Sahand Communications in Mathematical Analysis (SCMA) Vol. 3 No. 2 (2016), 67-74
http://scma.maragheh.ac.ir

ON THE TOPOLOGICAL CENTERS OF MODULE ACTIONS

KAZEM HAGHNEJAD AZAR

ABSTRACT. In this paper, we study the Arens regularity properties of module actions. We investigate some properties of topological centers of module actions $Z^{\ell}_{B^{**}}(A^{**})$ and $Z^{\ell}_{A^{**}}(B^{**})$ with some conclusions in group algebras.

1. INTRODUCTION

As it is well-known the [1], the second dual A^{**} of a Banach algebra A endowed with Arens multiplications is a Banach algebra. The constructions of the two Arens multiplications in A^{**} lead us to definition of topological centers for A^{**} with respect to both Arens multiplications. The topological centers of Banach algebras, module actions and their applications were introduced and discussed in [2, 6, 8, 9, 10]. We introduce some notations and definitions that we used throughout this paper.

Let A be a Banach algebra and A^*, A^{**}, respectively, are the first and the second dual of A. For $a \in A$ and $a' \in A^*$, we denote by $a'a$ and aa' respectively, the functionals in A^* defined by $\langle a'a, b\rangle = \langle a', ab\rangle = a'(ab)$ and $\langle aa', b\rangle = \langle a', ba\rangle = a'(ba)$ for all $b \in A$. The Banach algebra A is embedded in its second dual via the identification $\langle a, a'\rangle = \langle a', a\rangle$ for every $a \in A$ and $a' \in A^*$.

Let X, Y, Z be normed spaces and $m : X \times Y \to Z$ be a bounded bilinear mapping. Arens in [1] offers two natural extensions m^{***} and m^{t***t} of m from $X^{**} \times Y^{**}$ into Z^{**} as following

(i) $m^* : Z^* \times X \to Y^*$, given by

$$\langle m^*(z', x), y\rangle = \langle z', m(x, y)\rangle,$$

2010 *Mathematics Subject Classification.* 46L06, 46L07, 46L10, 47L25.
Key words and phrases. Arens regularity, Topological centers, Module actions.
Received: 18 April 2015, Accepted: 08 February 2016.

where $x \in X$, $y \in Y$, $z' \in Z^*$,

(ii) $m^{**} : Y^{**} \times Z^* \to X^*$, given by

$$\langle m^{**}(y'', z'), x\rangle = \langle y'', m^*(z', x)\rangle,$$

where $x \in X$, $y'' \in Y^{**}$, $z' \in Z^*$,

(iii) $m^{***} : X^{**} \times Y^{**} \to Z^{**}$, given by

$$\langle m^{***}(x'', y''), z'\rangle = \langle x'', m^{**}(y'', z')\rangle,$$

where $x'' \in X^{**}$, $y'' \in Y^{**}$, $z' \in Z^*$.

The mapping m^{***} is the unique extension of m such that

$$x'' \to m^{***}(x'', y''),$$

from X^{**} into Z^{**} is $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$, but the mapping $y'' \to m^{***}(x'', y'')$ is not in general $weak^* - to - weak^*$ continuous from Y^{**} into Z^{**} which means that always $Z_1(m) = X$. Hence the first topological center of m may be defined as following

$$Z_1(m) = \{x'' \in X^{**} : \quad y'' \to m^{***}(x'', y'')$$

$$\text{is} \quad \text{weak}^* - \text{to} - \text{weak}^* \quad \text{continuous}\}.$$

Let now $m^t : Y \times X \to Z$ be the transpose of m defined by $m^t(y, x) = m(x, y)$ for every $x \in X$ and $y \in Y$. Then, m^t is a continuous bilinear map from $Y \times X$ to Z, and so it may be extended as above to $m^{t***} : Y^{**} \times X^{**} \to Z^{**}$. The mapping $m^{t***t} : X^{**} \times Y^{**} \to Z^{**}$ in general is not equal to m^{***}, see [1], if $m^{***} = m^{t***t}$, then, m is called Arens regular. The mapping $y'' \to m^{t***t}(x'', y'')$ is $weak^* - to - weak^*$ continuous for every $x'' \in X^{**}$, but the mapping $x'' \to m^{t***t}(x'', y'')$ from X^{**} into Z^{**} is not in general $weak^* - to - weak^*$ continuous for every $y'' \in Y^{**}$ so, we define the second topological center of m as follows

$$Z_2(m) = \{y'' \in Y^{**} : \quad x'' \to m^{t***t}(x'', y'')$$

$$\text{is} \quad \text{weak}^* - \text{to} - \text{weak}^* \quad \text{continuous}\}.$$

It is clear that m is Arens regular if and only if $Z_1(m) = X^{**}$ or $Z_2(m) = Y^{**}$. Arens regularity of m is equivalent to the following

$$\lim_i \lim_j \langle z', m(x_i, y_j)\rangle = \lim_j \lim_i \langle z', m(x_i, y_j)\rangle,$$

whenever both limits exist for all bounded sequences $(x_i)_i \subseteq X$, $(y_i)_i \subseteq Y$ and $z' \in Z^*$, see [1, 10].

2. Topological centers of module actions

In this section, the notations WSC is used for weakly sequentially complete Banach space A, i.e., A is said to be weakly sequentially complete, if every weakly Cauchy sequence in A has a weak limit in A.

Suppose that A is a Banach algebra and B is a Banach $A-bimodule$. Let

$$\pi_\ell : A \times B \to B \quad \text{and} \quad \pi_r : B \times A \to B,$$

be the right and the left module actions of A on B. Then B^{**} is a Banach $A^{**}-bimodule$ with the following module actions where A^{**} is equipped with the left Arens product

$$\pi_\ell^{***} : A^{**} \times B^{**} \to B^{**} \quad \text{and} \quad \pi_r^{***} : B^{**} \times A^{**} \to B^{**}.$$

Similarly, B^{**} is a Banach $A^{**}-bimodule$ with the following module actions where A^{**} is equipped with the right Arens product

$$\pi_\ell^{t***t} : A^{**} \times B^{**} \to B^{**} \quad \text{and} \quad \pi_r^{t***t} : B^{**} \times A^{**} \to B^{**}.$$

According to [6], B^{**} is a Banach $A^{**}-bimodule$, where A^{**} is equipped with the first Arens product. We define B^*B as a subspace of A^*, that is, for all $b' \in B^*$ and $b \in B$, we define

$$\langle b'b, a\rangle = \langle b', ba\rangle.$$

We similarly define $B^{**}B^*$ as a subspace of A^{**} and we take $A^{(0)} = A$ and $B^{(0)} = B$.
When there is not any confusion, we set $\pi_\ell(a, b) = ab$, $\pi_r(b, a) = ba$ and we use the notions $Z^\ell_{B^{**}}(A^{**})$ and $Z^\ell_{A^{**}}(B^{**})$ for topological centers of module actions as follows:

$$Z^\ell_{B^{**}}(A^{**}) = \{a'' \in A^{**} : \quad \text{the map} \quad b'' \to a''b'' : B^{**} \to B^{**}$$
$$\text{is} \quad \text{weak}^* - \text{to} - \text{weak}^* \quad \text{continuous}\}.$$
$$Z^\ell_{A^{**}}(B^{**}) = \{b'' \in B^{**} : \quad \text{the map} \quad a'' \to b''a'' : A^{**} \to B^{**}$$
$$\text{is} \quad \text{weak}^* - \text{to} - \text{weak}^* \quad \text{continuous}\}.$$

In the following, we give the generalized definition for weakly completely continuous that has been introduced in [12], and study its relationships with topological centers of module actions.

Definition 2.1. Let B be a left Banach $A-module$. Then B^* is said to be left weakly completely continuous $= (\widetilde{Lwcc})$, if for each $b' \in B^*$, the mapping $a \to \pi_\ell^*(b', a)$ from A into B^*, be a weakly Cauchy sequence into weakly convergence, i.e., if $\{a_n\}$ is a weakly Cauchy in A, then, $\{\pi_\ell^*(b', a)\}$ converges to some points of B^*, see [13]. The definition of right weakly completely continuous ($=\widetilde{Rwcc}$) is similar. $b'\in B^*$ is weakly completely continuous ($=\widetilde{wcc}$) when b' is $\widetilde{Lwcc}$ and $\widetilde{Rwcc}$.

Theorem 2.2. *Let B be a left Banach $A-module$. Then by one of the following conditions, B^* is $\widetilde{Lwcc}$.*

(i) *A is WSC;*
(ii) *B^* is WSC;*
(iii) *$Z^{\ell}_{B^{**}}(A^{**})=A^{**}$.*

Proof. (i) Let $(a_n)_n\subseteq A$ be a weakly Cauchy sequence. Since A is WSC, there exists $a\in A$ such that $a_n\overset{w}{\to}a$. Now, let $b'\in B^*$ and $b''\in B^{**}$. Then

$$\begin{aligned}\langle b'',\pi_\ell^*(b',a_n)\rangle&=\langle\pi_\ell^{**}(b'',b'),a_n\rangle\\ \to\langle\pi_\ell^{**}(b'',b'),a\rangle&=\langle b'',\pi_\ell^*(b',a)\rangle.\end{aligned}$$

Thus $\pi_\ell^*(b',a_n)\overset{w}{\to}\pi_\ell^*(b',a)$.

(ii) Proof is the same of the one of (i).

(iii) Let $(a_n)_n\subseteq A$ be a weakly Cauchy sequence. Since the sequence $(a_n)_n\subseteq A$ is weakly bounded in A, it has a subsequence such as $(a_{n_k})_k\subseteq A$ which $a_{n_k}\overset{w^*}{\to}a''$ for some $a''\in A^{**}$. Then, for each $b'\in B^*$ and $b''\in B^{**}$, we have

$$\begin{aligned}\langle b'',\pi_\ell^*(b',a_{n_k})\rangle&=\langle\pi_\ell^{***}(a_{n_k},b''),b'\rangle\\ \to\langle\pi_\ell^{***}(a'',b''),b'\rangle&=\langle a'',\pi_\ell^{**}(b'',b')\rangle\\ &=\langle\pi_\ell^{*****}(b'',b'),a''\rangle\\ &=\langle b'',\pi_\ell^{****}(b',a'')\rangle.\end{aligned}$$

It is enough to we show that $\pi_\ell^{****}(b',a'')\in B^*$. Suppose that $(b''_\alpha)_\alpha\subseteq B^{**}$ such that $b''_\alpha\overset{w^*}{\to}b''$. Since $Z_{B^{**}}(A^{**})=A^{**}$, we have

$$\begin{aligned}\langle\pi_\ell^{****}(b',a''),b''_\alpha\rangle&=\langle b',\pi_\ell^{***}(a'',b''_\alpha)\rangle\\ &=\langle\pi_\ell^{***}(a'',b''_\alpha),b'\rangle\\ \to\langle\pi_\ell^{***}(a'',b''),b'\rangle&=\langle\pi_\ell^{****}(b',a''),b''\rangle.\end{aligned}$$

It follows that $\pi_\ell^{****}(b',a'')\in(B^{**},weak^*)^*=B^*$.

□

Example 2.3. (i) Suppose that G is a compact group and $1\le p\le\infty$. Take $L^p(G)$ and $M(G)$ as $L^1(G)-bimodule$. Since $L^1(G)$ is WSC, by using 2.2, $L^p(G)$ and $M(G)^*$ are $\widetilde{Lwcc}$. We know that $c_0^* = \ell^1$ and c_0 is a $\ell^1 - bimodule$. Since ℓ^1 is WSC, by using 2.2, ℓ^1 is $\widetilde{Lwcc}$.

(ii) Let B be a reflexive Banach space. Then, by using Theorem 2.6.23 from [4], $B\widehat{\otimes}B^*$, $N(B)$ (the space of nuclear operator on B), $K(B)$ (the space of compact operators on B), and $W(B)$ (the space of weakly compact operators on B) are Arens regular, and so by preceding theorem, they are $\widetilde{Lwcc}$.

Definition 2.4. Let A be a Banach space. An element a'' of A^{**} is said to be $Baire-1$ if there exists a sequence $(a_n)_n$ in A that converges to a'' in the $weak^*$ topology of A^{**}. The collection of $Baire-1$ elements of A^{**} is denoted by $\mathfrak{B}_1(A)$.

Theorem 2.5. *Let B be a left Banach $A-module$. Then, $\mathfrak{B}_1(A) \subseteq Z^{\ell}_{B^{**}}(A^{**})$ if and only if B^* is $\widetilde{Lwcc}$.*

Proof. Suppose that B^* is $\widetilde{Lwcc}$ and $a'' \in \mathfrak{B}_1(A)$. Then, there is a sequence $(a_n)_n \subseteq A$ such that $a_n \overset{w^*}{\rightarrow} a''$. It follows that $(a_n)_n$ is a weakly Cauchy sequence in A. Then, there is a $y' \in B^*$ such that $\pi_\ell^*(b', a_n) \overset{w}{\rightarrow} y'$. On the other hand, for every $b'' \in B^{**}$, we have the following equality,

$$\begin{aligned}\langle b'', y'\rangle &= \lim_n \langle b'', \pi_\ell^*(b', a_n)\rangle \\ &= \lim_n \langle \pi_\ell^{**}(b'', b'), a_n\rangle \\ &= \lim_n \langle a_n, \pi_\ell^{**}(b'', b')\rangle \\ &= \langle a'', \pi_\ell^{**}(b'', b')\rangle \\ &= \langle \pi_\ell^{*****}(b'', b'), a''\rangle \\ &= \langle b'', \pi_\ell^{****}(b', a'')\rangle.\end{aligned}$$

It follows that $\pi_\ell^{****}(b', a'') \in B^*$. Suppose that $(b''_\alpha)_\alpha \subseteq B^{**}$ where $b''_\alpha \overset{w^*}{\rightarrow} b''$. Then for each $b' \in B^*$, we have

$$\begin{aligned}\langle \pi_\ell^{***}(a'', b''_\alpha), b'\rangle &= \langle \pi_\ell^{****}(b', a''), b''_\alpha\rangle \\ &= \langle b''_\alpha, \pi_\ell^{****}(b', a'')\rangle \\ \rightarrow \langle b'', \pi_\ell^{****}(b', a'')\rangle &= \langle \pi_\ell^{***}(a'', b''), b'\rangle.\end{aligned}$$

Consequently $a'' \in Z_{B^{**}}(A^{**})$.

Conversely, Let $\mathfrak{B}_1(A) \subseteq Z_{B^{**}}(A^{**})$ and suppose that $(a_n)_n \subseteq A$ is a weakly Cauchy sequence in A. Then, it has a subsequence such as $(a_{n_k})_k \subseteq A$ such that $a_{n_k} \overset{w^*}{\rightarrow} a''$ for some $a'' \in A^{**}$. It follows that

$a'' \in \mathfrak{B}_1(A)$, and so $a'' \in Z_{B^{**}}(A^{**})$. Similar to Theorem 2.2, we have $\pi_\ell^{****}(b', a'') \in B^*$ for each $b' \in B^*$. Then, for every $b'' \in B^{**}$, we have

$$\begin{aligned}\lim_n \langle b'', \pi_\ell^*(b', a_n)\rangle &= \lim_n \langle \pi_\ell^{**}(b'', b'), a_n\rangle \\ &= \lim_k \langle \pi_\ell^{**}(b'', b'), a_{n_k}\rangle \\ &= \lim_k \langle \pi_\ell^{***}(a_{n_k}, b''), b'\rangle \\ &= \langle \pi_\ell^{***}(a'', b''), b'\rangle \\ &= \langle \pi_\ell^{*****}(b'', b'), a''\rangle \\ &= \langle b'', \pi_\ell^{****}(b', a'')\rangle.\end{aligned}$$

It follows that $\pi_\ell^*(b', a_n) \stackrel{w}{\to} \pi_\ell^{****}(b', b'')$ in B^*. Thus, B^* is $\widetilde{Lwcc}$.

□

Definition 2.6. Let B be a Banach $A-$*bimodule* and $a'' \in A^{**}$. We define the locally topological center of the left and the right module actions of a'' on B^{**}, respectively, as follows

$$Z^t_{a''}(B^{**}) = Z^t_{a''}(\pi^t_\ell) = \{b'' \in B^{**} : \pi_\ell^{t***t}(a'', b'') = \pi_\ell^{***}(a'', b'')\},$$
$$Z_{a''}(B^{**}) = Z_{a''}(\pi^t_r) = \{b'' \in B^{**} : \pi_r^{t***t}(b'', a'') = \pi_r^{***}(b'', a'')\}.$$

It is clear that

$$\bigcap_{a'' \in A^{**}} Z^t_{a''}(B^{**}) = Z^t_{A^{**}}(B^{**}) = Z_1(\pi^t_\ell),$$

$$\bigcap_{a'' \in A^{**}} Z_{a''}(B^{**}) = Z_{A^{**}}(B^{**}) = Z_1(\pi_r).$$

An element e'' of A^{**} is said to be a mixed unit if e'' is a right unit for the first Arens multiplication and a left unit for the second Arens multiplication. That is, e'' is a mixed unit if and only if, for each $a'' \in A^{**}$, $a''e'' = e''oa'' = a''$ where o is a notion for the second Arens multiplication. By [3], page 146, an element e'' of A^{**} is a mixed unit if and only if it is a *weak** cluster point of some BAI $(e_\alpha)_{\alpha\in I}$ in A, for another detail see [9].

Theorem 2.7. *Let B be a left Banach $A-module$. Then, we have the following assertions:*

(i) *Suppose that B has a sequential BAI $(e_n)_n \subseteq A$ and $Z_{e''}(B^{**})A \subseteq B$ where e'' is a mixed unit for A^{**} and $e_n \stackrel{w^*}{\to} e''$. If B is WSC, then $Z_{e''}(B^{**}) = B$;*

(ii) *If B^* is WSC, then, $\mathfrak{B}_1(A) \subseteq Z_{B^{**}}(A^{**})$;*

(iii) *Assume that B has a sequential $LBAI$ $(e_n)_n \subseteq A$ and B^* is WSC. If A is a left ideal in A^{**}, then, $Z_{B^{**}}(A^{**}) = A^{**}$;*

(iv) *Assume that B^* is WSC and A is a right ideal in A^{**}. If for each mixed unit $e'' \in A**$, $Z^{\ell}_{e''}(A^{**}) = A^{**}$, then,*

$$Z^{\ell}_{B^{**}}(A^{**}) = A^{**}.$$

Proof. (i) Since $B \subseteq Z_{e''}(B^{**})$ for every $b \in B$, $\pi_r^{***}(b, e'') = w^* - \lim_n \pi_r(b, e_n) = b$. Let $b'' \in Z_{e''}(B^{**})$ and $(b_\alpha)_\alpha \subseteq B$ with $b_\alpha \stackrel{w^*}{\to} b''$. Then, for every $b' \in B^*$, we have

$$\begin{aligned}\lim_n \langle \pi_r^{***}(b'', e_n), b' \rangle &= \langle \pi_r^{***}(b'', e''), b' \rangle \\ &= \lim_\alpha \langle \pi_r^{***}(b_\alpha, e''), b' \rangle \\ &= \lim_\alpha \langle b', b_\alpha \rangle = \langle b'', b' \rangle.\end{aligned}$$

It follows that $w^* - \lim \pi_r^{***}(b'', e_n) = b''$. Since $\pi_r^{***}(b'', e_n) \in B$ and B is WSC, $b'' \in B$.

(ii) Assume that B^* is WSC. Then, by Theorem 2.2, B^* is $\widetilde{Lwcc}$, and so by using Theorem 2.5, we have $\mathfrak{B}_1(A) \subseteq Z_{B^{**}}(A^{**})$.

(iii) Assume that B^* is WSC. Then, by using part (2), we have $\mathfrak{B}_1(A) \subseteq Z_{B^{**}}(A^{**})$. Let $a'' \in A^{**}$ and suppose that $e'' \in A^{**}$ is a left unit for A^{**} such that $e_n \stackrel{w^*}{\to} e''$. Then, for every $a'' \in A^{**}$, we have $e_n a'' \stackrel{w^*}{\to} e'' o a'' = a''$. Since $AA^{**} \subseteq A$, $a'' \in \mathfrak{B}_1(A)$. Consequently we have $a'' \in Z_{B^{**}}(A^{**})$.

(iv) Proof is the same as above.

□

Example 2.8. (i) Let G be a compact group. We know that $L^1(G)$ has a sequential BAI and it is WSC Banach algebra. Assume that e'' is a mixed unit for $L^1(G)^{**}$. Since

$$\begin{aligned}Z_{e''}(L^1(G)^{**})L^1(G) &\subseteq L^1(G)^{**}L^1(G) \\ &\subseteq L^1(G),\end{aligned}$$

by using the preceding theorem, we have

$$Z_{e''}(L^1(G)^{**}) = L^1(G).$$

(ii) Let G be a locally compact group. In the preceding theorem, if we take $B = c_0(G)$ and $A = \ell^1(G)$, then, it is clear that B is a Banach $A - bimodule$. Since $\ell^1(G) = c_0(G)^*$ is a WSC,

$$\mathfrak{B}_1(\ell^1(G)) \subseteq Z^{\ell}_{(\ell^1(G))^*}(\ell^\infty(G)^*).$$

Acknowledgment. I would like to thank the referee(s) for her/his careful reading of my paper and many valuable suggestions.

References

1. R.E. Arens, *The adjoint of a bilinear operation*, Proc. Amer. Math. Soc. 2 (1951), 839-848.
2. J. Baker, A.T. Lau, and J.S. Pym *Module homomorphism and topological centers associated with weakly sequentially compact Banach algebras*, Journal of Functional Analysis. 158 (1998), 186-208.
3. F.F. Bonsall and J. Duncan, *Complete normed algebras*, Springer-Verlag, Berlin 1973.
4. H.G. Dales, *Banach algebra and automatic continuity*, Oxford 2000.
5. H.G. Dales, F. Ghahramani, and N. Grønbæk *Derivation into iterated duals of Banach algebras* Studia Math. 128 1 (1998), 19-53.
6. M. Eshaghi Gordji and M. Filali, *Arens regularity of module actions*, Studia Math. 181 3 (2007), 237-254.
7. E. Hewitt and K.A. Ross, *Abstract harmonic analysis*, Springer, Berlin, Vol I 1963.
8. A.T. Lau and V. Losert, *On the second Conjugate Algebra of locally compact groups*, J. London Math. Soc. 37 (2)(1988), 464-480.
9. A.T. Lau and A. Ülger, *Topological center of certain dual algebras*, Trans. Amer. Math. Soc. 348 (1996), 1191-1212.
10. S. Mohamadzadeh and H.R.E. Vishki, *Arens regularity of module actions and the second adjoint of a derivation*, Bulletin of the Australian Mathematical Society 77 (2008), 465-476.
11. J.S. Pym, *The convolution of functionals on spaces of bounded functions*, Proc. London Math Soc. 15 (1965), 84-104.
12. A. Ülger, *Some stability properties of Arens regular bilinear operators*, Proc. Amer. Math. Soc. (1991) 34, 443-454.
13. A. Ülger, *Arens regularity of weakly sequentialy complte Banach algebras*, Proc. Amer. Math. Soc. 127 (11) (1999), 3221-3227.

[1] Department of Mathematics, University of Mohaghegh Ardabili, Ardabil, Iran.

E-mail address: haghnejad@uma.ac.ir