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# GROWTH ANALYSIS OF ENTIRE FUNCTIONS OF TWO COMPLEX VARIABLES

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Abstract. In this paper, we introduce the idea of generalized relative order (respectively generalized relative lower order) of entire functions of two complex variables. Hence, we study some growth properties of entire functions of two complex variables on the basis of the definition of generalized relative order and generalized relative lower order of entire functions of two complex variables.

## 1. Introduction, Definitions and Notations

Let $f$ be an entire function of two complex variables which is holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : |z_i| \leq r_i, i = 1, 2 \quad \text{for all} \quad r_1 \geq 0, r_2 \geq 0\},$$

and $M_f(r_1, r_2) = \max\{|f(z_1, z_2)| : |z_i| \leq r_i, i = 1, 2\}$. Then, in view of maximum principal and Hartogs theorem ([5], p. 2, p. 51), $M_f(r_1, r_2)$ is an increasing function of $r_1, r_2$. In the sequel, the following two notations are used:

$$\begin{aligned} &\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \quad \text{for} \quad k = 1, 2, 3, \ldots; \\ &\log^{[0]} x = x, \end{aligned}$$

and

$$\begin{aligned} &\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \quad \text{for} \quad k = 1, 2, 3, \ldots; \\ &\exp^{[0]} x = x. \end{aligned}$$

The following definition is well known:

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**Definition 1.1** ([5], p. 339, (see also [1])). The order ${}_{v_2}\rho_f$ and the lower order ${}_{v_2}\lambda_f$ of an entire function $f$ of two complex variables are defined as

$$ {}_{v_2}\rho_f = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)}, $$

and

$$ {}_{v_2}\lambda_f = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[2]} M_f(r_1, r_2)}{\log(r_1 r_2)}. $$

If we consider the above definiton for the case of single variable, then the definition coincides with the classical definition of order (see [12]) which is as follows:

**Definition 1.2** ([12]). The order $\rho_f$ and the lower order $\lambda_f$ of an entire function $f$ are defined in the following ways:

$$ \rho_f = \limsup_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}, $$

and

$$ \lambda_f = \liminf_{r \to \infty} \frac{\log^{[2]} M_f(r)}{\log r}, $$

where $M_f(r) = \max\{|f(z)| : |z| = r\}$.

If $f$ is non-constant then $M_f(r)$ is strictly increasing and continuous, and its inverse $M_f{}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s \to \infty} M_f{}^{-1}(s) = \infty$. Bernal ([2], [3]) introduced the definition of relative order of $g$ with respect to $f$, denoted by $\rho_f(g)$ as follows:

$$ \begin{aligned} \rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \quad \text{for all} \quad r > r_0, (\mu) > 0\} \\ &= \limsup_{r \to \infty} \frac{\log M_g^{-1} M_f(r)}{\log r}. \end{aligned} $$

The definition coincides with the classical one [12] if $g(z) = \exp z$.

During the past decades, several authors (see [6, 8, 9, 10]) made close investigations on the properties of relative order of entire functions of single variable. In the case of relative order, it was then natural for Banerjee and Dutta [4] to define the relative order of entire functions of two complex variables as follows:

**Definition 1.3** ([4]). The relative order between two entire functions of two complex variables denoted by ${}_{v_2}\rho_g(f)$ is defined as:

$$ \begin{aligned} {}_{v_2}\rho_g(f) &= \inf\{\mu > 0 : M_f(r_1, r_2) < M_g(r_1^\mu, r_2^\mu); r_1 \geq R(\mu), r_2 \geq R(\mu)\} \\ &= \limsup_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)}, \end{aligned} $$

where $f$ and $g$ are entire functions holomorphic in the closed polydisc

$$U = \{(z_1, z_2) : |z_i| \leq r_i, i = 1, 2 \quad \text{for all} \quad r_1 \geq 0, r_2 \geq 0\}$$

and the definition coincides with Definition 1.1 (see [4]) if

$$g(z_1, z_2) = \exp(z_1 z_2).$$

However, an entire function of two complex variables, for which order and lower order are the same, is said to be of regular growth. The function $\exp(z_1 z_2)$ is an example of regular growth of entire functions of two complex variables. Further, the functions which are not of regular growth are said to be of irregular growth.

Now in the line of Juneja, Kapoor and Bajpai [7], we would like to introduce the definitions of $(p, q)$-th order and $(p, q)$-th lower order of an entire function $f$ of two complex variables respectively as follows:

$$_{v_2}\rho_f(p, q) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]}(r_1 r_2)},$$

and

$$_{v_2}\lambda_f(p, q) = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[p]} M_f(r_1, r_2)}{\log^{[q]}(r_1 r_2)},$$

where $p, q$ are any two positive integers with $p \geq q$. In particular, if we consider $q = 1$, then the above definition is reduced to the following definitions of generalized order and generalized lower order in connection with two complex variables:

**Definition 1.4.** The generalized order $_{v_2}\rho_f^{[p]}$ and the generalized lower order $_{v_2}\lambda_f^{[p]}$ of an entire function $f$ of two complex variables are defined as

$$_{v_2}\rho_f^{[p]} = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[p]} M_f(r_1, r_2)}{\log(r_1 r_2)},$$

and

$$_{v_2}\lambda_f^{[p]} = \liminf_{r_1, r_2 \infty} \frac{\log^{[p]} M_f(r_1, r_2)}{\log(r_1 r_2)},$$

where $p \geq 1$.

These definitions extend the generalized order and generalized lower order of an entire function as considered in [11]. Further, an entire function $f$ of two complex variables is said to be of regular $(p, q)$-growth if its $(p, q)$-th order coincides with its $(p, q)$-th lower order, otherwise $f$ is said to be of irregular $(p, q)$-growth.

Now in the case of relative order (respectively relative lower order), it is then natural to define the generalized relative order (respectively generalized relative lower order) of entire functions of two complex variables as follows:

**Definition 1.5.** Let $f(z_1, z_2)$ and $g(z_1, z_2)$ be any two entire functions of two complex variables $z_1$ and $z_2$ with maximum modulus functions $M_f(r_1, r_2)$ and $M_g(r_1, r_2)$ respectively, then for any positive interger $p$, the generalized relative order (respectively generalized relative lower order) of $f$ with respect to $g$, denoted by ${}_{v_2}\rho_g^{[p]}(f)$ (respectively ${}_{v_2}\lambda_g^{[p]}(f)$) is defined as

$$ {}_{v_2}\rho_g^{[p]}(f) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[p]} M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} $$

$$ \left( \text{respectively } {}_{v_2}\lambda_g^{[p]}(f) = \liminf_{r_1, r_2 \to \infty} \frac{\log M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \right). $$

In this paper, we wish to prove some results related to the growth properties of composite entire functions of two complex variables on the basis of generalized relative order and generalized relative lower order of entire functions of two complex variables. We do not explain the standard definitions and notations in the theory of entire function of two complex variables as those are available in [5].

## 2. Theorems

In this section, we present the main results of the paper.

**Theorem 2.1.** *Let $f$ and $g$ be any two entire functions of two complex variables with $0 < {}_{v_2}\lambda_f^{[m]} \leq {}_{v_2}\rho_f^{[m]} < \infty$ and $0 < {}_{v_2}\lambda_f(m,p) \leq {}_{v_2}\rho_g(m,p) < \infty$ where $m$ and $p$ are any positive integers with $m \geq p$. Then,*

$$ \begin{aligned} \frac{{}_{v_2}\lambda_f^{[m]}}{{}_{v_2}\rho_g(m,p)} &\leq_{v_2} \lambda_g^{[p]}(f) \\ &\leq \min\left\{ \frac{{}_{v_2}\lambda_f^{[m]}}{{}_{v_2}\lambda_f(m,p)}, \frac{{}_{v_2}\rho_f^{[m]}}{{}_{v_2}\rho_g(m,p)} \right\} \\ &\leq \max\left\{ \frac{{}_{v_2}\lambda_f^{[m]}}{{}_{v_2}\lambda_f(m,p)}, \frac{{}_{v_2}\rho_f^{[m]}}{{}_{v_2}\rho_g(m,p)} \right\} \\ &\leq_{v_2} \rho_g^{[p]}(f) \end{aligned} $$

$$\leq \frac{_{v_2}\rho_f^{[m]}}{_{v_2}\lambda_f\left(m,p\right)}.$$

*Proof.* From the definitions of $_{v_2}\rho_f^{[m]}$ and $_{v_2}\lambda_f^{[m]}$, we have for all sufficiently large values of $r_1, r_2$ that

$$M_f\left(r_1,r_2\right) \leq \exp^{[m]}\left\{\left(_{v_2}\rho_f^{[m]}+\varepsilon\right)\log\left(r_1r_2\right)\right\}, \tag{2.1}$$

$$M_f\left(r_1,r_2\right) \geq \exp^{[m]}\left\{\left(_{v_2}\lambda_f^{[m]}-\varepsilon\right)\log\left(r_1r_2\right)\right\}, \tag{2.2}$$

and also for a sequence of values of $r_1, r_2$ tending to infinity, we get that

$$M_f\left(r_1,r_2\right) \geq \exp^{[m]}\left\{\left(_{v_2}\rho_f^{[m]}-\varepsilon\right)\log\left(r_1r_2\right)\right\}, \tag{2.3}$$

$$M_f\left(r_1,r_2\right) \leq \exp^{[m]}\left\{\left(_{v_2}\lambda_f^{[m]}+\varepsilon\right)\log\left(r_1r_2\right)\right\}. \tag{2.4}$$

Similarly, from the definitions of $_{v_2}\rho_g\left(m,p\right)$ and $_{v_2}\lambda_f\left(m,p\right)$, it follows for all sufficiently large values of $r_1, r_2, ..., r_n$ that

$$M_g\left(r_1,r_2\right) \leq \exp^{[m]}\left\{\left(_{v_2}\rho_g\left(m,p\right)+\varepsilon\right)\log^{[p]}\left(r_1r_2\right)\right\},$$

i.e.,

$$\left(r_1,r_2\right) \leq M_g^{-1}\left[\exp^{[m]}\left\{\left(_{v_2}\rho_g\left(m,p\right)+\varepsilon\right)\log^{[p]}\left(r_1r_2\right)\right\}\right],$$

i.e.,

$$M_g^{-1}\left(r_1,r_2\right) \geq \exp^{[p]}\left[\frac{\log^{[m]}\left(r_1r_2\right)}{\left(_{v_2}\rho_g\left(m,p\right)+\varepsilon\right)}\right], \tag{2.5}$$

$$M_g\left(r_1,r_2\right) \geq \exp^{[m]}\left\{\left(_{v_2}\lambda_g\left(m,p\right)-\varepsilon\right)\log^{[p]}\left(r_1r_2\right)\right\},$$

i.e.,

$$M_g^{-1}\left(r_1,r_2\right) \leq \exp^{[p]}\left[\frac{\log^{[m]}\left(r_1r_2\right)}{\left(_{v_2}\lambda_g\left(m,p\right)-\varepsilon\right)}\right], \tag{2.6}$$

and for a sequence of values of $r_1, r_2$ tending to infinity, we obtain that

$$M_g\left(r_1,r_2\right) \geq \exp^{[m]}\left\{\left(_{n_2}\rho_g\left(m,p\right)-\varepsilon\right)\log^{[p]}\left(r_1r_2\right)\right\},$$

i.e.,

$$M_g^{-1}\left(r_1,r_2\right) \leq \exp^{[p]}\left[\frac{\log^{[m]}\left(r_1r_2\right)}{\left(_{n_2}\rho_g\left(m,p\right)-\varepsilon\right)}\right], \tag{2.7}$$

$$M_g\left(r_1,r_2\right) \leq \exp^{[m]}\left\{\left(_{n_2}\lambda_g\left(m,p\right)+\varepsilon\right)\log^{[p]}\left(r_1r_2\right)\right\},$$

i.e.,

$$M_g^{-1}(r_1,r_2) \geq \exp^{[p]}\left[\frac{\log^{[m]}(r_1 r_2)}{({}_{n_2}\lambda_g(m,p)+\varepsilon)}\right]. \tag{2.8}$$

Now from $(2.3)$ and in view of $(2.5)$, we get for a sequence of values of $r_1, r_2$ tending to infinity we get that

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \geq \log^{[p]} M_g^{-1}\left[\exp^{[m]}\left\{\left({}_{v_2}\rho_f^{[m]}-\varepsilon\right)\log(r_1 r_2)\right\}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \geq \log^{[p]} \exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left({}_{v_2}\rho_f^{[m]}-\varepsilon\right)\log(r_1 r_2)\right\}}{({}_{v_2}\rho_g(m,p)+\varepsilon)}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \geq \frac{\left({}_{v_2}\rho_f^{[m]}-\varepsilon\right)}{({}_{v_2}\rho_g(m,p)+\varepsilon)}\log(r_1 r_2),$$

i.e.,

$$\frac{\log^{[p]} M_g^{-1} M_f(r_1,r_2)}{\log(r_1 r_2)} \geq \frac{\left({}_{v_2}\rho_f^{[m]}-\varepsilon\right)}{({}_{v_2}\rho_g(m,p)+\varepsilon)}.$$

As $\varepsilon(>0)$ is arbitrary, it follows that

$${}_{v_2}\rho_g^{[p]}(f) \geq \frac{{}_{v_2}\rho_f^{[m]}}{{}_{v_2}\rho_g(m,p)}. \tag{2.9}$$

Analogously from (2.2) and in view of (2.8), for a sequence of values of $r_1, r_2$ tending to infinity we get that

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \geq \log^{[p]} M_g^{-1}\left[\exp^{[m]}\left\{\left({}_{v_2}\lambda_f^{[m]}-\varepsilon\right)\log(r_1 r_2)\right\}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \geq \log^{[p]} \exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left({}_{v_2}\lambda_f^{[m]}-\varepsilon\right)\log(r_1 r_2)\right\}}{({}_{v_2}\lambda_g(m,p)+\varepsilon)}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \geq \frac{\left({}_{v_2}\lambda_f^{[m]}-\varepsilon\right)}{({}_{v_2}\lambda_g(m,p)+\varepsilon)}\log(r_1 r_2),$$

i.e.,

$$\frac{\log^{[p]} M_g^{-1} M_f(r_1,r_2)}{\log(r_1 r_2)} \geq \frac{\left({}_{v_2}\lambda_f^{[m]}-\varepsilon\right)}{({}_{v_2}\lambda_g(m,p)+\varepsilon)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$ {}_{v_2}\rho_g^{[p]}(f) \geq \frac{{}_{v_2}\lambda_f^{[m]}}{{}_{v_2}\lambda_g(m,p)}. \tag{2.10}$$

Again, in view of (2.6), we have from (2.1) for all sufficiently large values of $r_1, r_2$ that

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \leq \log^{[p]} M_g^{-1}\left[\exp^{[m]}\left\{\left({}_{v_2}\rho_f^{[m]}+\varepsilon\right)\log(r_1 r_2)\right\}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \leq \log^{[p]}\exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left({}_{v_2}\rho_f^{[m]}+\varepsilon\right)\log(r_1 r_2)\right\}}{\left({}_{v_2}\lambda_g(m,p)-\varepsilon\right)}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \leq \frac{\left({}_{v_2}\rho_f^{[m]}+\varepsilon\right)}{\left({}_{v_2}\lambda_g(m,p)-\varepsilon\right)}\log(r_1 r_2),$$

i.e.,

$$\frac{\log^{[p]} M_g^{-1} M_f(r_1,r_2)}{\log(r_1 r_2)} \leq \frac{\left({}_{v_2}\rho_f^{[m]}+\varepsilon\right)}{\left({}_{v_2}\lambda_g(m,p)-\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$ {}_{v_2}\rho_g^{[p]}(f) \leq \frac{{}_{v_2}\rho_f^{[m]}}{{}_{v_2}\lambda_g(m,p)}. \tag{2.11}$$

Again, from (2.2) and in view of (2.5), with the same reasoning, we get that

$$ {}_{v_2}\lambda_g^{[p]}(f) \geq \frac{{}_{v_2}\lambda_f^{[m]}}{{}_{v_2}\rho_g(m,p)}. \tag{2.12}$$

Also, in view of (2.7), we get from (2.1) for a sequence of values of $r_1, r_2$ tending to infinity that

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \leq \log^{[p]} M_g^{-1}\left[\exp^{[m]}\left\{\left({}_{v_2}\rho_f^{[m]}+\varepsilon\right)\log(r_1 r_2)\right\}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \leq \log^{[p]}\exp^{[p]}\left[\frac{\log^{[m]}\exp^{[m]}\left\{\left({}_{v_2}\rho_f^{[m]}+\varepsilon\right)\log(r_1 r_2)\right\}}{\left({}_{v_2}\rho_g(m,p)-\varepsilon\right)}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1,r_2) \leq \frac{\left({}_{v_2}\rho_f^{[m]}+\varepsilon\right)}{\left({}_{v_2}\rho_g(m,p)-\varepsilon\right)}\log(r_1 r_2),$$

i.e.,

$$\frac{\log^{[p]} M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \leq \frac{\left({}_{v_2}\rho_f^{[m]} + \varepsilon\right)}{\left({}_{v_2}\rho_g(m, p) - \varepsilon\right)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$(2.13) \qquad {}_{v_2}\lambda_g^{[p]}(f) \leq \frac{{}_{v_2}\rho_f^{[m]}}{{}_{v_2}\rho_g(m, p)}.$$

Similarly, from (2.4) and in view of (2.6), it follows for a sequence of values of $r_1, r_2$ tending to infinity we get that

$$\log^{[p]} M_g^{-1} M_f(r_1, r_2) \leq \log^{[p]} M_g^{-1}\left[\exp^{[m]}\left\{\left({}_{v_2}\lambda_f^{[m]} + \varepsilon\right)\log(r_1 r_2)\right\}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1, r_2) \leq \log^{[p]} \exp^{[p]}\left[\frac{\log^{[m]} \exp^{[m]}\left\{\left({}_{v_2}\lambda_f^{[m]} + \varepsilon\right)\log(r_1 r_2)\right\}}{\left({}_{v_2}\lambda_g(m, p) - \varepsilon\right)}\right],$$

i.e.,

$$\log^{[p]} M_g^{-1} M_f(r_1, r_2) \leq \frac{\left({}_{v_2}\lambda_f^{[m]} + \varepsilon\right)}{\left({}_{v_2}\lambda_g(m, p) - \varepsilon\right)}\log(r_1 r_2),$$

i.e.,

$$\frac{\log^{[p]} M_g^{-1} M_f(r_1, r_2)}{\log(r_1 r_2)} \leq \frac{\left({}_{v_2}\lambda_f^{[m]} + \varepsilon\right)}{\left({}_{v_2}\lambda_g(m, p) - \varepsilon\right)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain from above that

$$(2.14) \qquad {}_{v_2}\lambda_g^{[p]}(f) \leq \frac{{}_{v_2}\lambda_f^{[m]}}{{}_{v_2}\lambda_g(m, p)}.$$

The theorem follows from (2.9), (2.10), (2.11), (2.12), (2.13) and (2.14). □

**Corollary 2.2.** *Let $f$ be an entire function of two complex variables with generalized order ${}_{v_2}\rho_f^{[m]}$ and generalized lower order ${}_{v_2}\lambda_f^{[m]}$ where $m$ is any positive integer. Also, let $g$ be an entire function of two complex variables with regular $(m, p)$-growth where $p, m$ are all positive integers such that $m \geq p$. Then,*

$${}_{v_2}\lambda_g^{[p]}(f) = \frac{{}_{v_2}\lambda_f^{[m]}}{{}_{v_2}\rho_g(m, p)} \quad \textit{and} \quad {}_{v_2}\rho_g^{[p]}(f) = \frac{{}_{v_2}\rho_f^{[m]}}{{}_{v_2}\rho_g(m, p)}.$$

**Corollary 2.3.** *Let $f$ and $g$ be any two entire functions of two complex variables with regular generalized growth and regular $(m,p)$ growth respectively where $p,m$ are all positive integers with $m \geq p$. Then,*

$$_{v_2}\lambda_g^{[p]}(f) = _{v_2}\rho_g^{[p]}(f) = \frac{_{v_2}\rho_f^{[m]}}{_{v_2}\rho_g(m,p)}.$$

**Corollary 2.4.** *Let $f$ and $g$ be any two entire functions of two complex variables with regular generalized growth and regular $(m,p)$ growth respectively where $p,m$ are all positive integers with $m \geq p$. Also suppose that $_{v_2}\rho_f^{[m]} = {}_{v_2}\rho_g(m,p)$. Then,*

$$_{v_2}\lambda_g^{[p]}(f) = _{v_2}\rho_g^{[p]}(f) = 1.$$

**Corollary 2.5.** *Let $f$ be an entire function of two complex variables with generalized order $_{v_2}\rho_f^{[m]}$ and generalized lower order $_{v_2}\lambda_f^{[m]}$ where $m$ is any positive integer. Then for any entire function of two complex variables $g$,*

$$\begin{aligned} (i) &\quad _{v_2}\lambda_g^{[p]}(f) = \infty \text{ when } _{v_2}\rho_g(m,p) = 0, \\ (ii) &\quad _{v_2}\rho_g^{[p]}(f) = \infty \text{ when } _{v_2}\lambda_g(m,p) = 0, \\ (iii) &\quad _{v_2}\lambda_g^{[p]}(f) = 0 \text{ when } _{v_2}\rho_g(m,p) = \infty, \\ (iv) &\quad _{v_2}\rho_g^{[p]}(f) = 0 \text{ when } _{v_2}\lambda_g(m,p) = \infty, \end{aligned}$$

*where $p$ is any positive integer with $m \geq p$ .*

**Corollary 2.6.** *Let $g$ be an entire function of two complex variables with $(m,p)$-th order $_{v_n}\rho_g(m,p)$ and $(m,p)$-th lower order $_{v_n}\lambda_g(m,p)$ where $m,p$ are positive integers with $m \geq p$. Then for any entire function of two complex variables $f$,*

$$\begin{aligned} (i) &\quad _{v_2}\rho_g^{[p]}(f) = 0 \text{ when } _{v_2}\rho_f^{[m]} = 0, \\ (ii) &\quad _{v_2}\lambda_g^{[p]}(f) = 0 \text{ when } _{v_2}\lambda_f^{[m]} = 0, \\ (iii) &\quad _{v_2}\rho_g^{[p]}(f) = \infty \text{ when } _{v_2}\rho_f^{[m]} = \infty, \\ (iv) &\quad _{v_2}\lambda_g^{[p]}(f) = \infty \text{ when } _{v_2}\lambda_f^{[m]} = \infty. \end{aligned}$$

**Theorem 2.7.** *Let $f$, $g$ and $h$ be any three entire functions of two complex variables such that $_{v_2}\rho_h^{[p]}(f) < \infty$ and $_{v_2}\lambda_h^{[p]}(f \circ g) = \infty$ where $p$ is any positive integer. Then,*

$$\lim_{r_1,r_2\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r_1,r_2)}{\log^{[p]} M_h^{-1} M_f(r_1,r_2)} = \infty.$$

*Proof.* Let us suppose that the conclusion of the theorem do not hold. Then, we can find a constant $\beta > 0$ such that for a sequence of values of $r_1, r_2$ tending to infinity

$$\log^{[p]} M_h^{-1} M_{f\circ g}(r_1, r_2) \leq \beta \log^{[p]} M_h^{-1} M_f(r_1, r_2). \tag{2.15}$$

Again from the definition of ${}_{v_2}\rho_h^{[p]}(f)$, it follows for all sufficiently large values of $r_1, r_2$ that

$$\log^{[p]} M_h^{-1} M_f(r_1, r_2) \leq \left({}_{v_2}\rho_h^{[p]}(f) + \epsilon\right) \log(r_1 r_2). \tag{2.16}$$

Thus from (2.15) and (2.16), we have for a sequence of values of $r_1, r_2$ tending to infinity that

$$\log^{[p]} M_h^{-1} M_{f\circ g}(r_1, r_2) \leq \beta \left({}_{v_2}\rho_h^{[p]}(f) + \epsilon\right) \log(r_1 r_2),$$

i.e.,

$$\frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r_1, r_2)}{\log(r_1 r_2)} \leq \frac{\beta \left({}_{v_2}\rho_h^{[p]}(f) + \epsilon\right) \log(r_1 r_2)}{\log(r_1 r_2)},$$

i.e.,

$$\liminf_{r_1, r_2 \to \infty} \frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r_1, r_2)}{\log(r_1 r_2)} = {}_{v_2}\lambda_h^{[p]}(f \circ g) < \infty.$$

This is a contradiction.
Thus the theorem follows. □

*Remark* 2.8. Theorem 2.7 is also valid with "limit superior" instead of "limit" if ${}_{v_2}\lambda_h^{[p]}(f \circ g) = \infty$ is replaced by ${}_{v_2}\rho_h^{[p]}(f \circ g) = \infty$ and the other conditions remain the same.

**Corollary 2.9.** *Under the assumptions of Theorem 2.7 and Remark 2.8,*

$$\lim_{r_1, r_2 \to \infty} \frac{\log^{[p-1]} M_h^{-1} M_{f\circ g}(r_1, r_2)}{\log^{[p-1]} M_h^{-1} M_f(r_1, r_2)} = \infty,$$

*and*

$$\limsup_{r_1, r_2 \to \infty} \frac{\log^{[p-1]} M_h^{-1} M_{f\circ g}(r_1, r_2)}{\log^{[p-1]} M_h^{-1} M_f(r_1, r_2)} = \infty,$$

*respectively hold.*

*Proof.* The proof is omitted. □

Analogously, one may also state the following theorem, remark and corollary without their proofs as those may be carried out in the line of Remark 2.8, Theorem 2.7 and Corollary 2.9 respectively.

**Theorem 2.10.** *Let $f$, $g$ and $h$ be any three entire functions of two complex variables with ${}_{v_2}\rho_h^{[p]}(g) < \infty$ and ${}_{v_2}\rho_h^{[p]}(f \circ g) = \infty$ where $p$ is any integer. Then,*

$$\limsup_{r_1, r_2 \to \infty} \frac{\log^{[p]} M_h^{-1} M_{f \circ g}(r_1, r_2)}{\log^{[p]} M_h^{-1} M_g(r_1, r_2)} = \infty.$$

*Remark* 2.11. Theorem 2.10 is also valid with "limit" instead of "limit superior" if ${}_{v_2}\rho_h^{[p]}(f \circ g) = \infty$ is replaced by ${}_{v_2}\lambda_h^{[p]}(f \circ g) = \infty$ and the other conditions remain the same.

**Corollary 2.12.** *Under the assumptions of Theorem 2.10 and Remark 2.11,*

$$\limsup_{r_1, r_2 \to \infty} \frac{\log^{[p-1]} M_h^{-1} M_{f \circ g}(r_1, r_2)}{\log^{[p-1]} M_h^{-1} M_g(r_1, r_2)} = \infty,$$

*and*

$$\lim_{r_1, r_2 \to \infty} \frac{\log^{[p-1]} M_h^{-1} M_{f \circ g}(r_1, r_2)}{\log^{[p-1]} M_h^{-1} M_g(r_1, r_2)} = \infty,$$

*respectively hold.*

## References


1. A.K. Agarwal, *On the properties of entire function of two complex variables*, Canadian Journal of Mathematics, 20 (1968) 51–57.
2. L. Bernal, *Crecimiento relativo de funciones enteras. Contribución al estudio de lasfunciones enteras con índice exponencial finito*, Doctoral Dissertation, University of Seville, Spain, 1984.
3. L. Bernal, *Orden relativo de crecimiento de funciones enteras*, Collect. Math., 39 (1988) 209–229.
4. D. Banerjee and R. K. Dutta, *Relative order of entire functions of two complex variables*, International J. of Math. Sci. & Engg. Appls. (IJMSEA), 1(1) (2007) 141–154.
5. A.B. Fuks, *Theory of analytic functions of several complex variables*, Moscow, 1963.
6. S. Halvarsson, *Growth properties of entire functions depending on a parameter*, Annales Polonici Mathematici, 14(1) (1996) 71–96.
7. O.P. Juneja, G.P. Kapoor, and S.K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math., 282 (1976) 53–67.
8. C.O. Kiselman, *Order and type as measure of growth for convex or entire functions*, Proc. Lond. Math. Soc., 66(3) (1993) 152–186.
9. C.O. Kiselman, *Plurisubharmonic functions and potential theory in several complex variable*, a contribution to the book project, Development of Mathematics, 1950-2000, edited by Hean-Paul Pier.
10. B.K. Lahiri and D. Banerjee, *A note on relative order of entire functions*, Bull. Cal. Math. Soc., 97(3) (2005) 201–206.
11. D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., 69 (1963) 411–414.

12. E.C. Titchmarsh, *The theory of functions , 2nd ed. Oxford University Press* , Oxford, 1968.

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