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MULTISTEP COLLOCATION METHOD FOR NONLINEAR DELAY INTEGRAL EQUATIONS

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Abstract. The main purpose of this paper is to study the numerical solution of nonlinear Volterra integral equations with constant delays, based on the multistep collocation method. These methods for approximating the solution in each subinterval are obtained by fixed number of previous steps and fixed number of collocation points in current and next subintervals. Also, we analyze the convergence of the multistep collocation method when used to approximate smooth solutions of delay integral equations. Finally, numerical results are given showing a marked improvement in comparison with exact solution.

1. Introduction

The problems under consideration are nonlinear Volterra integral equations with constant delays $\tau > 0$,

$$y(t) = \begin{cases} g(t) + (Vy)(t) + (V_\tau y)(t) & t \in I = [0, T], \\ \phi(t) & t \in [-\tau, 0). \end{cases} \tag{1.1}$$

Here the classical and delay Volterra integral operator are defined by

$$(Vy)(t) = \int_0^t k_1(t, s, y(s))ds, \tag{1.2}$$

and

$$(V_\tau y)(t) = \int_0^{t-\tau} k_2(t, s, y(s))ds, \tag{1.3}$$

respectively. The aim of this paper is to introduce a multistep collocation method to approximate the solution of equation (1.1). It will be assumed

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that the given functions, $\phi : [-\tau, 0] \to \mathbb{R}$, $g : I \to \mathbb{R}$, $k_1 : D\times\mathbb{R} \to \mathbb{R}$, $D = \{(t, s) : 0 \le s \le t \le T\}$ and $k_2 : D_\tau \times \mathbb{R} \to \mathbb{R}$, $\quad D_\tau = I \times [-\tau, T - \tau]$ are at least continuous on their domains. Existence and uniqueness result for (1.1) can be found in [3]. We note that for $t_0 = 0$ there are no primary discontinuity points, in this case smooth data lead to smooth solutions on I. (See [3]).

There are many literature to study delay functional equations frequently encountered in physical and biological processes, see, for example, [9]. The analysis to the Volterra integral equations with proportional delays dates back to the works in [11] and [12]. Some more recent results on this subject can be found in [3]-[8]. During the past decade, numerical methods for (1.1) has attracted wide attention of many researchers. Various numerical methods for (1.1) have been introduced such as collocation method [2], iterated collocation method [5] and spectral method [7, 10]. Numerical methods for functional integral and integro-differential equations of Volterra type have been summarized in [3].

The layout of this paper is as follows. In Section 2, we briefly introduce the new families of multistep collocation method which has presented in [1]. In Section 3, the multistep collocation method develop for delay integral equations and in Section 4, we analyze the convergence of the multistep collocation method. Finally in Section 5, some numerical experiments are reported to clarify the method and some comparisons are made with exact solution.

2. Preliminaries

For convenience of the reader, we will present a review of the multistep collocation method from [1].

The multistep collocation methods are obtained by introducing in the collocation polynomial the dependence from r previous time steps; namely we seek for a collocation polynomial, whose restriction to the interval $[t_n, t_{n+1}]$ takes the form

(2.1)
$$u_n(t_n+sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j}, \quad s \in [0, 1], \quad n = r, \ldots, N-1,$$

where

$$U_{n,j} = u_n(t_{n,j}), \tag{2.2}$$

and $\varphi_k(s), \psi_j(s)$ are polynomials of degree $m + r - 1$ to be determined by imposing the interpolation conditions at the points t_{n-k}, that is

$u_n(t_{n-k}) = y_{n-k}$, and by satisfying (2.2). The method is then constructed by imposing the collocation conditions, which will be described in the following. The starting values $y_1, y_2, \dots, y_r$, needed in (2.1), may be obtained by using a suitable starting procedure, based on a classical one step method.

The interpolation conditions at $t_{n-k},\ k = 0, 1, \dots, r-1$, together with the condition (2.2), lead to the following linear system:

$$\begin{aligned} &\varphi_l(-k) = \delta_{lk}, \qquad \varphi_l(c_j) = 0, \qquad l, k = 0, 1, \dots, r-1, \\ &\psi_i(-k) = 0, \qquad \psi_i(c_j) = \delta_{ij}, \qquad i, j = 1, 2, \dots, m. \end{aligned} \tag{2.3}$$

The (2.3) represents a linear system of $(r+m)^2$ equations where the $(r+m)^2$ unknowns are the coefficients of the polynomials $\varphi_k(s)$ and $\psi_j(s)$, considering the $c_j, j = 1, 2, \dots, m$, as fixed parameters.

Assuming that $c_i \neq c_j$ and $c_1 \neq 0$, then the unique solution of the system (2.3) assumes the form

$$\begin{aligned} \varphi_k(s) &= \prod_{i=1}^{m} \frac{s - c_i}{-k - c_i} \cdot \prod_{\substack{i=0 \\ i \neq k}}^{r-1} \frac{s + i}{-k + i}, \\ \psi_j(s) &= \prod_{i=0}^{r-1} \frac{s + i}{c_j + i} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{m} \frac{s - c_i}{c_j - c_i}. \end{aligned} \tag{2.4}$$

The expression (2.4) follows immediately by observing that $\{\varphi_k(s),\ \psi_j(s)\}$ defined by means of (2.3) represent the fundamental Lagrange polynomials with respect to nodes $\{-k, c_j :\ k = 0, 1, \dots, r-1,\ j = 1, 2, \dots, m\}$. More details can be found in [1].

3. Main results

Let $t_n = nh, (n = 0, \dots, N, t_N = T)$ define a uniform partition for $I = [0, T]$, and let $\Omega_N := \{0 = t_0 < t_1 < \cdots < t_N = T\}, \sigma_0 := [t_0, t_1], \sigma_n := (t_n, t_{n+1}], (1 \leq n \leq N-1)$. The mesh Ω_N is constrained in the following sense:

$$h = \frac{\tau}{\tilde{r}} \quad \text{for some } \tilde{r} \in \mathbb{N}.$$

With a given mesh Ω_N, we associate the set of its interior points, $Z_N := \{t_n : n = 1, \dots, N-1\}$. For a fixed $N \geq 1$ and, for given integers $d \geq -1$ and $m \geq 1$, the piecewise polynomial space $S^{(d)}_{m+d}(Z_N)$ is defined by

$$S^{(d)}_{m+d}(Z_N) := \left\{ u : C^d(I) \to \mathbb{R}; u|_{\sigma_n} = u_n \in \Pi_{m+d}, \quad 0 \leq n \leq N-1 \right\},$$

where Π_{m+d} denotes the set of (real) polynomials of a degree not exceeding $m+d$. The dimension of this space is given by $\dim S^{(d)}_{m+d}(Z_N) =$

$Nm+d+1$. For integral equation, we have $d=-1$, hence, the collocation space will be

$$S_{m-1}^{(-1)}(Z_N) := \{u : u|_{\sigma_n} = u_n \in \Pi_{m-1}, \quad 0 \le n \le N-1\}.$$

Let $u_n = u|_{\sigma_n}, u \in S_{m-1}^{(-1)}(Z_N)$, for all $t \in \sigma_n$ we have

$$u_n(t_n+sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) U_{n,j}, \quad s \in [0,1], \quad r \le n \le N-1,$$

where $U_{n,j} = u_n(t_{n,j})$.

Now, we consider the set of collocation parameters $\{c_j\}_{j=1}^m$, where $0 \le c_1 < \cdots < c_m \le 1$, and define the set $X_N = \{t_{n,j} = t_n + c_j h\}$ of collocation points. The collocation solution $u \in S_{m-1}^{(-1)}(Z_N)$ will be determined by imposing the condition that u satisfies the integral equation (1.1) on the finite set X_N

$$u(t) = \begin{cases} g(t) + (Vu)(t) + (V_\tau u)(t), & t \in [0,T], \\ \\ \phi(t), & t \in [-\tau, 0). \end{cases} \tag{3.1}$$

In the equations (1.2) and (1.3) by substituting $t = t_{n,j}$, after some computations, for $t_{n,j} - \tau < 0$, we obtain

$$\begin{aligned} (V_\tau u_n)(t_{n,j}) = -h \Bigg[& \int_{c_j}^{1} k_2(t_{n,j}, t_{n-\tilde{r}} + sh, \phi(t_{n-\tilde{r}} + sh)) ds \\ & + \sum_{i=n-\tilde{r}}^{-1} \int_0^1 k_2(t_{n,j}, t_i + sh, \phi(t_i + sh)) ds \Bigg], \end{aligned}$$

and for $t_{n,j} - \tau \ge 0$, we obtain

$$\begin{aligned} (V_\tau u_n)(t_{n,j}) = h \Bigg[& \sum_{i=0}^{n-\tilde{r}-1} \int_0^1 k_2(t_{n,j}, t_i + sh, u_i(t_i + sh)) ds \\ & + \int_0^{c_j} k_2(t_{n,j}, t_{n-\tilde{r}} + sh, u_{n-\tilde{r}}(t_{n-\tilde{r}} + sh)) ds \Bigg]. \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} (Vu_n)(t_{n,j}) =& h \sum_{i=0}^{n-1} \int_0^1 k_1(t_{n,j}, t_i + sh, u_i(t_i + sh)) ds \\ & + h \int_0^{c_j} k_1(t_{n,j}, t_n + sh, u_n(t_n + sh)) ds. \end{aligned} \tag{3.3}$$

By substituting the equations (3.2) and (3.3) in the equation (3.1), we have

$$u_n(t_{n,j}) = D(t_{n,j}),$$

where

(3.4)

$$D(t_{n,j}) = g(t_{n,j}) + \begin{cases} (Vu_n)(t_{n,j}) + \Phi(t_{n,j}), & t_{n,j} - \tau < 0, \\ (Vu_n)(t_{n,j}) + (V_\tau u_n)(t_{n,j}), & t_{n,j} - \tau \geq 0, \end{cases}$$

and

$$\Phi(t_{n,j}) = \int_0^{t_{n,j}-\tau} k_2(t_{n,j}, s, \phi(s))ds, \quad j = 1, 2, \ldots, m, \quad n = 0, \ldots, \tilde{r}-1.$$

The exact multistep collocation method is then obtained by imposing the collocation conditions, i.e., that the collocation polynomials (2.1) exactly satisfies the equation (3.1) at the collocation points $t_{n,q}$, and by computing $y_{n+1} = u_n(t_{n+1})$:

$$\begin{cases} U_{n,q} = D_{n,q}, & q = 1, 2, \ldots, m, \\ y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) U_{n,j}, & n = r, r+1, \ldots, N-1, \end{cases}$$

where $D_{n,q} = D(t_{n,q})$. Generally, the integrals in $D(t_{n,q})$ cannot be evaluated analytically, but have to be approximate by suitable quadrature formulae.

Now, let μ_0 and μ_1 be given positive integers. Suppose that the quadrature parameters $\{d_l\}$ and $\{d_{j,l}\}$ satisfy $0 \leq d_1 < \cdots < d_{\mu_1} \leq 1$ and $0 \leq d_{j,1} < \cdots < d_{j,\mu_0} \leq c_j, (j = 1, \ldots, m)$, respectively. The quadrature weights are then given by

$$w_l := \int_0^1 \prod_{\substack{b=1 \\ b\neq l}}^{\mu_1} \frac{s - d_b}{d_l - d_b} ds, \qquad l = 1, \ldots, \mu_1,$$

$$w_{j,l} := \int_0^{c_j} \prod_{\substack{b=1 \\ b\neq l}}^{\mu_0} \frac{s - d_{j,b}}{d_{j,l} - d_{j,b}} ds, \quad l = 1, \ldots, \mu_0, \ j = 1, \ldots, m.$$

Then the $D(t_{n,q})$ approximate as follows

$$\bar{D}(t_{n,q}) = g(t_{n,q}) + (\bar{V}u_n)(t_{n,q}) + (\bar{V}_\tau u_n)(t_{n,q}), \tag{3.5}$$

where $P_n(t)$ is given by (3.8), and

$$(\bar{V}u_n)(t_{n,q}) = h \sum_{i=0}^{n-1} \sum_{l=1}^{\mu_1} w_l k_1(t_{n,q}, t_i + d_l h, P_i(t_i + d_l h)) \tag{3.6}$$

$$+ h \sum_{l=1}^{\mu_0} w_{q,l} k_1 \left(t_{n,q}, t_n + d_{q,l}h, P_n(t_n + d_{q,l}h)\right),$$

and for $t_{n,q} - \tau < 0$, we have

$$(\bar{V}_\tau u_n)(t_{n,q}) = -h\Big(\sum_{i=n-\tilde{r}}^{-1} \sum_{l=1}^{\mu_1} w_l k_2(t_{n,q}, t_i + d_l h, \phi(t_i + d_l h)) + \sum_{l=1}^{\mu_1} \bar{w}_{q,l} k_2(t_{n,q}, t_{n-\tilde{r}} + \xi_{q,l}h, \phi(t_{n-\tilde{r}} + \xi_{q,l}h))\Big),$$

and for $t_{n,q} - \tau \geq 0$, we have

(3.7)
$$(\bar{V}_\tau u_n)(t_{n,q}) = h\Big(\sum_{i=0}^{n-\tilde{r}-1} \sum_{l=1}^{\mu_1} w_l k_2(t_{n,q}, t_i + d_l h, P_i(t_i + d_l h)) + \sum_{l=1}^{\mu_0} w_{q,l} k_2(t_{n,q}, t_{n-\tilde{r}} + d_{q,l}h, P_{n-\tilde{r}}(t_{n-\tilde{r}} + d_{q,l}h))\Big),$$

and $\xi_{q,l} := c_q + (1-c_q)d_l, \bar{w}_{q,l} := (1-c_q)w_l, \quad q = 1, \ldots, m,\ l = 1, \ldots, \mu_1$.

The discretized multistep collocation polynomial, denoted by $P_n(t)$, is of the form

(3.8)
$$P_n(t_n + sh) = \sum_{k=0}^{r-1} \varphi_k(s) y_{n-k} + \sum_{j=1}^{m} \psi_j(s) Y_{n,j}, \quad s \in [0,1], \quad n = r, \ldots, N-1,$$

where the functions $\varphi_k(s)$ and $\psi_j(s)$ are given by (2.4) and $Y_{n,j} := P_n(t_{n,j})$ are determined by the solution of the following nonlinear system

(3.9)
$$\begin{cases} Y_{n,q} = \bar{D}_{n,q}, & q = 1, 2, \ldots, m, \\ y_{n+1} = \sum_{k=0}^{r-1} \varphi_k(1) y_{n-k} + \sum_{j=1}^{m} \psi_j(1) Y_{n,j}, & n = r, r+1, \ldots, N-1. \end{cases}$$

Note that, in the same manner in [1] we have:

Remark 3.1. The discretized multistep collocation method (3.8) and (3.9) provides a continuous approximation $P(t)$ of the solution $y(t)$ of the equation (1.1) in $[0, T]$, by considering

$$P(t)|_{(t_n, t_{n+1}]} = P_n(t),$$

where $P_n(t)$ is given by (3.8).

Theorem 3.2. *The discretized multistep collocation method (3.8) and (3.9) can be regarded as a multistep Runge-Kutta method for Volterra functional integral equations with non-vanishing delays:*

$$\begin{cases} Y_{n,q} = g(t_{n,q}) + (\bar{V}y)(t_{n,q}) + (\bar{V}_\tau y)(t_{n,q}), \\ y_{n+1} = \sum_{k=0}^{r-1} \theta_k y_{n-k} + \sum_{j=1}^{m} \lambda_j Y_{n,j}, \end{cases}$$

where

$$(\bar{V}y)(t_{n,q}) = h\left[\sum_{i=0}^{n-1}\sum_{l=1}^{\mu_1} w_l k_1\left(t_{n,q}, t_i + d_l h, \sum_{k=0}^{r-1}\nu_{k,l} y_{i-k} + \sum_{j=1}^{m}\gamma_{j,l} Y_{i,j}\right)\right.$$
$$\left. + \sum_{l=1}^{\mu_0} w_{q,l} k_1\left(t_{n,q}, t_n + d_{q,l} h, \sum_{k=0}^{r-1}\beta_{k,q,l} y_{n-k} + \sum_{j=1}^{m}\alpha_{j,q,l} Y_{n,j}\right)\right],$$

$$(\bar{V}_\tau y)(t_{n,q}) = -h\left[\sum_{i=n-\tilde{r}}^{-1}\sum_{l=1}^{\mu_1} w_l k_2\left(t_{n,q}, t_i + d_l h, \sum_{k=0}^{r-1}\nu_{k,l}\phi_{i-k} + \sum_{j=1}^{m}\gamma_{j,l}\phi_{i,j}\right)\right.$$
$$+ \sum_{l=1}^{\mu_1} \bar{w}_{q,l} k_2\left(t_{n,q}, t_{n-\tilde{r}} + \xi_{q,l} h, \sum_{k=0}^{r-1}\hat{\beta}_{k,q,l}\phi_{n-\tilde{r}-k}\right.$$
$$\left.\left. + \sum_{j=1}^{m}\alpha_{j,q,l}\phi_{n-\tilde{r},j}\right)\right], \qquad t_{n,q} - \tau < 0,$$

$$(\bar{V}_\tau y)(t_{n,q}) = h\left[\sum_{i=0}^{n-\tilde{r}-1}\sum_{l=1}^{\mu_1} w_l k_2\left(t_{n,q}, t_i + d_l h, \sum_{k=0}^{r-1}\nu_{k,l} y_{i-k} + \sum_{j=1}^{m}\gamma_{j,l} Y_{i,j}\right)\right.$$
$$+ \sum_{l=1}^{\mu_0} w_{q,l} k_2\left(t_{n,q}, t_{n-\tilde{r}} + d_{q,l} h, \sum_{k=0}^{r-1}\beta_{k,q,l} y_{n-\tilde{r}-k}\right.$$
$$\left.\left. + \sum_{j=1}^{m}\alpha_{j,q,l} Y_{n-\tilde{r},j}\right)\right], \qquad t_{n,q} - \tau \geq 0,$$

and

$$\phi(t_{n,q}) = \phi_{n,q}, \quad \phi(t_i) = \phi_i, \quad \varphi_k(d_l) = \nu_{k,l}, \quad \psi_j(d_l) = \gamma_{j,l}, \qquad \varphi_k(d_{q,l}) = \beta_{k,q,l},$$

$$\psi_j(d_{q,l}) = \alpha_{j,q,l}, \quad \theta_k = \varphi_k(1), \quad \lambda_j = \psi_j(1), \quad \hat{\beta}_{k,q,l} = \varphi_k(\xi_{q,l}).$$

Proof. By substituting the equation (3.8) in the equations (3.6), (3.7) and (3.9), and after some computations, the result is obtained. □

4. Convergence

Let $u_n \in S_{m-1}^{(-1)}(Z_N)$ denote the (exact) collocation solution to (1.1) defined by (3.8). In our convergence analysis, we examine the linear test equation

(4.1)
$$y(t)=\begin{cases} g(t)+\int_0^t k_1(t,s)y(s)ds+\int_0^{t-\tau} k_2(t,s)y(s)ds, & t\in I,\\ \phi(t), & t\in[-\tau,0),\end{cases}$$

where $k_1 \in C(D)$ and $k_2 \in C(D_\tau)$.

Theorem 4.1. *Let the given function in (4.1) satisfy* $g \in C^{m+r}(I), k_1 \in C^{m+r}(D), k_2 \in C^{m+r}(D_\tau), \phi \in C^{m+r}([-\tau,0])$, *and that for* $t\in[0,\tau]$ *the integral*

$$\Phi(t) := \int_0^{t-\tau} k_2(t,s)\phi(s)ds, \tag{4.2}$$

is known exactly. Also, suppose that the starting error is

$$\| y-u_n \|_{\infty,[0,t_r]} = O(h^{m+r}),$$

and

$$\rho(\mathbf{A}) < 1,$$

where ρ *denotes the spectral radius and*

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{0}_{(r-1)\times 1} & \mathbf{I}_{r-1} \\ \hline \varphi_{r-1}(1) & \varphi_{r-2}(1),\dots,\varphi_0(1)\end{array}\right]. \tag{4.3}$$

Then for all sufficiently small $h=\frac{\tau}{\tilde{r}}, (\tilde{r}\in\mathbb{N})$ *the constrained mesh collocation solution* $u_n \in S_{m-1}^{(-1)}(Z_N)$ *to (4.1), satisfies*

$$\| \mathcal{E} \|_\infty \le Ch^{m+r}, \tag{4.4}$$

where $\mathcal{E}(t)=y(t)-u(t)$ *be the error of the exact collocation method (3.5) and* C *is positive constant not depending on* h. *This estimate holds for all collocation parameters* $\{c_j\}$ *with* $0\le c_1<\dots<c_m\le 1$.

Proof. We will prove the estimate (4.4) by using the Peano Theorem to write (see [1],[3])

$$\begin{aligned} y(t_n+sh) &= \sum_{k=0}^{r-1}\varphi_k(s)y(t_{n-k}) \\ &+ \sum_{j=1}^{m}\psi_j(s)y(t_{n,j}) + h^{m+r}R_{m,r,n}(s), \quad s\in[0,1]. \end{aligned} \tag{4.5}$$

Here, the functions $\varphi_k(s)$ and $\psi_j(s)$ are given by (2.4) and

$$R_{m,r,n}(s) := \int_{1-r}^{1} K_{m,r}(s,z)y^{(m+r)}(t_n+zh)dz,$$

and

$$K_{m,r}(s,z) = \frac{1}{(m+r-1)!}\left\{(s-z)_+^{m+r-1} - \sum_{k=0}^{r-1}\varphi_k(s)(-k-z)_+^{m+r-1} - \sum_{j=1}^{m}\psi_j(s)(c_j-z)_+^{m+r-1}\right\}, \qquad z \in [0,1].$$

Thus, it follows from (4.5) that the collocation error $\mathcal{E} := y - u_n$ possesses to the local representation

(4.6)
$$\mathcal{E}(t_n+sh) = \sum_{k=0}^{r-1}\varphi_k(s)\mathcal{E}_{n-k} + \sum_{j=1}^{m}\psi_j(s)\mathcal{E}_{n,j} + h^{m+r}R_{m,r,n}(s), \quad n \geq r,$$

with $\mathcal{E}_{n-k} = \mathcal{E}(t_{n-k}), \mathcal{E}_{n,j} = \mathcal{E}(t_{n,j})$, and it satisfies

$$\mathcal{E}(t_{n,j}) = \int_0^{t_{n,j}} k_1(t_{n,j},s)\mathcal{E}(s)ds + \int_0^{t_{n-\tilde{r},j}} k_2(t_{n,j},s)\mathcal{E}(s)ds. \tag{4.7}$$

If $t < \tau$, then $u(t) = \phi(t)$ on $[-\tau, 0)$ and we have

$$\mathcal{E}(t_{n,j}) = \int_0^{t_{n,j}} k_1(t_{n,j},s)\mathcal{E}(s)ds. \tag{4.8}$$

Now, from (4.7) and (4.8) we have

(4.9)
$$\mathcal{E}_{n,j} = \begin{cases} h\left(\sum_{i=0}^{n-1}\int_0^1 k_1(t_{n,j},t_i+sh)\mathcal{E}(t_i+sh)ds + \int_0^{c_j} k_1(t_{n,j},t_n+sh)\mathcal{E}(t_n+sh)ds\right), & t_{n,j}-\tau < 0, \\ h\left(\sum_{i=0}^{n-1}\int_0^1 k_1(t_{n,j},t_i+sh)\mathcal{E}(t_i+sh)ds + \int_0^{c_j} k_1(t_{n,j},t_n+sh)\mathcal{E}(t_n+sh)ds + \sum_{i=0}^{n-\tilde{r}-1}\int_0^1 k_2(t_{n,j},t_i+sh)\mathcal{E}(t_i+sh)ds + \int_0^{c_j} k_2(t_{n,j},t_{n-\tilde{r}}+sh)\mathcal{E}(t_{n-\tilde{r}}+sh)ds\right), & t_{n,j}-\tau \geq 0. \end{cases}$$

By the hypothesis on the starting error it follows that

$$\mathcal{E}(t_i+sh) = h^{m+r}q_i(s), \quad i = 0,1,\ldots,r-1, \quad s \in [0,1], \tag{4.10}$$

with $\|q_i\|_\infty \leq C_1$ independent of h.

Define the matrices $\mathbf{B}_n^{(i)}, \mathbf{E}_n^{(i)} \in \mathbb{R}^{m\times r}$ and $\mathbf{D}_n^{(i)}, \mathbf{D}_n, \mathbf{F}_n^{(i)} \in \mathbb{R}^{m\times m}$

(4.11)
$$(\mathbf{B}_n^{(i)})_{jk} = \begin{cases} \displaystyle\int_0^1 k_1(t_{n,j}, t_i + sh)\varphi_k(s)ds, & i = r, \ldots, n-1, \\ \displaystyle\int_0^{c_j} k_1(t_{n,j}, t_n + sh)\varphi_k(s)ds, & i = n, \end{cases}$$

$$(\mathbf{D}_n^{(i)})_{jl} = \int_0^1 k_1(t_{n,j}, t_i + sh)\psi_l(s)ds, \quad i = r, \ldots, n-1, \tag{4.12}$$

$$(\mathbf{D}_n)_{jl} = \int_0^{c_j} k_1(t_{n,j}, t_n + sh)\psi_l(s)ds, \quad i = n, \tag{4.13}$$

(4.14)
$$(\mathbf{E}_n^{(i)})_{jk} = \begin{cases} \displaystyle\int_0^1 k_2(t_{n,j}, t_i + sh)\varphi_k(s)ds, & i = r, \ldots, n-\tilde{r}-1, \\ \displaystyle\int_0^{c_j} k_2(t_{n,j}, t_{n-\tilde{r}} + sh)\varphi_k(s)ds, & i = n-\tilde{r}, \end{cases}$$

(4.15)
$$(\mathbf{F}_n^{(i)})_{jl} = \begin{cases} \displaystyle\int_0^1 k_2(t_{n,j}, t_i + sh)\psi_l(s)ds, & i = r, \ldots, n-\tilde{r}-1, \\ \displaystyle\int_0^{c_j} k_2(t_{n,j}, t_{n-\tilde{r}} + sh)\psi_l(s)ds, & i = n-\tilde{r}, \end{cases}$$

and the vectors in $\mathbb{R}^m$ by

(4.16)
$$(\boldsymbol{\rho}_{1n}^{(i)})_j = \begin{cases} \displaystyle\int_0^1 k_1(t_{n,j}, t_i + sh)q_i(s)ds, & i = 0, \ldots, r-1, \\ \displaystyle\int_0^1 k_1(t_{n,j}, t_i + sh)R_{m,r,i}(s)ds, & i = r, \ldots, n-r, \\ \displaystyle\int_0^{c_j} k_1(t_{n,j}, t_n + sh)R_{m,r,n}(s)ds, & i = n, \end{cases}$$

(4.17)

$$(\boldsymbol{\rho}_{2n}^{(i)})_j = \begin{cases} \int_0^1 k_2(t_{n,j}, t_i + sh) q_i(s) ds, & i = 0, \dots, r-1, \\ \int_0^1 k_2(t_{n,j}, t_i + sh) R_{m,r,i}(s) ds, & i = r, \dots, n - \tilde{r} - 1, \\ \int_0^{c_j} k_2(t_{n,j}, t_{n-\tilde{r}} + sh) R_{m,r,n-\tilde{r}}(s) ds, & i = n - \tilde{r}, \end{cases}$$

(4.18)
$$\mathcal{E}_i^{(1)} = [\mathcal{E}_{i-r+1}, \dots, \mathcal{E}_i]^T,$$

(4.19)
$$\mathcal{E}_i^{(2)} = [\mathcal{E}_{i,1}, \dots, \mathcal{E}_{i,m}]^T.$$

Now, by using the relations (4.11)-(4.19) and by substituting the (4.9) in the (4.6) and after some computations, we obtain

(4.20)

$$(\mathbf{I}_m - h\mathbf{D}_n)\mathcal{E}_n^{(2)} = \begin{cases} h\left(\sum_{i=r}^{n} \mathbf{B}_n^{(i)} \mathcal{E}_i^{(1)} + \sum_{i=r}^{n-1} \mathbf{D}_n^{(i)} \mathcal{E}_i^{(2)}\right) \\ \quad + h^{m+r+1} \sum_{i=0}^{n} \boldsymbol{\rho}_{1n}^{(i)}, & t_{n,j} - \tau < 0, \\ h\left(\sum_{i=r}^{n-1} \mathbf{D}_n^{(i)} \mathcal{E}_i^{(2)} + \sum_{i=r}^{n} \mathbf{B}_n^{(i)} \mathcal{E}_i^{(1)} \right. \\ \quad \left. + \sum_{i=r}^{n-\tilde{r}} \mathbf{E}_n^{(i)} \mathcal{E}_i^{(1)} + \sum_{i=r}^{n-\tilde{r}} \mathbf{F}_n^{(i)} \mathcal{E}_i^{(2)}\right) \\ \quad + h^{m+r+1} \left(\sum_{i=0}^{n} \boldsymbol{\rho}_{1n}^{(i)} + \sum_{i=0}^{n-\tilde{r}} \boldsymbol{\rho}_{2n}^{(i)}\right), & t_{n,j} - \tau \geq 0. \end{cases}$$

Now, consider the difference equation

$$\mathcal{E}_i^{(1)} = \mathbf{A}\mathcal{E}_{i-1}^{(1)} + \mathbf{S}\mathcal{E}_{i-1}^{(2)} + h^{m+r} \tilde{\boldsymbol{\rho}}_{m,r,i-1}, \quad i \geq r,$$

with solution

(4.21)
$$\mathcal{E}_i^{(1)} = \mathbf{A}^{i-r+1} \mathcal{E}_{r-1}^{(1)} + \sum_{j=r-1}^{i-1} \mathbf{A}^{i-j-1} (\mathbf{S}\mathcal{E}_j^{(2)} + h^{m+r} \tilde{\boldsymbol{\rho}}_{m,r,j}),$$

where $\mathbf{A}$ is given by (4.3) and

$$\mathbf{S} = \begin{bmatrix} \mathbf{0}_{(r-1)\times m} \\ \boldsymbol{\psi}^T(1) \end{bmatrix}, \qquad \boldsymbol{\psi}(1) = [\psi_0(1), \dots, \psi_m(1)]^T,$$
$$\tilde{\boldsymbol{\rho}}_{m,r,j} = \begin{bmatrix} \mathbf{0}_{(r-1)\times m} \\ R_{m,r,j}(1) \end{bmatrix}.$$

Also, from (4.10) we have

$$\| \mathcal{E}_{r-1}^{(1)} \|_1 \le rC_1 h^{m+r}, \qquad \| \mathcal{E}_{r-1}^{(2)} \|_1 \le mC_1 h^{m+r}.$$

(For further detail see [1]).

Now, by substituting the (4.21) in the (4.20), we obtain

$$\begin{aligned}(\mathbf{I}_m - h\mathbf{D}_n)\mathcal{E}_n^{(2)} = h \Bigg(& \sum_{i=r}^{n-1} \mathbf{D}_n^{(i)} \mathcal{E}_i^{(2)} + \sum_{i=r}^{n} \mathbf{B}_n^{(i)} \mathbf{A}^{i-r+1} \mathcal{E}_{r-1}^{(1)} \\ & + \sum_{i=r}^{n} \sum_{l=r-1}^{i-1} \mathbf{B}_n^{(i)} \mathbf{A}^{i-l-1} \mathbf{S} \mathcal{E}_l^{(2)} \Bigg) \\ & + h^{m+r+1} \Bigg(\sum_{i=r}^{n} \sum_{l=r-1}^{i-1} \mathbf{B}_n^{(i)} \mathbf{A}^{i-l-1} \tilde{\boldsymbol{\rho}}_{m,r,l} \\ & + \sum_{i=0}^{n} \boldsymbol{\rho}_{1n}^{(i)} \Bigg), \qquad t_{n,j} - \tau < 0,\end{aligned}$$

$$\begin{aligned}(\mathbf{I}_m - h\mathbf{D}_n)\mathcal{E}_n^{(2)} = h \Bigg(& \sum_{i=r}^{n-1} \mathbf{D}_n^{(i)} \mathcal{E}_i^{(2)} + \sum_{i=r}^{n-\tilde{r}} \mathbf{F}_n^{(i)} \mathcal{E}_i^{(2)} \\ & + \sum_{i=r}^{n} \mathbf{B}_n^{(i)} \mathbf{A}^{i-r+1} \mathcal{E}_{r-1}^{(1)} \\ & + \sum_{i=r}^{n} \sum_{l=r-1}^{i-1} \mathbf{B}_n^{(i)} \mathbf{A}^{i-l-1} \mathbf{S} \mathcal{E}_l^{(2)} \\ & + \sum_{i=r}^{n-\tilde{r}} \mathbf{E}_n^{(i)} \mathbf{A}^{i-r+1} \mathcal{E}_{r-1}^{(1)} \\ & + \sum_{i=r}^{n-\tilde{r}} \sum_{l=r-1}^{i-1} \mathbf{E}_n^{(i)} \mathbf{A}^{i-l-1} \mathbf{S} \mathcal{E}_l^{(2)} \Bigg) \\ & + h^{m+r+1} \Bigg(\sum_{i=r}^{n} \sum_{l=r-1}^{i-1} \mathbf{B}_n^{(i)} \mathbf{A}^{i-l-1} \tilde{\boldsymbol{\rho}}_{m,r,l}\end{aligned} \tag{4.22}$$

$$+\sum_{i=r}^{n-\tilde{r}}\sum_{l=r-1}^{i-1}\mathbf{E}_n^{(i)}\mathbf{A}^{i-l-1}\tilde{\boldsymbol{\rho}}_{m,r,l}$$
$$\left.+\sum_{i=0}^{n}\boldsymbol{\rho}_{1n}^{(i)}+\sum_{i=0}^{n-\tilde{r}}\boldsymbol{\rho}_{2n}^{(i)}\right), \qquad t_{n,j}-\tau\geq 0.$$

Since the kernels k_i are continuous on their domains, the elements of the matrixes $\mathbf{D}_n$, are all bounded. By using the Neumann Lemma the inverse of the matrix $\mathcal{B}_n = I_m - h\mathbf{D}_n$ exists whenever $h \parallel \mathbf{D}_n \parallel < 1$ for some matrix norm. This clearly holds whenever h is sufficiently small. In other words, there is an $\bar{h} > 0$ so that for any mesh Π_N with $h < \bar{h}$, each matrix $\mathcal{B}_n$ has a uniformly bounded inverse. Note that, from this assumptions, there exists a constant $P_0 < \infty$ so that for all mesh diameters $h \in (0, \bar{h})$, the uniform bound

$$\parallel (\mathbf{I}_m - h\mathbf{D}_n)^{-1} \parallel_1 \leq P_0,$$

holds.

Here, for $B \in L(\mathbb{R}^m), \|B\|_1$ denotes the matrix (operator) norm induced by the l^1-norm in $\mathbb{R}^m$. Assume that

$$\begin{array}{lll}\parallel \mathbf{D}_n^{(i)} \parallel_1\leq P_1, & \parallel \mathbf{F}_n^{(i)} \parallel_1\leq P_2, & \parallel \mathbf{B}_n^{(i)} \parallel_1\leq P_3,\\ \parallel \mathbf{E}_n^{(i)} \parallel_1\leq P_4, & \parallel \mathbf{S} \parallel_1\leq P_5, & \parallel y^{(m+r)} \parallel_1\leq M_{m,r},\end{array}$$

$$\parallel \tilde{\boldsymbol{\rho}}_{m,r,j} \parallel_1\leq \beta_{m,r} = K_{m,r}M_{m,r},$$
$$K_{m,r} = \max_{s\in[0,1]}\int_{1-r}^{1}|K_{m,r}(s,\nu)|d\nu,$$
$$\bar{K}_1 = \max_{t\in I}\int_0^t |k_1(t,s)|ds,$$
$$\bar{K}_2 = \max_{t\in I}\int_0^t |k_2(t,s)|ds \geq \max_{t\in I}\int_0^{t-\tau}|k_2(t,s)|ds,$$

$$\parallel \boldsymbol{\rho}_{1,n}^{(i)} \parallel_1\leq \begin{cases}\gamma_1 = m\bar{K}_1C_1, & i=0,1,\ldots,r-1,\\ \alpha_{m,r}^{(1)} = m\bar{K}_1K_{m,r}M_{m,r}, & i=r,\ldots,n,\end{cases}$$

$$\parallel \boldsymbol{\rho}_{2,n}^{(i)} \parallel_1\leq \begin{cases}\gamma_2 = m\bar{K}_2C_1, & i=0,1,\ldots,r-1,\\ \alpha_{m,r}^{(2)} = m\bar{K}_2K_{m,r}M_{m,r}, & i=r,\ldots,n-\tilde{r}.\end{cases}$$

Moreover, since $\rho(\mathbf{A}) < 1$, there exists a constant P_6 such that

$$\sum_{i=0}^{k} \| \mathbf{A}^i \|_1 \leq P_6,$$

independently of $k \in \mathbb{N}$. Then, from (4.22)

$$\| \mathcal{E}_n^{(2)} \| \leq \begin{cases} hJ_4 \sum_{i=r}^{n-1} \| \mathcal{E}_i^{(2)} \| + h^{m+r} J_5, & t_{n,j} - \tau < 0, \\ hJ_0 \sum_{i=r}^{n-1} \| \mathcal{E}_i^{(2)} \| + h^{m+r} J_1, & t_{n,j} - \tau \geq 0, \end{cases} \tag{4.23}$$

where

$$\begin{aligned} J_0 &= P_0(P_1 + P_2 + (P_3 + P_4)P_5 P_6), \\ J_1 &= hP_0\{rC_1(P_3 + P_4)P_6 + mC_1(P_3 + P_4)P_5 P_6 \\ &\qquad + \beta_{m,r}(P_3 + P_4)P_6 + \gamma_1 + \gamma_2 + \alpha_{m,r}^{(1)} + \alpha_{m,r}^{(2)}\}, \\ J_4 &= P_0(P_1 + P_3 P_5 P_6), \\ J_5 &= hP_0\{(C_1(r + mP_5) + \beta_{m,r})P_3 P_6 + \gamma_1 + \alpha_{m,r}^{(1)}\}. \end{aligned}$$

Now, by using the discrete Gronwall inequality, we have

$$\| \mathcal{E}_n^{(2)} \| \leq \begin{cases} h^{m+r} J_6, & t_{n,j} - \tau < 0, \\ h^{m+r} J_2, & t_{n,j} - \tau \geq 0, \end{cases}$$

with

$$J_2 \leq J_1 \exp(TJ_0) < \infty, \qquad J_6 \leq J_5 \exp(TJ_4) < \infty.$$

Also, from (4.21), we have

$$\| \mathcal{E}_n^{(1)} \| \leq J_3 h^{m+r},$$

where

$$J_3 = rC_1 P_6 + J_2 P_5 P_6 + \beta_{m,r} P_6.$$

Now, by using the local error representation (4.6) this yields

$$| \mathcal{E}(t_n + sh) | \leq C_2 h^{m+r},$$

where $C_2 = W_{m,r}(J_2 + J_3) + K_{m,r} M_{m,r}$, and

$$W_{m,r} := \max_{k,j} \{ \| \varphi_k \|_\infty, \| \psi_j \|_\infty, \quad k = 0, \ldots, r-1, j = 1, \ldots, m \},$$

uniformly for $s \in [0,1]$ and $n \geq r$. This is equivalent to the estimate $\| \mathcal{E} \|_\infty \leq Ch^{m+r}$, with $C = \max\{C_1, C_2\}$. □

Theorem 4.2. *Let the assumptions of Theorem 4.1 hold, except that the integrals*

$$\Phi(t) = \int_0^{t-\tau} k_2(t,s)\phi(s)ds, \qquad t = t_{n,j}, n = 0, 1, \ldots, \tilde{r}-1,$$

are now approximated by quadrature formulas $\bar{\Phi}(t)$*, with corresponding quadrature errors* $E_0(t) := \Phi(t) - \bar{\Phi}(t)$*, such that*

$$\|E_0(t)\| \leq h^q \tag{4.24}$$

for some $q > 0$*. Then the collocation solution* $u_n \in S_{m-1}^{(-1)}$ *satisfies, for all sufficiently small* $h > 0$*,*

$$\| \mathcal{E} \|_\infty \leq Ch^p,$$

with $p := \min\{m + r, q\}$*, where* C *are finite constants not depending on* h*.*

Proof. From (3.4), we have to calculate the integrals $\Phi(t)$ only if $t_n < t_{\tilde{r}}$. Thus, if $t_n \geq t_{\tilde{r}}$, the estimate (4.4) holds. So, we assume that $0 \leq n < \tilde{r}$. Then we have

$$\mathcal{E}(t_{n,j}) = \int_0^{t_{n,j}} k_1(t_{n,j}, s)\mathcal{E}(s)ds - E_0(t_{n,j}). \tag{4.25}$$

Substituting (4.6) in the (4.25) and by using the (4.23) and (4.24) we have

$$\| \mathcal{E}_n^{(2)} \| \leq hJ_4 \sum_{i=r}^{n-1} \| \mathcal{E}_i^{(2)} \| + h^p J_7,$$

where $J_7 = h^{m+r-p}J_5 + h^{q-p}$ and $p = \min\{m + r, q\}$. Hence, by using the Theorem 4.1 and the discrete Gronwall inequality, the statement of Theorem 4.2 follows. □

We conclude this section with a comment regarding the extension of the results of Theorem 4.1 to the nonlinear equation (1.1). Under the assumption of the existence of a (unique) solution $y(t)$ on I, the nonlinear analogue of the error equation (4.7) is

$$\begin{aligned} \mathcal{E}(t_{n,j}) = & \int_0^{t_{n,j}} \left[k_1(t_{n,j}, s, y(s)) - k_1(t_{n,j}, s, u(s))\right] ds \\ & + \int_0^{t_{n,j}-\tau} [k_2(t_{n,j}, s, y(s)) - k_2(t_{n,j}, s, u(s))]ds. \end{aligned} \tag{4.26}$$

If the partial derivatives $\frac{\partial k_i}{\partial y}$, $i = 1, 2$ are continuous and bounded on $D \times S$ and $D_\tau \times S_\tau$ with $S := \{y \in \mathbb{R} : | y - y(s) | < M, \ s \in I\}$ and $S_\tau := \{y \in \mathbb{R} : | y - y(s) | < M, \ s \in [-\tau, T - \tau]\}$, for some $M < \infty$, and if $h > 0$ is sufficiently small (assuring the existence of a unique collocation solution u_n), then (4.26) may again be written in the form (4.7). The roles of k_i are now assumed by

$$H_i(t,s) := \frac{\partial k_i(t,s,z_i(s))}{\partial y}, \quad i = 1,2$$

where $z_i(s) := \theta_i y(s) + (1-\theta_i)u_n(s),\ 0 \leq \theta_i(s) \leq 1$. Hence, the above proof is easily adapted to deal with the nonlinear case (1.1), and so the convergence results of Theorem 4.1 remain valid for nonlinear delay integral equations.

5. Numerical results

In this section, illustrative examples are given to show efficiency of proposed method.

Typical forms of collocation parameters c_j are:

Gauss points: Zeros of $P_k(2t-1)$;

RadouI points: Zeros of $P_k(2t-1) - P_{k-1}(2t-1)$;

Chelyshkov points: Zeros of $P_{k0}(t) = t^k P_k^{(2k,0)}(1-2t)$;

where $P_k(t)$ and $P_k^{(\alpha,\beta)}(t)$ are Legendre and Jacobi polynomials, respectively and $\mathrm{e}(t) = y(t) - u(t)$ be the error of the exact multistep collocation method.

For computational purposes, in the test problems we need quadrature rules to obtain numerical solutions. For this propose, we have to apply the rules that preserve the order of the main method. In each cases of Examples the obtained nonlinear equations was solved by the Newton's method. All the computations were carried out with Maple.

Example 5.1. Consider the linear Volterra integral equation with non vanishing delays

(5.1)

$$y(t) = \exp(-t)(2+\exp(\tau)) - 2 + \int_0^t y(s)ds + \int_0^{t-\tau} y(s)ds, \quad t \in [0,1].$$

Its exact solution is has the following analytical form

$$y(t) = \begin{cases} \exp(-t), & t \in [-\tau, 0), \\ \exp(-t), & t \geq 0, \end{cases} \tag{5.2}$$

and delay parameter τ.

Table 1. The results of Example 5.1 for $h = \frac{\tau}{\bar{r}} = 0.3 \times 10^{-3}$.

N	Gauss	Radau II	Chelyshkov
	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$
4	7.0×10^{-10}	9.0×10^{-10}	1.0×10^{-10}
8	1.6×10^{-9}	1.2×10^{-9}	1.2×10^{-9}
12	5.6×10^{-8}	1.3×10^{-9}	1.0×10^{-9}

Table 2. The results of Example 5.1 for $h = \frac{\tau}{\bar{r}} = 0.3 \times 10^{-7}$.

N	Gauss	Radau II	Chelyshkov
	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$
4	8.0×10^{-10}	5.0×10^{-10}	5.0×10^{-10}
8	1.0×10^{-9}	1.7×10^{-9}	8.0×10^{-10}
12	1.0×10^{-9}	1.7×10^{-9}	7.0×10^{-10}

Example 5.2. The nonlinear Volterra integral equation with non vanishing delays
(5.3)

$$y(t) = \exp(t) - \frac{1}{2}\exp(2t)(1-\exp(-2\tau)) - \tau + \int_{t-\tau}^{t} (1+y^2(s))ds, \quad t \in [0,1]$$

has the following analytical solution

$$y(t) = \begin{cases} \exp(t), & t \in [-\tau, 0), \\ \exp(t), & t \geq 0. \end{cases} \quad (5.4)$$

Table 3. The results of Example 5.2 for $h = \frac{\tau}{\bar{r}} = 0.3 \times 10^{-3}$.

N	Gauss	Radau II	Chelyshkov
	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$
4	2.1×10^{-7}	2.0×10^{-7}	2.0×10^{-7}
8	2.1×10^{-7}	2.1×10^{-7}	2.1×10^{-7}
12	2.1×10^{-7}	2.1×10^{-7}	2.1×10^{-7}

Table 4. The results of Example 5.2 for $h = \frac{\tau}{\bar{r}} = 0.3 \times 10^{-7}$.

N	Gauss	Radau II	Chelyshkov
	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$
4	0	0	0
8	0	0	0
12	0	0	0

Example 5.3 ([2])**.** The nonlinear Volterra integral equation with non vanishing delays

$$y(t) = \frac{1}{4}\left(\sin(2(t-\tau)) - \sin(2\tau)\right) + \cos t - \frac{\tau}{2} + \int_{t-\tau}^{t} y^2(s)ds, \quad t \in [0,1],$$

with exact solution

$$y(t) = \begin{cases} \cos t, & t \geq 0, \\ \cos t, & t \in [-\tau, 0). \end{cases}$$

Table 5. The results of Example 5.3 for $h = \frac{\tau}{\tilde{r}} = 0.3 \times 10^{-3}$.

N	Gauss	Radau II	Chelyshkov
	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$
4	1.3×10^{-4}	1.5×10^{-4}	1.3×10^{-4}
8	7.3×10^{-4}	7.5×10^{-4}	1.5×10^{-4}
12	9.1×10^{-4}	1.3×10^{-3}	1.3×10^{-3}

The results for collocation points c_j are presented in Tables 1-6 which indicate that the numerical solutions obtained from (3.2) and step sizes equal to $h = 0.3 \times 10^{-3}$ and $h = 0.3 \times 10^{-7}$ are nearly identical. These results indicate that the use of time steps smaller than about $h = \frac{\tau}{\tilde{r}} = 10^{-3}$, for som $\tilde{r} \in \mathbb{N}$, the error function approaches to zero.

Table 6 The results of Example 5.3 for $h = \frac{\tau}{\tilde{r}} = 0.3 \times 10^{-7}$.

N	Gauss	Radau II	Chelyshkov
	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$
4	1.3×10^{-8}	1.5×10^{-8}	1.4×10^{-8}
8	7.4×10^{-8}	7.5×10^{-8}	7.4×10^{-8}
12	1.3×10^{-7}	1.3×10^{-7}	1.3×10^{-7}

Example 5.4. The nonlinear Volterra integral equation with non vanishing delays

$$y(t) = 2 - \cos\tau - \int_{t-\tau}^{t} \sin(ty(s) - s)ds, \quad t \geq 0,$$

with exact solution

$$y(t) = \begin{cases} 1, & t \geq 0, \\ 1, & t \in [-\tau, 0), \end{cases}$$

it corresponds to setting $k_2 = -k_1(= k)$, in equation (1.1).

Table 7. The results of Example 5.4 for collocation parameters $c_1 = 0.212, c_2 = 0.590, c_3 = 0.911, c_4 = 1$.

N	$h = 0.5 \times 10^{-2}$	$h = 0.5 \times 10^{-4}$	$h = 0.5 \times 10^{-9}$	$h = 0.5 \times 10^{-13}$
	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$	$\parallel \mathcal{E} \parallel_\infty$
4	0.2×10^{-2}	0.2×10^{-4}	0.2×10^{-9}	0
8	0.2×10^{-3}	0.2×10^{-4}	0.2×10^{-9}	0
12	0.2×10^{-3}	0.2×10^{-4}	0.2×10^{-9}	0

6. Conclusion

We have shown that the multistep collocation method yields an efficient and very accurate numerical method for the approximation of solutions to nonlinear Volterra integral equations with non-vanishing delays. Numerical results show that this method is effective for nonlinear delay Volterra integral equations.

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