+\alpha)^n \geq 1+n\alpha$. Furthermore $(1+\alpha)^n = 1+n\alpha$ if and only if $\alpha = 0$ or $n = 1$, and this theorem indicates for every positive x and positive integer $n > 0$, there is one and only one positive real y such that $y^n = x$ (in this study we try to prove the above theorems in fuzzy real numbers).
Section 2 contains basic definitions and theorems. In Section 3, we prove Bernoulli's inequality and two other theorems in fuzzy real numbers.

2. Preliminaries and basic facts

In this section, we introduce some preliminaries in the theory of fuzzy real numbers. Let η be a fuzzy subset on $\mathbb{R}$, i.e., a mapping $\eta : \mathbb{R} \to [0,1]$ which associates with each real number t, its grade of membership $\eta(t)$. [3]

2010 *Mathematics Subject Classification.* 26E50, 03E72, 39B72.
Key words and phrases. Fuzzy real number, Bernoulli's inequality, Real number.
Received: 9 August 2015, Accepted: 30 November 2015.
* Corresponding author.

Definition 2.1 ([2]). A fuzzy subset η on $\mathbb{R}$ is called a fuzzy real number, whose its α-level set is denoted by $[\eta]_\alpha$, i.e., $[\eta]_\alpha = \{t : \eta(t) \geq \alpha\}$, if it satisfies two axioms:

(N_1) There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.
(N_2) For each $\alpha \in (0,1]$, $[\eta]_\alpha = [\eta^1_\alpha, \eta^2_\alpha]$ where $-\infty < \eta^1_\alpha \leq \eta^2_\alpha < +\infty$.

The set of all fuzzy real numbers denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers. The number $\overline{0}$ stands for the fuzzy real number as:

$$\overline{0}(t) = \begin{cases} 1, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Clearly, $\overline{0} \in F^*(\mathbb{R})$. Also the set of all real numbers can be embedded in $F(\mathbb{R})$, because, if $r \in (-\infty, \infty)$, then $\overline{r} \in F(\mathbb{R})$ satisfies $\overline{r}(t) = \overline{0}(t - r)$.

Theorem 2.2 ([1]). *Let $\{[a_\alpha, b_\alpha] : \alpha \in (0,1]\}$ be a family of nested bounded closed intervals and $\eta : \mathbb{R} \to [0,1]$ be a function defined by*

$$\eta(t) = \sup\{\alpha \in (0,1] : t \in [a_\alpha, b_\alpha]\}.$$

Then η is a fuzzy real number.

Theorem 2.3 ([1]). *Let $[a_\alpha, b_\alpha]$, $0 < \alpha \leq 1$, be a family of non-empty intervals. If*

(i) *$[a_{\alpha_1}, b_{\alpha_1}] \supset [a_{\alpha_2}, b_{\alpha_2}]$ for all $0 < \alpha_1 \leq \alpha_2 \leq 1$,*
(ii) *$[\lim_{k\to\infty} a_{\alpha_k}, \lim_{k\to\infty} b_{\alpha_k}] = [a_\alpha, b_\alpha]$ whenever $\{\alpha_k\}$ is an increasing sequence in $(0,1]$ converging to α,*

then the family $[a_\alpha, b_\alpha]$ represents the α-level sets of a fuzzy real number η such that $\eta(t) = \sup\{\alpha \in (0,1] : t \in [a_\alpha, b_\alpha]\}$ and $[\eta]_\alpha = [\eta_{\alpha_2}, \eta_{\alpha_2}] = [a_\alpha, b_\alpha]$.

Definition 2.4 ([2]). Fuzzy arithmetic operations $\oplus$, $\ominus$, $\otimes$ and $\oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:

(i) $(\eta \oplus \delta)(t) = \sup_{s\in R}\{\eta(s) \wedge \delta(t - s)\}, t \in \mathbb{R}$,
(ii) $(\eta \ominus \delta)(t) = \sup_{s\in R}\{\eta(s) \wedge \delta(s - t)\}, t \in \mathbb{R}$,
(iii) $(\eta \otimes \delta)(t) = \sup_{s\in R, s\neq 0}\{\eta(s) \wedge \delta(t/s)\}, t \in \mathbb{R}$,
(iv) $(\eta \oslash \delta)(t) = \sup_{s\in R}\{\eta(st) \wedge \delta(s)\}, t \in \mathbb{R}$.

Definition 2.5 ([2]). For $k \in R \setminus 0$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta) = \eta(t/k)$ and $0 \odot \eta$ is defined to be $\overline{0}$.

Lemma 2.6 ([5])**.** *Let η and δ be fuzzy real numbers. Then*

$$\forall t \in \mathbb{R}, \quad \eta(t) = \delta(t) \quad \Leftrightarrow \quad \forall \alpha \in (0,1], \quad [\eta]_\alpha = [\delta]_\alpha.$$

Lemma 2.7 ([2])**.** *Let $\eta, \delta \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$ and $[\delta]_\alpha = [\delta_\alpha^1, \delta_\alpha^2]$. Then*

(i) $[\eta \oplus \delta]_\alpha = [\eta_\alpha^1 + \delta_\alpha^1, \eta_\alpha^2 + \delta_\alpha^2]$,
(ii) $[\eta \ominus \delta]_\alpha = [\eta_\alpha^1 - \delta_\alpha^2, \eta_\alpha^2 - \delta_\alpha^1]$,
(iii) $[\eta \otimes \delta]_\alpha = [\eta_\alpha^1 \delta_\alpha^1, \eta_\alpha^2 \delta_\alpha^2], \quad \eta, \delta \in F^*(\mathbb{R})$,
(iv) $[1 \oslash \delta]_\alpha = [1/\delta_\alpha^2, 1/\delta_\alpha^1], \quad \delta_\alpha^1 > 0$.

Definition 2.8 ([5])**.** Let η be a non-negative fuzzy real number and $p \neq 0$ be a real number. We define η^p as:

$$\eta^p(t) = \begin{cases} \eta(t^{1/p}), & t \geq 0, \\ \eta^p(t) = 0, & t < 0. \end{cases}$$

Also we set $\eta^p = \overline{1}$, in the case $p = 0$.
In [5], Sadeqi and al. showed that η^p is a non-negative fuzzy real number, i.e., $\eta^p \in F^*(R)$, $\forall p \in R$. We need to investigate conditions (N_1) and (N_2) in the definition of fuzzy real numbers. For condition (N_1), since η is a fuzzy real number, there exists $t_0 \in [0, \infty)$, such that $\eta(t_0) = 1$. Set $t' = t_0^p$. Then

$$\eta^p(t') = \eta((t')^{1/p}) = \eta((t_0^p)^{1/p}) = \eta(t_0) = 1.$$

For condition (N_2), since $[\eta]_\alpha = [\eta_\alpha^1, \eta_\alpha^2]$, for all $\alpha \in (0,1]$, we have

$$\begin{aligned} [\eta^p]_\alpha &= \{t; \eta^p(t) \geq \alpha\} \\ &= \{t; \eta(t^{1/p}) \geq \alpha\} \\ &= \{s^p; \eta(s) \geq \alpha\} \\ &= (\{s; \eta(s) \geq \alpha\})^p \\ &= ([\eta_\alpha^1, \eta_\alpha^2])^p. \end{aligned}$$

Therefore, η^p is a fuzzy real number. Also it is clear that if $p > 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^1)^p, (\eta_\alpha^2)^p]$ and if $p < 0$, then $[\eta^p]_\alpha = [(\eta_\alpha^2)^p, (\eta_\alpha^1)^p]$.

Theorem 2.9 ([5])**.** *Let η be a non-negative fuzzy real number and p, q be non-zero integers. Then*

(i) $p > 0 \Rightarrow \eta^p = \otimes_{i=1}^p \eta$,
(ii) $p < 0 \Rightarrow \eta^p = \overline{1} \oslash (\otimes_{i=1}^{-p} \eta)$,
(iii) $\eta^p \otimes \eta^q = \eta^{p+q}; \quad pq > 0$,
(iv) $(\eta^p)^q = \eta^{pq}$.

Definition 2.10 ([2])**.** Define a partial ordering $\preceq$ in $F(R)$ by $\eta \preceq \delta$ if and only if $\eta_\alpha^1 \leq \delta_\alpha^1$ and $\eta_\alpha^2 \leq \delta_\alpha^2$ for all $\alpha \in (0,1]$. The strict inequality in $F(R)$ is defined by $\eta \prec \delta$ if and only if $\eta^1_\alpha < \delta^1_\alpha$ and $\eta^2_\alpha < \delta^2_\alpha$ for all $\alpha \in (0,1]$.

3. Main results

Using results of the last section and an idea of W. Rudin [4], we prove the following theorems.

Proposition 3.1. *Let η, δ be fuzzy real numbers and p be a non-zero integer. Then*

(i) $\eta^p \otimes \delta^p = (\eta \otimes \delta)^p = \otimes_{i=1}^{p}(\eta \otimes \delta)$ *if* $p > 0$,
(ii) $\eta^p \otimes \delta^p = 1 \oslash (\otimes_{i=1}^{-p}(\eta \otimes \delta))$ *if* $p < 0$,
(iii) $\eta^p \oslash \delta^p = (\eta \oslash \delta)^p = \otimes_{i=1}^{p}(\eta \oslash \delta)$.

Proof. We prove this proposition by using Lemmas 2.6, 2.7 and Theorem 2.9

(i) Let $p > 0$ and $\alpha \in (0,1]$. Then

$$\begin{aligned} [\eta^p \otimes \delta^p]_\alpha &= \left[(\eta^1_\alpha)^p(\delta^1_\alpha)^p, (\eta^2_\alpha)^p(\delta^2_\alpha)^p\right] \\ &= \left[\prod_{i=1}^{p}(\eta^1_\alpha\delta^1_\alpha), \prod_{i=1}^{p}(\eta^2_\alpha\delta^2_\alpha)\right] \\ &= \left[\otimes_{i=1}^{p}\eta \otimes \delta\right]_\alpha . \end{aligned} \tag{3.1}$$

(ii) For all $\alpha \in (0,1]$ and $p < 0$, we have

$$\begin{aligned} [\eta^p \otimes \delta^p]_\alpha &= \left([\eta^1_\alpha\delta^1_\alpha, \eta^2_\alpha\delta^2_\alpha]\right)^p \\ &= \left[(\eta^2_\alpha\delta^2_\alpha)^p, (\eta^1_\alpha\delta^1_\alpha)^p\right] \\ &= \left[1/(\eta^2_\alpha\delta^2_\alpha)^{-p}, 1/(\eta^1_\alpha\delta^1_\alpha)^{-p}\right] \\ &= \left[1/\prod_{i=1}^{-p}(\eta^2_\alpha\delta^2_\alpha), 1/\prod_{i=1}^{-p}(\eta^1_\alpha\delta^1_\alpha)\right] \\ &= \left[\overline{1} \oslash (\otimes_{i=1}^{-p}\eta \otimes \delta)\right]_\alpha . \end{aligned} \tag{3.2}$$

(iii) Let $p > 0$ and $\alpha \in (0,1]$. Then

$$[\eta^p \oslash \delta^p]_\alpha = \left[(\eta^1_\alpha/\delta^2_\alpha)^p, (\eta^2_\alpha/\delta^1_\alpha)^p\right] = \left[\prod_{i=1}^{p}(\eta^1_\alpha/\delta^2_\alpha), \prod_{i=1}^{p}(\eta^2_\alpha/\delta^1_\alpha)\right] = \left[\otimes_{i=1}^{p}(\eta \oslash \delta)\right]_\alpha . \tag{3.3}$$

In case $p < 0$, the proof is similar. It is easy to check the other cases.

□

Theorem 3.2. *For any non-negative fuzzy real number $\eta > \overline{0}$ and any integer $n > 0$, there is one and only one fuzzy real number $\delta > 0$ such that $\delta^n = \eta$.*

Proof. Let δ be a non-negative fuzzy real number. Then, according to Definition 2.8, δ^n is a fuzzy real number and also, according to the Theorem 2.9, $[\delta^n]_\alpha = [(\delta^1_\alpha)^n, (\delta^2_\alpha)^n]$, since both δ^1_α and δ^2_α are non-negative real numbers. Using [4], we have $(\delta^1_\alpha)^n = a_\alpha$ and $(\delta^2_\alpha)^n = b_\alpha$. Therefore, we have $[\delta^n]_\alpha = [a_\alpha, b_\alpha] = [\eta]_\alpha$. Now, let δ_1, $\delta_2 \in F^*(\mathbb{R})$, $(\delta_1)^n = \eta$ and $(\delta_2)^n = \eta$, then $\delta_1 = \delta_2$. Hence δ is a unique fuzzy real number. □

Corollary 3.3. *If a and b are two positive fuzzy real numbers and n is a positive integer, then $(a \otimes b)^{1/n} = (a^{1/n}) \otimes (b^{1/n})$.*

Proof. Put $\eta = a^{1/n}$ and $\delta = b^{1/n}$. According to Proposition 3.1 we have $a \otimes b = \eta^n \otimes \delta^n = (\eta \otimes \delta)^n$, so, according to Theorem 3.2, we have:

$$(a \otimes b)^{1/n} = \eta \otimes \delta = (a^{1/n}) \otimes (b^{1/n}).$$

□

Theorem 3.4. *(Bernoulli's inequality) Let α be a fuzzy real number such that $1 \oplus \alpha \succeq 0$. Then, for all $n \in N$, we have $(1 \oplus \alpha)^n \succeq 1 \oplus n\alpha$. Furthermore $(1 \oplus \alpha)^n = 1 \oplus n\alpha \quad \Leftrightarrow \quad \alpha = 0$ or $n = 1$.*

Proof. We prove this theorem by using Definition 2.10 and Theorem 2.9. Using mathematical induction, for $n = 1$, we have $(1 \oplus \alpha) \succeq 1 \oplus \alpha$. If the theorem is true for $n = k$, then for $n = k + 1$ we have

$$(1 \oplus \alpha)^{k+1} = (1 \oplus \alpha)^k \otimes (1 \oplus \alpha) \\ \succeq (1 \oplus k\alpha) \otimes (1 \oplus \alpha) \\ = 1 \oplus \alpha \oplus k\alpha \oplus k\alpha^2 \\ \succeq 1 \oplus \alpha \oplus k\alpha \\ = 1 \oplus (1 + k)\alpha, \tag{3.4}$$

hence, the theorem is true for $n = k + 1$ too.
If $\alpha = 0$ or $n = 1$, the proof is clear. Conversely, let $(1 \oplus n\alpha) = (1 \oplus \alpha)^n$ and $n \neq 0$, then we have $1 \oplus n\alpha = (1 \oplus \alpha)^n \succeq 1 \oplus n\alpha \oplus (n - 1)\alpha^2$ and hence

$$1 \succeq 1 \oplus (n - 1)\alpha^2 \qquad \Rightarrow \qquad (n - 1)\alpha^2 \preceq 0,$$

we know that $n - 1 > 0$, so we conclude that $\alpha^2 \preceq 0$. It follows that $\alpha^2 = 0$, hence $\alpha = 0$. Then equality is established. □

Corollary 3.5. *Let β be a fuzzy real number such that $1 \oplus \beta \succeq 0$. Then for all $n \in \mathbb{N}$, we have $\sqrt[n]{1 \oplus \beta} \preceq 1 \oplus \beta/n$. Therefore $\sqrt[n]{1 \oplus \beta} = 1 \oplus \beta/n$ if and only if $\beta = 0$ or $n = 1$.*

Proof. Let $\beta \oslash n = \alpha$. Using of Bernoulli's inequality, we have $(1 \oplus \alpha)^n \succeq 1 \oplus n\alpha$. So $\sqrt[n]{1 \oplus n\alpha} \preceq 1 \oplus \alpha$ or $\sqrt[n]{1 \oplus \beta} \preceq 1 \oplus \beta/n$. □

Theorem 3.6. *Let x, y be fuzzy real numbers and $x \oplus y \succeq 0$. Then for all $n \in \mathbb{N}$, we have*

$$(x \oplus y/2)^n \preceq x^n \oplus y^n/2.$$

Proof. We prove this theorem by using Definition 2.10 and Theorem 2.9. Using mathematical induction, for $n = 1$, we have $(x \oplus y/2) \leq x \oplus y/2$. If theorem is true for $n = k$, then for $n = k + 1$ we have

$$(x^1_\alpha + y^1_\alpha/2)^{k+1} \leq (x^1_\alpha)^k + (y^1_\alpha)^k/2 \times (x^1_\alpha + y^1_\alpha/2) \\ = (x^1_\alpha)^{k+1} + (x^1_\alpha)^k y^1_\alpha + x^1_\alpha (y^1_\alpha)^k + (y^1_\alpha)^{k+1}/4 \\ \leq (x^1_\alpha)^{k+1} + (y^1_\alpha)^{k+1}/2. \tag{3.5}$$

Similarly we have $(x^2_\alpha + y^2_\alpha/2)^{k+1} \leq (x^2_\alpha)^{k+1} + (y^2_\alpha)^{k+1}/2$. Hence from the above inequality and Definition 2.10, we have $(x \oplus y/2)^n \preceq x^n \oplus y^n/2$. □

References

1. T. Bag and S.K. Samanta, *A comperative study of fuzzy norms on a linear space*, Fuzzy Set and Systems, 159(6)(2008), 670–684.
2. C. Felbin, *Finite dimensional fuzzy normed linear spaces*, Fuzzy Sets and Systems, 48 (1992), 239–248.
3. M. Mizumoto and K. Tanaka, *Some properties of fuzzy numbers*, in: M.M. Gupta et al., Eds., Advances in Fuzzy Set Theory and Applications (North-Holland, New York, 1979), 153–164.
4. W. Rudin, *Principles of Mathetical Analysis*, Mcgraw-Hill, New York, 1976.
5. I. Sadeqi, F. Moradlou, and M. Salehi, *On approximate Cauchy equation in Felbins type fuzzy normed linear spaces*, to appear in Iran. J. Fuzzy Syst. 10: 3 (2013), 51-63.

---

[1] Department of Mathematics, Faculty of Science, University of Maragheh, Maragheh, Iran.

E-mail address: bdaraby@maragheh.ac.ir

[2] Department of Mathematics, Faculty of Science, University of Maragheh, Maragheh, Iran.

E-mail address: javad.jafari33333@gmail.com