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ON n-DERIVATIONS

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Abstract. In this article, the notion of $n-$derivation is introduced for all integers $n \geq 2$. Although all derivations are $n-$derivations, in general these notions are not equivalent. Some properties of ordinary derivations are investigated for $n-$derivations. Also, we show that under certain mild condition $n-$derivations are derivations.

1. Introduction

Let A be an algebra. By a derivation on A, we mean a linear map $D: A \to A$ satisfying $D(ab) = D(a).b + a.D(b)$ for any $a, b \in A$. Derivations on algebras and Banach algebras are studied in several aspects. In [10], it is shown that a continuous derivation on a Banach algebra leaves the primitive ideals of the algebra invariant. Also, it is known that on a commutative Banach algebra the range of a continuous derivation is contained in radical [8, 11]. Here we introduce the notion of $n-$derivations for $n \geq 2$, that extends the notion of derivation. Also, we show that if A is unital, then, $n-$derivations are derivations.

In this paper there are two main themes. In Section 2, among other things, general theory of derivation are generalized to $n-$derivations such as Leibnitz rule and Singer-Wermer's theorem. In Section 3, some relations between $n-$derivations and derivations are studied analogous to the way that $n-$homomorphisms are related to homomorphisms as [6]. It must be mentioned that the facts obtained here are concerend with $n-$derivations defined on algebras as well as Banach algebras.

2. relationships between n-derivations and derivations

This section is devoted to the extension of some results on derivations for $n-$ derivations such as Leibnitz rule and Singer-Wermer's theorem.

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As derivations, in general we can define a n-derivation from an algebra into a module on this algebra. Indeed, let A be an algebra and let E be an A-bimodule. A linear map $D : A \to E$ is a derivation if $D(ab) = D(a).b + a.D(b)$ for any $a, b \in A$. For $n \in \mathbb{N}$, a linear map $D : A \to E$ is called an n-derivation if

$$D(a_1a_2a_3 \dots a_n) = D(a_1).a_2a_3 \dots a_n + a_1.D(a_2).a_3 \dots a_n + \dots + \\ + a_1a_2a_3 \dots D(a_n), \quad (a_1, a_2, a_3, \dots a_n \in A). \tag{2.1}$$

It is clear that 2-derivations are a derivation, in the usual sense. A simple calculation shows that a 2-derivation is n-derivation for all n. The converse does not hold in general (see example 2.2). Essentially this section is devoted to the study of same properties of n-derivations that are the same as properties of derivation.

Proposition 2.1. *Let A be a unital algebra and E be a unital A-bimodule. Then any n-derivation is a derivation.*

Proof. By assumption we have $D(e^n) = D(e) + \dots + D(e) = nD(e)$, and so $D(e) = 0$. Now for any a and b in A,

$$\begin{aligned} D(ab) &= D(abe \dots e) \\ &= D(a).be \dots e + a.D(b).e \dots e + 0 + \dots + 0 \\ &= D(a).b + a.D(b). \end{aligned}$$

□

The following example shows that 2.1 does not hold in general.

Example 2.2. Let A be the subalgebra of $\mathbb{M}_3(\mathbb{C})$ having zero on and below the diagonal. Since $A^3 = 0$, the identity map on A is a 3-derivation, while is not a derivation.

As derivations, n-derivations satisfy in the Leibnitz rule, that can be proved by induction.

Proposition 2.3 (Leibnitz rule)**.** *Let D be a derivation on an algebra A. Then for any $m \in \mathbb{N}$,*

$$D^m(a_1a_2a_3...a_n) =$$

$$= \sum_{k_1+k_2+\dots+k_n=m} \binom{m}{k_1, k_2, \dots, k_n} D^{k_1}(a_1).D^{k_2}(a_2) \dots D^{k_n}(a_n), \\ (a_1, a_2, a_3, \dots a_n \in A).$$

where $\binom{m}{k_1,k_2,\dots,k_n} = \frac{m!}{k_1!k_2!\dots k_n!}$ *and* $D^0 = I$ *is the identity operator.*

Proof. The conclusion follows by induction on m. For $m = 1$ the result holds by definition of an n-derivation. Assume that it holds for m. Then,

$$\begin{aligned}
D^{m+1} &(a_1a_2a_3\dots a_n)\\
&= D(D^m(a_1a_2a_3\dots a_n))\\
&= \sum_{k'_1+k'_2+\dots+k'_n=m} \binom{m}{k'_1,k'_2,\dots,k'_n} D(D^{k'_1}(a_1).D^{k'_2}(a_2)\dots D^{k'_n}(a_n))\\
&= \sum_{j=1}^{n} \sum_{k'_1+k'_2+\dots+k'_n=m} \binom{m}{k'_1,k'_2,\dots,k'_n} D^{k'_1}(a_1)\dots D^{k'_j+1}(a_2)\dots D^{k'_n}(a_n)\\
&= \sum_{k_1+k_2+\dots+k_n=m+1} \binom{m+1}{k_1,k_2,\dots,k_n} D^{k_1}(a_1)\dots D^{k_j}(a_2)\dots D^{k_n}(a_n),\\
&\qquad\qquad (a_1,a_2,a_3,\dots a_n\in A).
\end{aligned}$$

Thus the result holds for all m. □

As in the complex plane, in a Banach space X we can define a rearrangement of a series. Let $\sum_{k=1}^{\infty} a_{\phi(k)}$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$, where $\phi : \mathbb{N} \to \mathbb{N}$ is a bijection. A similar argument to complex numbers case, shows that if $\sum_{k=1}^{\infty} \|a_k\| < \infty$, then $\sum_{k=1}^{\infty} a_{\phi(k)}$ converges to $\sum_{k=1}^{\infty} a_k$ in X. There are some relations between derivations and homomorphisms. For example, if $D : A \to A$ is a bounded derivation on a Banach algebra A, then $\exp D$ is an invertible homomorphism on A, see [1]. According to [6] we recall that a linear operator $\phi : A \to A$ is an n-homomorphism if $\phi(a_1a_2a_3\dots a_n) = \phi(a_1).\phi(a_2)\dots\phi(a_n)$ for any $a_1,a_2,a_3,\dots a_n \in A$. By an $n-$character we mean an n-homomorphism from A to $\mathbb{C}$. The set of all characters on A is denoted by Φ_A.

Proposition 2.4. *Let D be a bounded n-derivation on a Banach algebra A. Then $\exp D$ is a bounded n-homomorphism on A, where*

$$\exp D(a) = \sum_{k=1}^{\infty} \frac{1}{k!} D^k(a), \quad (a \in A).$$

Proof. For $a_1, a_2, a_3, \dots a_n \in A$, by Leibnitz rule we have,

$$\begin{aligned}
&\exp D\,(a_1 a_2 a_3 \dots a_n) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} D^k (a_1 a_2 a_3 \dots a_n) \\
&= \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{k_1+\cdots+k_n=k} \binom{k}{k_1, k_2, \dots, k_n} D^{k_1}(a_1) \dots D^{k_n}(a_n) \\
&= \sum_{k=1}^{\infty} \sum_{k_1+\dots+k_n=k} \frac{1}{k_1!}\, D^{k_1}(a_1)\, \frac{1}{k_2!}\, D^{k_2}(a_2)\, \dots\, \frac{1}{k_n!}\, D^{k_n}(a_n) \\
&= \sum_{k_1=1}^{\infty} D^{k_1}(a_1) \sum_{k_2=1}^{\infty} D^{k_2}(a_2) \dots \sum_{k_n=1}^{\infty} D^{k_n}(a_n) \\
&= \exp D(a_1) \exp D(a_2)\ \dots \exp D(a_n).
\end{aligned}$$

Note that all series appearing above are absolutely convergent. □

According to Singer-Wermer's theorem, in commutative Banach algebras, the range of any continuous derivation is contained in its radical [11]. There is an analogous result for n-derivations under an additional assumption.

Theorem 2.5. *Let D be a bounded n-derivation on a commutative Banach algebra A. If all $n-$characters of A are norm decreasing, then $D(A) \subseteq rad(A)$. In particular, if A is semisimple, $D = 0$.*

Proof. Fix $\varphi \in \Phi_A$ and $\lambda \in \mathbb{C}$. Since $\lambda D : A \to A$ is a bounded n-derivation, $\exp(\lambda D)$ will be a bounded n-homomorphism on A. Let $\phi_\lambda(a) = \varphi(\exp(\lambda D)(a))$, for each $a \in A$, thus ϕ is a $n-$character on A. By hypothesis, we have $|\phi_\lambda(a)| \le \|a\|$ and

$$\begin{aligned}
\phi_\lambda(a) &= \varphi(\sum_{k=1}^{\infty} \frac{\lambda^k}{k!} D^k(a)) \\
&= \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \varphi(D^k(a)) \quad (a \in A).
\end{aligned}$$

Thus the mapping $\lambda \to \phi_\lambda(a)$ is a bounded entire function for a fix $a \in A$. Hence it is a constant function and so $\varphi(D^k(a)) = 0$ for all $k \ge 1$. In particular $\varphi(D(a)) = 0$. Since φ was an arbitrary character on A, $D(a)$ lies in $rad(A)$, as we desired. □

3. Common results with $n-$homomorphism

There is a natural connection between derivations and homomorphisms. This makes us enable to investigate $n-$derivations and obtain some results that are studied for $n-$homomorphisms in [6]. It is shown that a certain multiple of a $n-$homomorphism is a homomorphism. We have a similar result for an n-derivation.

Theorem 3.1. *Let A be a unital algebra with identity e, and let E be an A-bimodule (not necessarily unital). If $D : A \to E$ is an n-derivation, then the map $\Delta : A \to E$, defined by $\Delta(a) = D(a).e \quad (a \in A)$ is a derivation.*

Proof. For any $a, b \in A$ we have

$$\begin{aligned} a.D(e).b &= a.D(\underbrace{e\dots e}_{n-times}).b \\ &= \underbrace{a.D(e).b + \cdots + a.D(e).b}_{n-times}, \end{aligned}$$

which implies $a.D(e).b = 0$, and so

$$\begin{aligned} \Delta(ab) &= D(ab \quad \underbrace{e\dots e}_{n-2-times} \quad).e \\ &= D(a).b + a.D(b).e + 0 + \cdots + 0 \\ &= \Delta a.b + a.\Delta b. \end{aligned}$$

□

We now return to $n-$derivations on Banach algebras. An important problem concerning derivations is extension of a derivation to a larger algebra. Perhaps the most important one appears in the notable theorem of Johnson on amenability of group algebra, where a derivation on $L^1(G)$ can be extended to $M(G)$. From now on, we suppose that A is a Banach algebra and E is a Banach $A-$bimodule. For the next result, we recall the first Arens product. The first Arens product on A^{**} is obtained by making in turn the definitions:

$$\begin{aligned} \langle b, f.a\rangle &= \langle ab, f\rangle\,, \\ \langle a, G.f\rangle &= \langle f.a, G\rangle\,, \\ \langle f, F\Box G\rangle &= \langle G.f, F\rangle\,, \quad (a, b \in A, f \in A^*, F, G \in A^{**}). \end{aligned} \tag{3.1}$$

By natural embedding of A into A^{**}, A is a subalgebra of A^{**}. The identities in 3.1 show, that $F\Box G$ is continuous in F in $\sigma(A^{**}, A^*)$, the w^*-topology of A^{**}, for fixed G. But in general it is not continuous in

G for fixed F, unless $F \in A \subseteq A^{**}$. Since A is w^*-dense in A^{**} (see Theorem V.4.5 in [4]), we get

$$F \square G = w^* - \lim_{\alpha} w^* - \lim_{\beta} a_\alpha b_\beta, \tag{3.2}$$

where (a_α) and (b_β) are nets in A such that $a_\alpha \to F$ and $b_\beta \to G$ in w^*-topology of A^{**}. The second Arens product, denoted by $\diamondsuit$, is defined by:

$$(A^{**}, \diamondsuit) = ((A^{op})^{**}, \square)^{op},$$

where A^{op} is the algebra formed by reversing the order of the product in A. As first Arens product, the second Arens product makes A^{**} a Banach algebra containing A as a closed subalgebra. In general, the two products $\square$ and $\diamondsuit$ are distinct on A^{**}. A Banach algebra A is Arens regular if these two products coincide on A^{**}.

Applying n-times 3.2 shows that if $F_1, F_2, \ldots, F_n \in A^{**}$ and $a_{\alpha_i} \to F_i$ for $i = 1, 2, \ldots, n$, then

$$F_1 \square F_2 \square \ldots \square F_n = w^* - \lim_{\alpha_1} w^* - \lim_{\alpha_2} \ldots w^* - \lim_{\alpha_n} \quad a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_n}. \tag{3.3}$$

The above process can be used to the module action of A on E. According [3], E^{**} is a Banach $A^{**}-$bimodule, where $A^{**}$is equipped with the first Arens product, by the action defined as follows.

$$\begin{aligned}
\langle a, \Phi.\phi\rangle &= \langle \phi.a, \Phi\rangle\,, \\
\langle a, \phi.x\rangle &= \langle xa, \phi\rangle\,, \\
\langle x, F.\phi\rangle &= \langle \phi.x, F\rangle\,, \\
\langle \phi, F.\Phi\rangle &= \langle \Phi.\phi, F\rangle\,, \\
\langle \phi, \Phi.F\rangle &= \langle F.\phi, \Phi\rangle\,, \quad (x \in E, \phi \in E^*, \Phi \in E^{**}, a \in A, F \in A^{**}).
\end{aligned}$$

As Arens product, the module action concerning second duals can be determined by w^*-limits. Let (a_α) be a net in A, and let (x_β) be a net in E such that $a_\alpha \to F$ and $x_\beta \to \Phi$ in A^{**} and E^{**}, respectively, in the w^*-topology. Then one has

$$F.\Phi = w^* - \lim_{\alpha} w^* - \lim_{\beta} \quad a_\alpha x_\beta,$$

$$\Phi.F = w^* - \lim_{\alpha} w^* - \lim_{\beta} \quad x_\beta a_\alpha.$$

In [3] the extension of a continuous derivation $D : A \to E$ to a continuous derivation from A^{**} into E^{**} is discussed. This has a main role in extension of a continuous derivation on $L^1(\omega)$ to a continuous derivation on $M(\omega)$ in [5], where the inclusion $M(\omega) \subset L^1(\omega)^{**}$ makes sense as inclusion of Banach algebras. Here we have the same result for a continuous n-derivation. We need a lemma that is a module version of Proposition 3.1 in [12]. We recall that a bounded operator T from a Banach space E_1 into a Banach space E_2 is weakly compact if $T(B)$ is relatively weakly compact in E_2, where B is the closed unit ball of E_1. For a bounded operator $T : E_1 \to E_2$, T is weakly compact if and only if, for every bounded net (x_α) in E_1, $(T(x_\alpha))$ contains a weakly convergent subnet in E_2.

Lemma 3.2. *Let A be a Banach algebra, E be a left Banach $A-$module, and let the map $L_x : A \to E$ defined by $L_x(a) = a.x$ $(a \in A)$, be weakly compact for all $x \in E$. Then $A.E^{**} \subseteq E$.*

Proof. Let $a \in A$ and $\Phi \in E^{**}$. Then there exists a bounded net (x_α) in E such that $\| x_\alpha \| \leq \| \Phi \|$ for all α and $x_\alpha \to \Phi$ in E^{**}, in the w^*-topology. The w^*-continuity of the map $\Psi \to a.\Psi$ on E^{**} yields $a.x_\alpha \to a.\Phi$, in the w^*-topology of E^{**}. Since the net (x_α) in E is norm bounded, weak compactness of the map $x \to a.x$ on E implies that there exists a subnet (x_β) of (x_α) and an element $x \in E$ such that $a.x_\beta \to x$ in the weak topology of E. Since the w^*-topology of E^{**} restricted to E is the weak topology, we have $a.x_\beta \to x$ in E^{**}, in the w^*-topology of E^{**}. Thus $\Phi = x$ belong to E.

□

A similar result holds for a right Banach $A-$module E. So we have:

Corollary 3.3. *Let A be a Banach algebra, E be a Banach $A-$bimodule, and let the maps $L_x : A \to E$ and $R_x : A \to E$ defined by $L_x(a) = a.x$ and $R_x(a) = x.a$ $(a \in A)$, be weakly compact for all $x \in E$. Then $A.E^{**} \subseteq E$ and $E^{**}.A \subseteq E$.*

Theorem 3.4. *Let A be a Banach algebra, E be a Banach $A-$bimodule, and $D : A \to E$ be a continuous n-derivation. Then the second dual $D^{**} : A^{**} \to E^{**}$ of D is also an n-derivation.*
If, in addition, A is Arens regular with bounded approximate identity and the map $L_x : A \to E$, is weakly compact for all $x \in E$, then there are a continuous derivation $\delta : A \to E$ and an element $\Phi \in E^{**}$ such that

$$D(a) = \delta(a) + a.\Phi \quad (a \in A).$$

Proof. Let $F_1, F_2, \ldots, F_n \in A^{**}$. Then there exist nets (a_{α_i}) in A for $i = 1, 2, \ldots, n$ such that $a_{\alpha_i} \to F_i$. Now 3.3 and w^*-continuity of D^{**}

imply that

$$\begin{aligned} D^{**}(F_1 \Box F_2 \Box \ldots \Box F_n) \\ &= w^* - \lim_{\alpha_1} \ldots w^* - \lim_{\alpha_n} D(a_{\alpha_1} a_{\alpha_2} \ldots a_{\alpha_n}) \\ &= w^* - \lim_{\alpha_1} \ldots w^* - \lim_{\alpha_n} \sum_{j=1}^{n} a_{\alpha_1} \ldots a_{\alpha_{j-1}} D(a_{\alpha_j}) a_{\alpha_{j+1}} \ldots a_{\alpha_n} \\ &= \sum_{j=1}^{n} F_1 \Box \ldots \Box F_{j-1}.D^{**}(F_j).F_{j+1} \Box \ldots \Box F_n. \end{aligned}$$

Now let A be Arens regular with bounded approximate identity. Proposition 28.7 in [1] implies that A^{**} has identity, e say. Then Theorem 3.1 yields a continuous derivation $\Delta : A^{**} \to E^{**}$, where $\Delta(F) = D^{**}(F).e$. Since the restriction of $D^{**} : A^{**} \to E^{**}$ on A is $D : A \to E$, the map δ defined through $\delta(a) = D(a).e \quad (a \in A)$ is a derivation. On the other hand, taking $\Phi = D^{**}(e)$, $D^{**}(F) = \Delta(F) + F.D^{**}(e)$ yield

$$D(a) = \delta(a) + a.\Phi \quad (a \in A).$$

By Lemma 3.2, weak compactness of the maps L_x for all $x \in E$ implies that $a.\Phi \in E$, so $\delta(a) = a.\Phi - D(a) \in E$. □

Let E be a Banach A-bimodule. An operator $D : A \to E$ is called a local (respectively, approximately local) derivation if for each $a \in A$ there is a derivation $D_a : A \to E$ (respectively, a sequence of derivations $(D_{a,n})$ such that $D(a) = D_a(a)$ (respectively, $D(a) = \lim_{n\to\infty} D_{a,n}(a)$). If, in addition, D is bounded, then we say that D is a bounded local derivation (respectively, bounded approximately local derivation).

In [9], it is shown that when A is a C^*-algebra, a Banach algebra generated by idempotents, a semisimple annihilator Banach algebra, or the group algebra of a SIN or a totally disconnected group, bounded approximately local derivations from A into X are derivations. This result extends a result of B. E. Johnson about local derivations on C^*-algebras [7]. After investigation of approximately local derivations by E. Samei, it is proved that under certain conditions, a bounded approximately local derivation is a sum of a derivation and a multiplier [9]. We have a similar result for $n-$derivations.

Corollary 3.5. *Let A be an Arens regular Banach algebra with a bounded approximate identity, E be a Banach $A-$bimodule such that the maps $L_x : A \to E$, are weakly compact for all $x \in E$, and $D : A \to E$ be a continuous n-derivation. Then there is a continuous derivation δ and a* *continuous multiplier T form A into E such that*

$$D = \delta + T.$$

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