Sahand Communications in Mathematical Analysis (SCMA) Vol. 5 No. 1 (2017), 31-40
http://scma.maragheh.ac.ir

A FAMILY OF POSITIVE NONSTANDARD NUMERICAL METHODS WITH APPLICATION TO BLACK-SCHOLES EQUATION

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Abstract. Nonstandard finite difference schemes for the Black-Scholes partial differential equation preserving the positivity property are proposed. Computationally simple schemes are derived by using a nonlocal approximation in the reaction term of the Black-Scholes equation. Unlike the standard methods, the solutions of new proposed schemes are positive and free of the spurious oscillations.

1. Introduction

Recently, option valuation has been one of the most important problem in the market of financial derivatives [11, 12, 13, 17, 21]. European call (put) option and American call (put) option are two types of known options. American option can be exercised at any time before expiry and European only at expiry. A partial differential equation that models the option pricing is the well-known Black-Scholes equation from mathematical finance [1, 2]. This model relies on stochastic differential equation which has received major attention by specialities in the financial mathematics and numerical analysis methods. Black-Scholes equation can be approximated and integrated numerically by various methods [3, 10, 11, 12, 13, 14, 18, 19, 20]. The basic idea for solving the Black-Scholes equation is using of the finite difference approximations which represent an important class of numerical procedures employed in finance. Traditionally, important requirements in this context are the investigation of the consistency of the discrete scheme with the original

2010 *Mathematics Subject Classification.* 65M06, 65M12, 65L12.
Key words and phrases. Black-Scholes equation, Option pricing, Finite difference scheme, Positivity-preserving.
Received: 27 October 2015, Accepted: 07 March 2016.
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differential equation and linear stability analysis with smooth solutions. These requirements are important, because they guarantee convergence of the discrete solution to the exact one, but the essential qualitative properties of the solution are not transferred to the numerical solution. Thus, the stated disadvantage might be catastrophic. One approach to avoid this qualitatively instability is to use the nonstandard discretization technique [6, 11, 12, 13, 17]. Here, we are interested in constructing a family of nonstandard finite difference schemes which preserve the positivity as well as stability and consistency [4, 5, 7, 8, 9].
The rest of the paper is organized as follows: In Section 2, we review the numerical results of the classical standard methods in the presence of discontinuities in the initial conditions of the Black-Scholes equation. In Section 3, we propose our new family of methods and investigate the positivity and stability requirements and we present the numerical results. Finally, we end the paper with some conclusions in Section 4.

2. Classical methods for the Black-Scholes equation

In this paper, we are interested in the following modified version of the Black-Scholes equation for the European option pricing with initial and boundary conditions as:

$$-\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-rV=0, \tag{2.1}$$

$$V(S,0)=\max(S-K,0)1_{[L,U]}(S),$$

$$V(S,t)\to 0 \quad \text{as} \quad S\to 0 \quad \text{or} \quad S\to\infty.$$

We need to update the initial condition at the monitoring dates $0 = t_0 < t_1 < \cdots < t_F = T$:

$$V(S,t_i)=V(S,t_i^-)1_{[L,U]}(S), \tag{2.2}$$

where $1_{[L,U]}(S)$ is the indicator function:

$$1_{[L,U]}=\begin{cases} 1 & if \quad L\le S\le U \\ 0 & if \quad S\notin[L,U]. \end{cases}$$

To obtain the finite difference approximation for equation (2.1), let the computational domain $\Omega=[0,S_{\max}]\times[0,T]$ is discretized by a uniform mesh with steps ΔS, Δt in order to obtain grid points $(j\Delta S, n\Delta t)$, $j = 0,1,\ldots,$ and $n=0,1,\ldots,X$ so that $S_{\max}=M\Delta S$ and $T=X\Delta t$. Let V_j^n denotes the approximation of $V(S_j,t_n)$. Replacing the derivatives with respect to S by

$$\frac{\partial V}{\partial S} \approx \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta S},$$

$$\frac{\partial^2 V}{\partial S^2} \approx (1-\theta)\frac{V_{j-1}^{n} - 2V_j^{n} + V_{j+1}^{n}}{\Delta S^2} + \theta\frac{V_{j-1}^{n+1} - 2V_j^{n+1} + V_{j+1}^{n+1}}{\Delta S^2},$$

and the derivative with respect to t by

$$\frac{\partial V}{\partial t} \approx \frac{V_j^{n+1} - V_j^{n}}{\Delta t},$$

lead to a family of the classical finite difference methods which builds a system $AV^{n+1} = BV^n$, where A and B are the following tridiagonal matrices:

$$A = \left\{ \frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2\theta}{2\Delta S^2}; \quad \frac{1}{\Delta t} + \frac{\sigma^2 S_j^2 \theta}{\Delta S^2} + r; \quad -\frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2\theta}{2\Delta S^2} \right\},$$

$$B = \left\{ \frac{\sigma^2 S_j^2(1-\theta)}{2\Delta S^2}; \quad \frac{1}{\Delta t} - \frac{\sigma^2 S_j^2(1-\theta)}{\Delta S^2}; \quad \frac{\sigma^2 S_j^2(1-\theta)}{2\Delta S^2} \right\}.$$

For the appropriately chosen values of the implicitness parameter $\theta \in (0\,;1]$, we obtain different schemes, for example, for $\theta = 1$, we obtain the fully implicit scheme, that in the presence of discontinuous payoff and low volatility arises spurious oscillations. In this scheme, if the

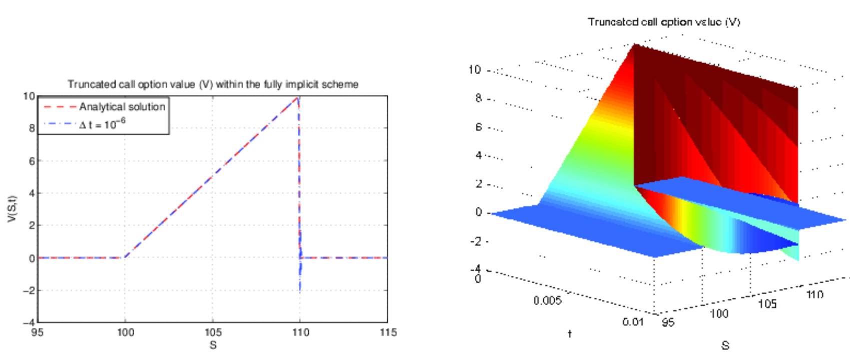

FIGURE 1. Numerical solutions for the fully implicit scheme with $\Delta S = 0.02, \Delta t = 10^{-6}$. Parameters: $L = 90, K = 100, U = 110, r = 0.05, \sigma = 0.001, T = 0.01, Smax = 120$.

condition $\sigma^2 > r$ is violated, spurious oscillations and negative values of V can occur, (as it is illustrated in Figure 1, also in the right figure the numerical and analytical solutions are compared), (see [14]).
Also the classical methods provide spurious oscillations and negative

values for different values of θ, see Figure 2.

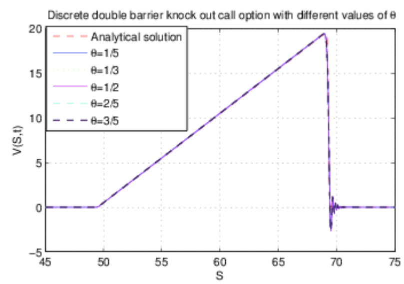

FIGURE 2. Numerical oscillations and negative values in the solution of the classical methods for different values of θ. Parameters: $r = 0.01, \sigma = 0.001, T = 1, U = 70, K = 50, L = 30, S_{max} = 140, \Delta S = 0.05, \Delta t = 10^{-3}$.

3. Construction of the new scheme

3.1. **The new scheme.** To overcome the drawbacks mentioned above, we develop the classical methods within the strategy suggested by Milev-Tagliani [10 – 14]. We replace the reaction term in equation (2.1) by

$$V(S, t + \Delta t) = (aV_{j+1}^{n+1} + V_j^{n+1} + bV_{j-1}^{n+1}) - (a + b)V_j^n, \tag{3.1}$$

the corresponding finite difference approximation provides the difference equation

$$PV^{n+1} = NV^n, \tag{3.2}$$

where P and N are the following tridiagonal matrices:

$$P = \left\{ rb + \frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2 \theta}{2\Delta S^2}; \quad \frac{1}{\Delta t} + \frac{\sigma^2 S_j^2 \theta}{\Delta S^2} + r; \quad ra - \frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2 \theta}{2\Delta S^2} \right\}, \tag{3.3}$$

$$N = \left\{ \frac{\sigma^2 S_j^2 (1-\theta)}{2\Delta S^2}; \quad \frac{1}{\Delta t} - \frac{\sigma^2 S_j^2 (1-\theta)}{\Delta S^2} + ra + rb; \quad \frac{\sigma^2 S_j^2 (1-\theta)}{2\Delta S^2} \right\}. \tag{3.4}$$

Here a and b are arbitrary parameters to be determined below.

3.2. **Positivity.** The parameters a and b are chosen according to the following theorem:

Theorem 3.1. *Sufficient conditions for scheme (3.2) to be positivity-preserving are*

$$b \leq -\frac{r}{8\sigma^2\theta}, \qquad a \leq -\frac{r}{8\sigma^2\theta}, \qquad \Delta t < \frac{1}{(1-\theta)(\sigma M)^2 - r(a+b)}, \tag{3.5}$$

Proof. From (3.2) it is enough to show that $P^{-1} > 0$ and $N \geq 0$.

- Following [22], the condition $P^{-1} > 0$ holds if P is an M-matrix. Being P an M-matrix then

$$rb + \frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2 \theta}{2\Delta S^2} \leq 0, \tag{3.6}$$

$$ra - \frac{rS_j}{2\Delta S} - \frac{\sigma^2 S_j^2 \theta}{2\Delta S^2} \leq 0. \tag{3.7}$$

From (3.6) we can write

$$\begin{aligned} & rb \leq \frac{\theta}{2}\left(\frac{\sigma S_j}{\Delta S}\right)^2 - \frac{r}{2}\frac{S_j}{\Delta S}, \\ \Leftrightarrow\quad & rb \leq \frac{\sigma^2\theta}{2}\left[\left(\frac{S_j}{\Delta S}\right)^2 - \frac{r}{\sigma^2\theta}\frac{S_j}{\Delta S}\right], \\ \Leftrightarrow\quad & rb \leq \frac{\sigma^2\theta}{2}\left[\left(\frac{S_j}{\Delta S}\right)^2 - \frac{r}{\sigma^2\theta}\frac{S_j}{\Delta S} + \frac{r^2}{4\sigma^4\theta^2} - \frac{r^2}{4\sigma^4\theta^2}\right], \\ \Leftrightarrow\quad & rb \leq \frac{\sigma^2\theta}{2}\left[\left(\frac{S_j}{\Delta S} - \frac{r}{2\sigma^2\theta}\right)^2 - \frac{r^2}{4\sigma^4\theta^2}\right], \\ \Leftrightarrow\quad & rb \leq \frac{\sigma^2\theta}{2}\left(\frac{S_j}{\Delta S} - \frac{r}{2\sigma^2\theta}\right)^2 - \frac{r^2}{8\sigma^2\theta}, \end{aligned} \tag{3.8}$$

now, the last inequality in (3.8) shows sufficiency of $b \leq -\frac{r}{8\sigma^2\theta}$ for (3.6). Similarly, from (3.7) we conclude that $a \leq -\frac{r}{8\sigma^2\theta}$.

- The condition $N \geq 0$ is satisfied, provided that

$$\begin{aligned} & \frac{1}{\Delta t} - \frac{\sigma^2 S_j^2(1-\theta)}{\Delta S^2} + ra + rb > 0, \\ \Leftrightarrow\quad & \Delta t < \frac{1}{(1-\theta)(\sigma j)^2 - r(a+b)}, \end{aligned}$$

a way to ensure this is to demand

$$\Delta t < \frac{1}{(1-\theta)(\sigma M)^2 - r(a+b)},$$

and this completes the proof. □

3.3. **Stability.**

Theorem 3.2. *The new method is conditionally stable and convergent with local truncation error* $O(\Delta t, \Delta S^2)$.

Proof. Under conditions (3.5), $P = [P_{ij}]$ is similar to a symmetric tridiagonal matrix, so that the eigenvalues of P, $\lambda_i(P)$, $i = 1, \dots, N$ are real [15, 16, 22]. Also P is row diagonally dominant with

$$\delta_i = |P_{ii}| - \sum_{j \neq i} |P_{ij}| = \frac{1}{\Delta t} + r + r(a+b),$$

which yields $\|P^{-1}\|_\infty \leq \max \frac{1}{\delta_i}$. So, following [22]

$$\|P^{-1}\|_\infty \leq \frac{1}{\frac{1}{\Delta t} + r + r(a+b)},$$

and by combining with $\|N\|_\infty = \frac{1}{\Delta t} + r(a+b)$, we have

$$\begin{aligned} \rho(P^{-1}N) &\leq \|(P^{-1}N)\|_\infty \\ &= \|P^{-1}\|_\infty \|N\|_\infty \\ &\leq \frac{\frac{1}{\Delta t} + r(a+b)}{\frac{1}{\Delta t} + r + r(a+b)} \\ &< 1. \end{aligned}$$

where $\rho(P^{-1}N)$ is the spectral radius of the matrix $P^{-1}N$. Therefore the scheme is stable and then via the Lax-theorem convergent with local truncation error

$$\begin{aligned} T_{j,n} = &-\frac{V_j^{n+1} - V_j^n}{\Delta t} + rS_j \frac{V_{j+1}^{n+1} - V_{j-1}^{n+1}}{2\Delta S} \\ &+ \frac{1}{2}\sigma^2 S_j^2 (1-\theta) \frac{V_{j-1}^n - 2V_j^n + V_{j+1}^n}{\Delta S^2} \\ &+ \frac{1}{2}\sigma^2 S_j^2 \theta \frac{V_{j-1}^{n+1} - 2V_j^{n+1} + V_{j+1}^{n+1}}{\Delta S^2} \\ &- r\left(\left(aV_{j+1}^{n+1} + V_j^{n+1} + bV_{j-1}^{n+1}\right) - (a+b)V_j^n\right), \end{aligned}$$

by Taylor's expansion

$$V_j^{n+1} = V_j^n + \Delta t \left(\frac{\partial V}{\partial t}\right)_j^n + \frac{1}{2}\Delta t^2 \left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \frac{1}{6}\Delta t^3 \left(\frac{\partial^3 V}{\partial t^3}\right)_j^n + \cdots,$$

$$V_{j+1}^{n+1} = V_j^n + \Delta S \left(\frac{\partial V}{\partial S}\right)_j^n + \Delta t \left(\frac{\partial V}{\partial t}\right)_j^n + \frac{1}{2}\Delta S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \frac{1}{2}\Delta t^2 \left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \Delta S \Delta t \left(\frac{\partial^2 V}{\partial S \partial t}\right)_j^n + \frac{1}{6}\Delta S^3 \left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

$$V_{j-1}^{n+1} = V_j^n - \Delta S \left(\frac{\partial V}{\partial S}\right)_j^n + \Delta t \left(\frac{\partial V}{\partial t}\right)_j^n + \frac{1}{2}\Delta S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \frac{1}{2}\Delta t^2 \left(\frac{\partial^2 V}{\partial t^2}\right)_j^n - \Delta S \Delta t \left(\frac{\partial^2 V}{\partial S \partial t}\right)_j^n - \frac{1}{6}\Delta S^3 \left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

$$V_{j+1}^{n} = V_j^n + \Delta S \left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)_j^n + \frac{1}{6}\Delta S^3 \left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

$$V_{j-1}^{n} = V_j^n - \Delta S \left(\frac{\partial V}{\partial S}\right)_j^n + \frac{1}{2}\Delta S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)_j^n - \frac{1}{6}\Delta S^3 \left(\frac{\partial^3 V}{\partial S^3}\right)_j^n + \cdots,$$

substitution into the expression for $T_{j,n}$ then gives

$$T_{j,n} = \left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right)_j^n - r(1+2a)\Delta t \left(\frac{\partial V}{\partial t}\right)_j^n - ra\Delta S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)_j^n - \frac{1}{2}\Delta t \left(\frac{\partial^2 V}{\partial t^2}\right)_j^n - r(\frac{1}{2}+a)\Delta t^2 \left(\frac{\partial^2 V}{\partial t^2}\right)_j^n + \cdots.$$

But V is the solution of the differential equation, so,

$$\left(-\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right)_j^n = 0.$$

Therefore the principle part of the local truncation error is

$$-r(1+2a)\Delta t \left(\frac{\partial V}{\partial t}\right)_j^n - ra\Delta S^2 \left(\frac{\partial^2 V}{\partial S^2}\right)_j^n.$$

Hence $T_{j,n} = O(\Delta t) + O(\Delta S^2)$. These conclude the theorem. □

The proposed scheme for different values of θ specially $\theta = 1$ guarantees a solution being positivity preserving and free of spurious oscillations (see Figures 3-4).

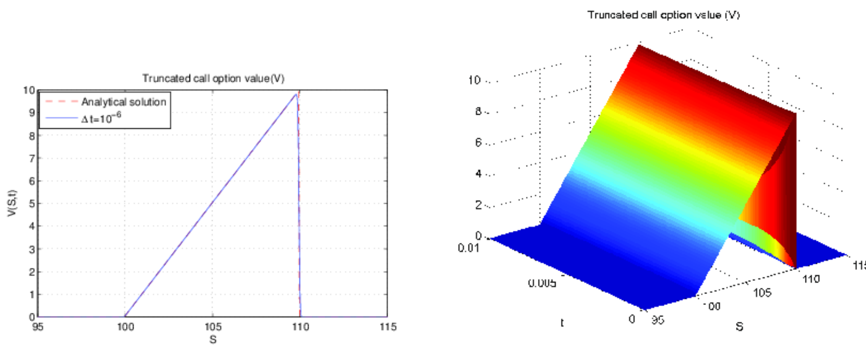

FIGURE 3. Positivity preserving of the solution of the nonstandard fully implicit scheme. Parameters: $r = 0.05, \sigma = 0.001, T = 0.01, U = 110, K = 100, L = 90, S_{max} = 120, \Delta S = 0.02, \Delta t = 10^{-6}$.

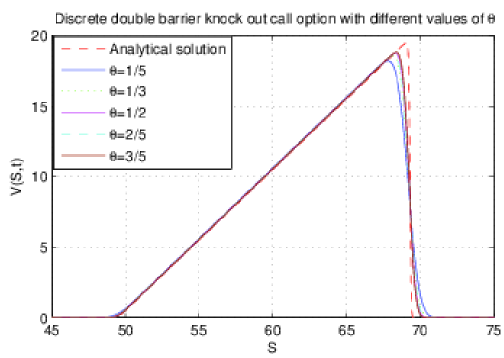

FIGURE 4. Truncated call option value for nonstandard proposed schems with different values of θ. Parameters: $r = 0.01, \sigma = 0.001, T = 1, U = 70, K = 50, L = 30, S_{max} = 140, \Delta S = 0.05, \Delta t = 10^{-3}$.

4. Conclusions

Schemes preserving the positivity property of approximated solution are of great importance in solving the famous Black-Scholes equation. Such schemes are free of spurious oscillations in the presence of discontinuous payoff and low volatility. In this paper, we have discussed nonstandard finite difference schemes which have this qualitative stability property. We used the nonstandard discretization of the reaction term in a nonlocal way. The obtained schemes are computationally simple. Furthermore, they are positivity-preserving and oscillations free. Future works will include extending the methods to multi-asset options.

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