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# ON CERTAIN FRACTIONAL CALCULUS OPERATORS INVOLVING GENERALIZED MITTAG-LEFFLER FUNCTION

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Abstract. The object of this paper is to establish certain generalized fractional integration and differentiation involving generalized Mittag-Leffler function defined by Salim and Faraj [25]. The considered generalized fractional calculus operators contain the Appell's function $F_3$ [2, p.224] as kernel and are introduced by Saigo and Maeda [23]. The Marichev-Saigo-Maeda fractional calculus operators are the generalization of the Saigo fractional calculus operators. The established results provide extensions of the results given by Gupta and Parihar [3], Saxena and Saigo [30], Samko et al. [26]. On account of the general nature of the generalized Mittag-Leffler function and generalized Wright function, a number of known results can be easily found as special cases of our main results.

## 1. Introduction and Preliminaries

During the last two decades, Mittag-Leffler function has come into prominence after about nine decades of its discovery by a Swedish Mathematician G.M. Mittag-Leffler, due to the vast potential of its applications in solving the problems of physical, biological, engineering and earth sciences etc. In this survey paper, nearly all types of Mittag-Leffler type functions existing in the literature are presented (for example, see [4, 5, 11, 12, 19, 31]). Mittag-Leffler function naturally occurs as the solution of fractional order differintegral equations.

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The function $E_\alpha(z)$ is introduced by the Swedish mathematician Gosta Mittag-Leffler[17, 18], and defined as:

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad (\alpha \in \mathbb{C}, \Re(\alpha) > 0),$$

the Mittag-Leffler function is a direct generalization of $\exp(z)$ in which $\alpha = 1$.

A generalization of $E_\alpha(z)$ was given and studied by Wiman [32], defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad (\alpha, \beta \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0). \tag{1.1}$$

The M-L functions are generalization of the exponential, hyperbolic and trigonometric functions since $E_{1,1}(z) = e^z$, $E_{2,1}(z^2) = \cosh(z)$, $E_{2,1}(-z^2) = \cos(z)$ and $E_{2,2}(-z^2) = \sin(z)/z$.

In 1971, the generalization of (1.1) was introduced by Prabhakar [21] in terms of the series representation as given following (see also, [6]):

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta) n!}, \quad (\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \Re(\beta) > 0), \tag{1.2}$$

where $(\gamma)_n = \Gamma(\gamma + n)/\Gamma(\gamma)$ is the Pochhammer symbol

$$((\gamma)_0 = 1, (\gamma)_n = \gamma(\gamma+1)\cdots(\gamma+k-1)),$$

for $n = k \in \mathbb{N}$. Also, note that $E^1_{\alpha,\beta} = E_{\alpha,\beta}$.

The basic properties, different functional and recurrence relations, integral representations and asymptotic behaviors of the three parameter M-L functions (1.2) and their applications are studied in [4, 15] in detail. Detailed analysis on the asymptotic behavior, inequalities and convergence of the three parameter M-L functions were studied in [19, 20].

Further generalization of M-L function (four parameter M-L function) was defined by Salim [24], as following:

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(\alpha n + \beta)\,(\delta)_n},$$

where $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $\Re(\delta) > 0$.

Recently, a new generalization of Mittag-Leffler function introduced by Salim and Faraj [25] in the following manner:

$$E^{\delta,\xi,q}_{\alpha,\beta,p}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn} z^n}{\Gamma(\alpha n + \beta)\,(\xi)_{pn}}, \tag{1.3}$$

where $\alpha, \beta, \delta, \xi \in \mathbb{C}; \Re(\alpha), \Re(\beta), \Re(\delta), \Re(\xi) > 0; p, q > 0, q \leq \Re(\alpha)+p$ and

$$(\gamma)_{qn} = \frac{\Gamma(\gamma+qn)}{\Gamma(\gamma)}. \tag{1.4}$$

Here, $(\gamma)_{qn}$ denotes the generalized Pochhammer symbol.
For $\xi = p = q = 1$ and $\delta = \xi = p = q = 1$, equation (1.3) is reduced to generalized Mittag-Leffler function $E^{\delta}_{\alpha,\beta}(z)$ and Mittag-Leffler function $E_{\alpha,\beta}(z)$ respectively.

The generalized Wright hypergeometric function is introduced by Wright [33] and given by

$$\begin{aligned} {}_p\Psi_q(z) &= {}_p\Psi_q\left[z\left|\begin{array}{c}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{array}\right.\right] \\ &= \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+n\alpha_i)}{\prod_{j=1}^{q}\Gamma(b_j+n\beta_j)}\frac{z^n}{n!}, \end{aligned} \tag{1.5}$$

where $z, a_i, b_j \in \mathbb{C}$ and $\alpha_i, \beta_j \in \mathbb{R} - \{0\}, \; (i = 1, \ldots, p; \, j = 1, \ldots, q)$. Wright proved several theorems on the asymptotic expansion of generalized Wright function ${}_p\Psi_q(z)$ for all values of the argument $z$ under the condition

$$\sum_{i=1}^{p}\alpha_i - \sum_{j=1}^{q}\beta_j \leq 1. \tag{1.6}$$

When $\alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1$, then (1.5) is reduced to a generalized hypergeometric function ${}_pF_q(.)$ as shown below

$${}_p\Psi_q\left[z\left|\begin{array}{c}(a_1,1),\ldots,(a_p,1)\\(b_1,1),\ldots,(b_q,1)\end{array}\right.\right] = \frac{\prod_{i=1}^{p}\Gamma(a_i)}{\prod_{j=1}^{q}\Gamma(b_j)}{}_pF_q\left(a_1,\ldots,a_p;b_1,\ldots,b_q;z\right),$$

where $z, a_i, b_j \in \mathbb{C}, (i = 1, \ldots, p; j = 1, \ldots, q)$; and

$$Re\left(\sum_{j=1}^{q}b_j - \sum_{i=1}^{p}a_i\right) > 0.$$

Properties of this generalized Wright function were investigated in [7]. In particular, it was proved that ${}_p\Psi_q(z)$, $z \in \mathbb{C}$ in an entire function under the condition (1.6)

## 2. Generalized fractional calculus operators

The fractional integral operator has many interesting applications in various subfields in applicable mathematical analysis; for example, [8], it has applications related to a certain class of complex analytic functions. The results given in [1, 9, 10, 13, 16, 27, 29] can be referred to some

basic results on fractional calculus.
Let $\alpha, \alpha', \beta, \beta', \gamma \in C$, $x > 0$ and $\Re(\gamma) > 0$, then, the generalized fractional integral operators involving Appell's function or Horns function $F_3$ are introduced by Marichev [14] and later extended and studied by Saigo and Maeda [23], as follows (see, [23, p.393, eq.(4.12) and (4.13)])

(2.1)
$$\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\, t^{-\alpha'}\, F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\mathrm{d}t,$$

(2.2)
$$\left(I_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1}\, t^{-\alpha}\, F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x}\right) f(t)\mathrm{d}t.$$

These operators are reduced to the Saigo fractional integral operators [11, 22] due to the following relations:
$$I_{0+}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{0+}^{\gamma,\alpha-\gamma,-\beta} f(x) \ (\gamma \in C),$$
and
$$I_{-}^{\alpha,0,\beta,\beta',\gamma} f(x) = I_{-}^{\gamma,\alpha-\gamma,-\beta} f(x) \ (\gamma \in C).$$
The generalized fractional differentiation operators [23] involving the Appell function $F_3$ as a kernel are defined by
$$\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{2.3}$$
$$= \left(\frac{d}{dx}\right)^k \left(I_{0+}^{-\alpha',-\alpha,-\beta'+k,-\beta,-\gamma+k} f\right)(x), \tag{2.4}$$
where
$$(\mathrm{Re}(\gamma) > 0;\ k = [\mathrm{Re}(\gamma)] + 1)$$
$$\left(D_{-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\gamma} f\right)(x) \tag{2.5}$$
$$= \left(-\frac{d}{dx}\right)^k \left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta+k,-\gamma+k} f\right)(x), \tag{2.6}$$
where
$$(\mathrm{Re}(\gamma) > 0;\ k = [\mathrm{Re}(\gamma)] + 1).$$
These operators reduce to the Saigo fractional derivative operators [22, 28] as
$$\left(D_{0+}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{0+}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x), \qquad (\mathrm{Re}(\gamma) > 0); \tag{2.7}$$

$$\left(D_{-}^{0,\alpha',\beta,\beta',\gamma} f\right)(x) = \left(D_{-}^{\gamma,\alpha'-\gamma,\beta'-\gamma} f\right)(x), \qquad (\mathrm{Re}(\gamma) > 0). \tag{2.8}$$

Further [p. 394, eq.(4.18) and (4.19)], [23] we also have

$$I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\ x^{\rho-1} = \Gamma\begin{bmatrix} \rho,\ \rho+\gamma-\alpha-\alpha'-\beta,\ \rho+\beta'-\alpha' \\ \rho+\gamma-\alpha-\alpha',\ \rho+\gamma-\alpha'-\beta,\ \rho+\beta' \end{bmatrix} x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{2.9}$$

where $\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\rho) > \max[0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), \mathrm{Re}(\alpha'-\beta')]$, and

$$I_{-}^{\alpha,\alpha',\beta,\beta',\gamma}\ x^{\rho-1} = \Gamma\begin{bmatrix} 1+\alpha+\alpha'-\gamma-\rho,\ 1+\alpha+\beta'-\gamma-\rho,\ 1-\beta-\rho \\ 1-\rho,\ 1+\alpha+\alpha'+\beta'-\gamma-\rho,\ 1+\alpha-\beta-\rho \end{bmatrix} x^{\rho-\alpha-\alpha'+\gamma-1}, \tag{2.10}$$

where
$\mathrm{Re}(\gamma) > 0$, $\mathrm{Re}(\rho) < 1 + \min[Re(-\beta),\ Re(\alpha+\alpha'-\gamma),\ Re(\alpha+\beta'-\gamma)]$.
Here, the symbol $\Gamma\begin{bmatrix} a,b,c \\ d,e,f \end{bmatrix}$ will be used to represent the ratio of product of gamma functions as $\frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}$.

## 3. Generalized Fractional Integration of the generalized Mittag-Leffler function

In this section, we establish image formulas for the generalized Mittag-Leffler function involving Marichev-Saigo-Meada fractional integral operators (2.1) and (2.2), in term of the generalized Wright function. These formulas are given by the following theorems:

**Theorem 3.1.** *Let $\alpha, \alpha', \beta, \beta', \mu, \delta, \xi, \rho \in \mathbb{C}$, $x > 0$, $\nu > 0$, $p, q > 0$, $q \leq \Re(\nu) + p$ and $a \in \Re$ be such that $\Re(\mu) > 0$,*

$$\Re(\nu n + \rho) > \max\left[0, \Re\left(\alpha+\alpha'+\beta-\mu\right), \Re\left(\alpha'-\beta'\right)\right],$$

*then, there holds the formula*

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x) = \frac{x^{\rho-\alpha-\alpha'+\mu-1}\Gamma(\xi)}{\Gamma(\delta)}$$
$$\times\ {}_4\Psi_4\left[ax^{\nu}\left|\begin{matrix} (1,1),(\delta,q),(\rho-\alpha-\alpha'-\beta+\mu,\nu),(\rho-\alpha'+\beta',\nu) \\ (\xi,p),(\rho-\alpha-\alpha'+\mu,\nu),(\rho-\alpha'-\beta+\mu,\nu),(\rho+\beta',\nu) \end{matrix}\right.\right].$$

*Proof.* By using series representation of generalized Mittag-Leffler function (1.3) and left-sided Saigo-Maeda fractional integration power function formula (2.9), we have

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)$$
$$= \left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}}{\Gamma(\nu n+\rho)(\xi)_{pn}}\left(at^{\nu}\right)^{n}\right]\right\}(x),$$

by interchanging the order of integration and summation, we arrive at

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ (a)^{n}}{\Gamma\left(\nu n+\rho\right)(\xi)_{pn}}\left(I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}t^{\nu n+\rho-1}\right)(x)$$

$$=x^{\rho-\alpha-\alpha'+\mu-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ (ax^{\nu})^{n}}{(\xi)_{pn}}$$

$$\times\frac{\Gamma\left(\rho-\alpha-\alpha'-\beta+\mu+\nu n\right)\Gamma\left(\rho-\alpha'+\beta'+\nu n\right)}{\Gamma\left(\rho-\alpha-\alpha'+\mu+\nu n\right)\Gamma\left(\rho-\alpha'-\beta+\mu+\nu n\right)\Gamma\left(\rho+\beta'+\nu n\right)},$$

next, using (1.4) and (1.5), we get

$$\left\{I_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)=\frac{x^{\rho-\alpha-\alpha'+\mu-1}\Gamma\left(\xi\right)}{\Gamma\left(\delta\right)}$$

$$\times\ {}_4\Psi_4\left[ax^{\nu}\left|\begin{array}{c}(1,1),(\delta,q),(\rho-\alpha-\alpha'-\beta+\mu,\nu),(\rho-\alpha'+\beta',\nu)\\(\xi,p),(\rho-\alpha-\alpha'+\mu,\nu),(\rho-\alpha'-\beta+\mu,\nu),(\rho+\beta',\nu)\end{array}\right.\right].$$

This completes the proof of theorem. □

If we take $\alpha=\alpha+\beta,\alpha'=\beta'=0,\beta=-\eta$ and $\mu=\alpha$, then, Theorem 3.1 is reduced to the following result given by Gupta and Parihar [3, p.140, eq.(2.1)]:

**Corollary 3.2.** *Let* $\alpha,\beta,\eta,\delta,\xi,\rho\in\mathbb{C}$, $x>0,\nu>0$, $p,q>0,q\leq\Re(\nu)+p$ *and* $a\in\Re$ *be such that* $\Re(\alpha)>0$, $\Re\left(\rho+\eta-\beta\right)>0$,

$$\Re\left(\nu n+\rho\right)>\max\left[0,\Re\left(\beta-\eta\right)\right],$$

*then, there holds the formula*

$$\left\{I_{0+}^{\alpha,\beta,\eta}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)=\frac{x^{\rho-\beta-1}\Gamma\left(\xi\right)}{\Gamma\left(\delta\right)}\tag{3.1}$$

$$\times\ {}_3\Psi_3\left[ax^{\nu}\left|\begin{array}{c}(1,1),(\delta,q),(\rho-\beta+\eta,\nu)\\(\xi,p),(\rho-\beta,\nu),(\rho+\alpha+\eta,\nu)\end{array}\right.\right].$$

*Remark* 3.3. If we set $\beta=-\alpha$ and $\xi=p=q=1$ in (3.1), then, we obtain the known result given by Saxena and Saigo [30, p.145, eq.(14)]; further, if we also set $\delta=1$, then we get the known result given by Samko et al. [26, table (9.1), formula (23)] (see also, [30, p.146, eq.(15)]).

**Theorem 3.4.** *Let* $\alpha,\alpha',\beta,\beta',\mu,\delta,\xi,\rho\in\mathbb{C}$, $x>0$, $\nu>0$, $p,q>0$, $q\leq\Re(\nu)+p$ *and* $a\in\Re$ *be such that* $\Re(\mu)>0$,

$$\Re\left(1-\nu n-\rho\right)<1+\min\left[\Re\left(-\beta\right),\Re\left(\alpha+\alpha'-\mu\right),\Re\left(\alpha+\beta'-\mu\right)\right],$$

*then, there holds the formula*

(3.2)

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x)=\frac{x^{-\rho-\alpha-\alpha'}\Gamma(\xi)}{\Gamma(\delta)}$$

$$\times\ {}_5\Psi_5\left[ax^{-\nu}\left|\begin{matrix}(1,1),(\delta,q),(\rho+\alpha+\alpha',\nu),(\rho+\alpha+\beta',\nu),(\rho-\beta+\mu,\nu)\\(\xi,p),(\rho,\nu),(\rho+\mu,\nu),(\rho+\alpha+\alpha'+\beta',\nu),(\rho+\alpha-\beta+\mu,\nu)\end{matrix}\right.\right].$$

*Proof.* By using series representation of generalized Mittag-Leffler function (1.3) and left-sided Saigo-Maeda fractional integration power function formula (2.10), we get

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x)$$

$$=\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ \left(at^{-\nu}\right)^n}{\Gamma(\nu n+\rho)\,(\xi)_{pn}}\right]\right\}(x),$$

by interchanging the order of integration and summation, we arrive at the following:

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ (a)^n}{\Gamma(\nu n+\rho)\,(\xi)_{pn}}\left(I_{-}^{\alpha,\alpha',\beta,\beta',\mu}t^{(1-\nu n-\rho-\mu)-1}\right)(x)$$

$$=x^{-\rho-\alpha-\alpha'}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ \left(ax^{-\nu}\right)^n}{\Gamma(\rho+\nu n)\,(\xi)_{pn}}$$

$$\times\ \frac{\Gamma(\rho+\alpha+\alpha'+\nu n)\ \Gamma(\rho+\alpha+\beta'+\nu n)\ \Gamma(\rho-\beta+\mu+\nu n)}{\Gamma(\rho+\mu+\nu n)\ \Gamma(\rho+\alpha+\alpha'+\beta'+\nu n)\ \Gamma(\rho+\alpha-\beta+\mu+\nu n)},$$

next, using (1.4), (1.5) and rearranging the terms, we have

$$\left\{I_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{-\mu-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x)=\frac{x^{-\rho-\alpha-\alpha'}\Gamma(\xi)}{\Gamma(\delta)}$$

$$\times\ {}_5\Psi_5\left[ax^{-\nu}\left|\begin{matrix}(1,1),(\delta,q),(\rho+\alpha+\alpha',\nu),(\rho+\alpha+\beta',\nu),(\rho-\beta+\mu,\nu)\\(\xi,p),(\rho,\nu),(\rho+\mu,\nu),(\rho+\alpha+\alpha'+\beta',\nu),(\rho+\alpha-\beta+\mu,\nu)\end{matrix}\right.\right].$$

This completes the proof of theorem. □

If we take $\alpha=\alpha+\beta$, $\alpha'=\beta'=0$, $\beta=-\eta$ and $\mu=\alpha$ then (3.2) is reduced to the following result given by Gupta and Parihar [3, p.141, eq.(2.3)]:

**Corollary 3.5.** *Let* $\alpha,\beta,\eta,\delta,\xi,\rho\in\mathbb{C}$, $x>0$, $\nu>0$, $p,q>0$, $q\leq\Re(\nu)+p$ *and* $a\in\Re$ *be such that*

$$\Re(\alpha)>0,\quad \Re(\alpha+\rho)>\max\left[-\Re(\beta),-\Re(\eta)\right],\quad \Re(\beta)\neq\Re(\eta)$$

*and*

$$\Re(1-\nu n-\rho)<1+\min\left[\Re(\beta),\Re(\eta)\right],$$

*then, there holds the formula*

(3.3)

$$\left\{I_{-}^{\alpha,\beta,\eta}\left[t^{-\alpha-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x)=\frac{x^{-\alpha-\beta-\rho}\Gamma\left(\xi\right)}{\Gamma\left(\delta\right)}$$
$$\times\ {}_4\Psi_4\left[ax^{-\nu}\left|\begin{array}{c}(1,1),(\delta,q),(\rho+\alpha+\beta,\nu),(\rho+\alpha+\eta,\nu)\\(\xi,p),(\rho,\nu),(\rho+\alpha,\nu),(\rho+2\alpha+\beta+\eta,\nu)\end{array}\right.\right].$$

*Remark* 3.6. If we set $\beta=-\alpha$ and $\xi=p=q=1$ in (3.3), then, we obtain the known result given by Saxena and Saigo [30, p.147, eq.(23)]; further, if we also set $\delta=1$, then, we obtain the known result [30, p.148, eq.(24)]).

## 4. Generalized Fractional Differentiation of the generalized Mittag-Leffler function

In this section, we establish image formulas for the generalized Mittag-Leffler function involving Marichev-Saigo-Meada fractional derivative operators (2.3) and (2.5), in term of the generalized Wright function. These formulas are given by the following theorems:

**Theorem 4.1.** *Let* $\alpha,\alpha',\beta,\beta',\mu,\delta,\xi,\rho\in\mathbb{C}$*,* $x>0$*,* $\nu>0$*,* $p,q>0$*,* $q\leq\Re(\nu)+p$ *and* $a\in\Re$ *be such that* $\Re(\mu)>0$*,*

$$\Re\left(\nu n+\rho\right)+\min\left[0,\Re\left(\beta-\alpha\right),\Re\left(\mu-\alpha-\alpha'-\beta\right)\right]>0,$$

*then, there holds the formula*

(4.1)

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)=\frac{x^{\rho+\alpha+\alpha'-\mu-1}\Gamma\left(\xi\right)}{\Gamma\left(\delta\right)}$$
$$\times\ {}_4\Psi_4\left[ax^{\nu}\left|\begin{array}{c}(1,1),(\delta,q),(\rho+\alpha+\alpha'+\beta'-\mu,\nu),(\rho+\alpha-\beta,\nu)\\(\xi,p),(\rho+\alpha+\alpha'-\mu,\nu),(\rho+\alpha+\beta'-\mu,\nu),(\rho-\beta,\nu)\end{array}\right.\right].$$

*Proof.* By using (1.3) and (2.4) with the help of (2.9), we have

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)$$
$$=\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}}{\Gamma\left(\nu n+\rho\right)(\xi)_{pn}}\left(at^{\nu}\right)^{n}\right]\right\}(x),$$

by interchanging the order of differentiation and summation, we have

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ (a)^{n}}{\Gamma(\nu n+\rho)(\xi)_{pn}}\left(D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}t^{\nu n+\rho-1}\right)(x)$$

$$=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ (a)^{n}}{\Gamma(\nu n+\rho)(\xi)_{pn}}\left(I_{0+}^{-\alpha',-\alpha,-\beta',-\beta,-\mu}t^{\nu n+\rho-1}\right)(x)$$

$$=x^{\rho+\alpha+\alpha'-\mu-1}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ (ax^{\nu})^{n}}{(\xi)_{pn}}$$

$$\times\frac{\Gamma(\rho+\alpha+\alpha'+\beta'-\mu+\nu n)\ \Gamma(\rho+\alpha-\beta+\nu n)}{\Gamma(\rho+\alpha+\alpha'-\mu+\nu n)\ \Gamma(\rho+\alpha+\beta'-\mu+\nu n)\ \Gamma(\rho-\beta+\nu n)},$$

using (1.4), (1.5) and rearranging the terms, we have

$$\left\{D_{0+}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\rho-1}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)=\frac{x^{\rho+\alpha+\alpha'-\mu-1}\Gamma(\xi)}{\Gamma(\delta)}$$

$$\times\ {}_4\Psi_4\left[ax^{\nu}\left|\begin{array}{c}(1,1),(\delta,q),(\rho+\alpha+\alpha'+\beta'-\mu,\nu),(\rho+\alpha-\beta,\nu)\\(\xi,p),(\rho+\alpha+\alpha'-\mu,\nu),(\rho+\alpha+\beta'-\mu,\nu),(\rho-\beta,\nu)\end{array}\right.\right].$$

This completes the proof of theorem. □

If we take (2.7) into account, then (4.1) is reduced to the following result given by Gupta and Parihar [3, p.142, eq.(2.4)]:

**Corollary 4.2.** *Let* $\alpha,\beta,\eta,\delta,\xi,\rho\in\mathbb{C}$, $x>0$, $\nu>0$, $p,q>0$, $q\leq\Re(\nu)+p$ *and* $a\in\Re$ *be such that* $\Re(\alpha)>0$, $\Re(\rho+\beta+\eta)>0$,

$$\Re(\nu n+\rho)+\min[0,\Re(\eta-\beta)]>0,$$

*then, there holds the formula*

(4.2)

$$\left\{D_{0+}^{\alpha,\beta,\eta}\left[t^{\rho-1}E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{\nu}\right)\right]\right\}(x)$$

$$=\frac{x^{\rho+\beta-1}\Gamma(\xi)}{\Gamma(\delta)}\,{}_3\Psi_3\left[ax^{\nu}\left|\begin{array}{c}(1,1),(\delta,q),(\rho+\alpha+\beta+\eta,\nu)\\(\xi,p),(\rho+\beta,\nu),(\rho+\eta,\nu)\end{array}\right.\right].$$

*Remark* 4.3. If we put $\beta=-\alpha$ and $\xi=p=q=1$ in (4.2), then we obtain the known result given by Saxena and Saigo [30, p.149, eq.(29)]; further, if we also set $\delta=1$, then we get the known result [30, p.149, eq.(30)].

**Theorem 4.4.** *Let $\alpha, \alpha', \beta, \beta', \mu, \delta, \xi, \rho \in \mathbb{C}$, $x > 0$, $\nu > 0$, $p, q > 0$, $q \leq \Re(\nu) + p$ and $a \in \Re$ be such that $\Re(\mu) > 0$,*

$\Re(1 - \nu n - \rho) < \min\left[0, \Re\left(\mu - \alpha - \alpha' - n\right), \Re\left(-\alpha' - \beta + \mu\right), -\Re\left(\beta'\right)\right]$,
$(n = [\Re(\mu)] + 1)$,

*then, there holds the formula*

$$\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\mu-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x) = \frac{x^{\alpha+\alpha'-\rho}\Gamma(\xi)}{\Gamma(\delta)} \times {}_5\Psi_5\left[ax^{-\nu}\left|\begin{matrix}(1,1),(\delta,q),(\rho-\alpha-\alpha',\nu),(\rho-\alpha'-\beta,\nu),(\rho+\beta'-\mu,\nu)\\(\xi,p),(\rho,\nu),(\rho-\mu,\nu),(\rho-\alpha-\alpha'-\beta,\nu),(\rho-\alpha'+\beta'-\mu,\nu)\end{matrix}\right.\right]. \tag{4.3}$$

*Proof.* By using (1.3) and (2.6) with the help of (2.10), then we arrive at

$$\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\mu-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x) = \left\{D_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\mu-\rho}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\left(at^{-\nu}\right)^n}{\Gamma(\nu n+\rho)(\xi)_{pn}}\right]\right\}(x),$$

by interchanging the order of integration and summation, also using the relation (2.5), we have

$$\begin{aligned}&\left\{D_{-}^{\alpha,\alpha',\beta,\beta',\mu}\left[t^{\mu-\rho}\ E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x)\\&\quad=\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\ (a)^n}{\Gamma(\nu n+\rho)(\xi)_{pn}}\left(I_{-}^{-\alpha',-\alpha,-\beta',-\beta,-\mu}t^{(1-\nu n-\rho+\mu)-1}\right)(x)\\&\quad=x^{-\rho+\alpha+\alpha'}\sum_{n=0}^{\infty}\frac{(\delta)_{qn}\left(ax^{-\nu}\right)^n}{\Gamma(\rho+\nu n)(\xi)_{pn}}\\&\times\ \frac{\Gamma(\rho-\alpha-\alpha'+\nu n)\ \Gamma(\rho-\alpha'-\beta+\nu n)\ \Gamma(\rho+\beta'-\mu+\nu n)}{\Gamma(\rho-\mu+\nu n)\ \Gamma(\rho-\alpha-\alpha'-\beta+\nu n)\ \Gamma(\rho-\alpha'+\beta'-\mu+\nu n)},\end{aligned}$$

by using (1.4) and the definition of generalized Wright hypergeometric as given in (1.5), then we easily get the R.H.S. of (4.3). This completes the proof of the Theorem 4. □

If we take (2.8) into account, then, (4.3) is reduced to the following result given by Gupta and Parihar [3, p.143, eq. (2.5)]:

**Corollary 4.5.** *Let $\alpha, \beta, \eta, \delta, \xi, \rho \in \mathbb{C}$, $x > 0$, $\nu > 0$, $p, q > 0$, $q \leq \Re(\nu) + p$ and $a \in \Re$ be such that*

$\Re(\alpha) > 0, \quad \Re(\rho) > \max\left[\Re(\alpha+\beta) + n, -\Re(\eta)\right], \quad \Re(\alpha+\beta+\eta) + n \neq 0$

*(where $n = [\Re(\alpha)] + 1$), and*

$$\Re(1 - \nu n - \rho) + \max\left[\Re(\beta) + [\Re(\beta)] + 1, -\Re(\alpha+\eta)\right],$$

*then, there holds the formula*

(4.4)

$$\left\{D_{-}^{\alpha,\beta,\eta}\left[t^{\alpha-\rho}\,E_{\nu,\rho,p}^{\delta,\xi,q}\left(at^{-\nu}\right)\right]\right\}(x)=\frac{x^{\alpha+\beta-\rho}\Gamma(\xi)}{\Gamma(\delta)}$$

$$\times\ {}_4\Psi_4\left[ax^{-\nu}\left|\begin{array}{c}(1,1),(\delta,q),(\rho-\alpha-\beta,\nu),(\rho+\eta,\nu)\\(\xi,p),(\rho,\nu),(\rho-\alpha,\nu),(\rho-\alpha-\beta+\eta,\nu)\end{array}\right.\right].$$

*Remark* 4.6. If we put $\beta=-\alpha$ and $\xi=p=q=1$ in (4.4), then, we obtain the known result given by Saxena and Saigo [30, p.150, eq.(35)]; further, if we also set $\delta=1$, then we obtain the known result [30, p.151, eq.(36)]).

## 5. Concluding Remarks

In the present paper, we derive a new generalization of generalized Mittag-Leffler function and obtain the relations between the generalized M-L-function and Marichev-Saigo-Maeda (also known as Saigo-Maeda) fractional calculus operators. The obtained results are extension of the work done by many authors, for example Kumar and Kumar [11], Kumar and Purohit [12], Gupta and Parihar [3], Salim and Faraj [25], Saxena and Saigo [30], and many more. The provided results are new and have unique identity in the literature. On account of the general nature of the generalized Mittag-Leffler function and generalized Wright function, a number of known results can be easily found as special cases of our main results.

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