Sahand Communications in Mathematical Analysis (SCMA) Vol. 4 No. 1 (2016), 15-27
http://scma.maragheh.ac.ir

A SPECTRAL METHOD BASED ON THE SECOND KIND CHEBYSHEV POLYNOMIALS FOR SOLVING A CLASS OF FRACTIONAL OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we consider the second-kind Chebyshev polynomials (SKCPs) for the numerical solution of the fractional optimal control problems (FOCPs). Firstly, an introduction of the fractional calculus and properties of the shifted SKCPs are given and then operational matrix of fractional integration is introduced. Next, these properties are used together with the Legendre-Gauss quadrature formula to reduce the fractional optimal control problem to solving a system of nonlinear algebraic equations that greatly simplifies the problem. Finally, some examples are included to confirm the efficiency and accuracy of the proposed method.

1. Introduction

Fractional order dynamics have been received considerable recent attention and have been proved to model many real life problems such as, mechanical systems [11], solid mechanics [22], continuum and statistical mechanics [21], fluid-dynamics [12], finance [15], viscoelastic dampers [17], viscoelasticity [4, 5], bioengineering [20], electromagnetic waves [13], control theory [7], etc.

FOCPs are one of the fractional dynamic systems that can be appeared in several problems in science and engineering. FOCP refers

2010 *Mathematics Subject Classification.* 49J15, 26A33.

Key words and phrases. Fractional optimal control problems, Caputo fractional derivative, Riemann-Liouville fractional integral, Second-kind Chebyshev polynomials, Operational matrix.

Received: 31 January 2016, Accepted: 14 May 2016.

to the minimization of an objective functional subject to dynamic constraints, on state and control variables, which have fractional order models. Although there are some different definitions of fractional derivatives, two types of them have been used very often for FOCP which are Riemann–Liouville and Caputo fractional derivatives. There are some numerical methods to solve FOCPs with these two types of definitions, for example see [1, 2, 3, 6, 14, 16, 18, 19, 26, 27, 28]. In the present paper, we consider the following optimal control problem with the Caputo fractional derivative [19]:

$$\min J = \int_0^1 f(t, x(t), u(t))dt, \tag{1.1}$$

subject to:

$$D_t^{\alpha} x(t) = g\left(t, x(t)\right) + b(t)u(t), \qquad n-1 < \alpha \leq n, \quad b(t) \neq 0, \tag{1.2}$$

$$D^{(i)}x(0) = x_i, \quad i = 0, 1, \ldots, n-1, \tag{1.3}$$

where f and g are smooth functions of their arguments. The existence and uniqueness of the solution for the dynamical system (1.2) have been discussed in [10]. In this paper, we use the shifted second-kind Chebyshev orthogonal basis for solving problem (1.1)–(1.3). To do this, we use operational matrix of fractional integration and Legendre-Gauss quadrature formula. The main advantage of this method is that the problem is reduced to a system of algebraic equations. It can be seen that the operational matrix of fractional integration for Chebyshev basis needs much fewer computational efforts compared with that for Legendre basis introduced in [19] (see Section 2) and it makes our method more computationally attractive.

The structure of this paper is arranged in the following way: In Section 2, an introduction of fractional calculus and properties of the shifted SKCPs are given and also, the operational matrix of fractional integration is introduced. In Section 3, a numerical method is considered to solve problem (1.1)–(1.3). In Section 4, illustrative examples are included to demonstrate the applicability and efficiency of the method. Finally, a brief conclusion is given in Section 5.

2. Preliminaries and notations

In this section, we give some preliminaries which will be used further in this work.

Definition 2.1. The fractional derivative of $x(t)$ in the Caputo sense is defined as follows

$$D_t^\alpha x(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n}{d\tau^n} x(\tau) d\tau, & n-1 < \alpha < n, \\ x^{(n)}(t) & \alpha = n, \end{cases}$$

where n is the ceiling function of α and

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Definition 2.2. The Riemann-Liouville fractional integral operator I_t^α of order α is given by

$$I_t^\alpha x(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} x(\tau) d\tau, & \alpha > 0, \\ x(t), & \alpha = 0. \end{cases}$$

Some properties of the Riemann-Liouville fractional integral operator I_t^α and the Caputo fractional differential operator D_t^α are as follows:

$$I_t^\alpha t^k = \frac{\Gamma(k+1)}{\Gamma(k+1+\alpha)} t^{k+\alpha}, \quad \alpha \geq 0, \quad k > -1, \tag{2.1}$$

$$D_t^\alpha I_t^\alpha x(t) = x(t),$$

$$I_t^\alpha D_t^\alpha x(t) = x(t) - \sum_{i=0}^{n-1} x^{(i)}(0) \frac{t^i}{i!}, \qquad n-1 < \alpha \leq n, \quad t > 0. \tag{2.2}$$

Definition 2.3. The shifted SKCP of order i is defined on $[0,1]$ as

$$\psi_i(t) = U_i(2t-1), \qquad i = 0, 1, 2, \ldots,$$

where $U_i(t)$ is the well-known SKCP of order i. We note that the SKCPs are orthogonal functions on the interval $[-1,1]$ and can be determined with the aid of the following recursive formula:

$$U_i(t) = 2tU_{i-1}(t) - U_{i-2}(t), \qquad i \geq 2,$$

with $U_0(t) = 0$ and $U_1(t) = 2t$.

The orthogonal property for the shifted SKCPs is as follows:

$$\int_0^1 w(t)\psi_i(t)\psi_j(t) dt = \begin{cases} \frac{\pi}{4}, & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

where $w(t) = \sqrt{4t - 4t^2}$.

The shifted SKCP of order i could be written as:

$$\psi_i(t) = \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k+1)!2^{2k}}{(i-k)!(2k+1)!} t^k. \tag{2.3}$$

A function $x(t)$, square integrable on $[0,1]$, can be expanded using the shifted SKCPs as follows:

$$x(t) \simeq \sum_{i=0}^{\infty} c_i \psi_i(t), \tag{2.4}$$

where

$$c_i = \frac{4}{\pi} \int_0^1 w(t) x(t) \psi_i(t) dt, \qquad i = 0, 1, 2, \ldots. \tag{2.5}$$

If we consider the first $N+1$ terms in (2.4), an approximation of the function $x(t)$ is obtained as:

$$x(t) \simeq \sum_{i=0}^{N} c_i \psi_i(t) = C^T \psi(t),$$

in which

$$C = [c_0, c_1, \ldots, c_N]^T,$$

and

$$\psi(t) = [\psi_0(t), \psi_1(t), \ldots, \psi_N(t)]^T. \tag{2.6}$$

Considering the vector $\psi(t)$ in equation (2.6), we get the following result:

Theorem 2.4. *If $\psi(t)$ is the shifted SKCPs vector defined by (2.6), then the fractional Integral of order α of this vector is given by*

$$I_t^{\alpha} \psi(t) \simeq P^{(\alpha)} \psi(t), \tag{2.7}$$

where $P^{(\alpha)}$ is the $(N+1) \times (N+1)$ operational matrix of fractional Integration as

$$P^{(\alpha)} = \begin{bmatrix} \theta_\alpha(0,0) & \theta_\alpha(0,1) & \ldots & \theta_\alpha(0,N) \\ \theta_\alpha(1,0) & \theta_\alpha(1,1) & \ddots & \theta_\alpha(1,N) \\ \vdots & \vdots & \ddots & \vdots \\ \theta_\alpha(N,0) & \theta_\alpha(N,1) & \ldots & \theta_\alpha(N,N) \end{bmatrix},$$

in which n is the ceiling function of α and

$$\theta_\alpha(i,j) = \sum_{k=0}^{i} \frac{(-1)^{i-k}(j+1)2^{2(k+1)} k!(i+k+1)!\Gamma\left(k+\alpha+\frac{3}{2}\right)}{\sqrt{\pi}(2k+1)!(i-k)!\Gamma(k+\alpha+j+3)\Gamma(k+\alpha-j+1)}. \tag{2.8}$$

Proof. Using equations (2.1) and (2.3), for $i = 0, 1, \ldots, N$ we have

$$\begin{aligned} I_t^\alpha \psi_i(t) &= \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k+1)!2^{2k}}{(i-k)!(2k+1)!} I_t^\alpha t^k \\ &= \sum_{k=0}^{i} (-1)^{i-k} \frac{(i+k+1)!2^{2k}k!}{(i-k)!(2k+1)!\Gamma(k+\alpha+1)} t^{k+\alpha}. \end{aligned} \tag{2.9}$$

Approximating $t^{k+\alpha}$ using the shifted second-kind Chebyshev series gives

$$t^{k+\alpha} = \sum_{j=0}^{N} a_{kj} \psi_j(t), \tag{2.10}$$

where a_{kj} are obtained from (2.5) as follows

$$a_{kj} = \frac{4(j+1)\Gamma(k+\alpha+\frac{3}{2})\Gamma(k+\alpha+1)}{\sqrt{\pi}\Gamma(k+\alpha+j+3)\Gamma(k+\alpha-j+1)}, \qquad j = 0, 1, 2, \ldots, N.$$

Substituting (2.10) into equation (2.9), we have

$$I_t^\alpha \psi_i(t) = \sum_{j=0}^{N} \theta_\alpha(i,j) \psi_j(t),$$

where $\theta_\alpha(i,j)$ is defined by (2.8). Therefore, for $i = 0, 1, \ldots, N$ we obtain

$$I_t^\alpha \psi_i(t) = [\theta_\alpha(i,0), \theta_\alpha(i,1), \ldots, \theta_\alpha(i,N)]\, \psi(t). \tag{2.11}$$

Finally, equation (2.7) is established using equation (2.11). □

3. Numerical method

In this section, we present a numerical method to solve FOCP (1.1)–(1.3) using properties of the shifted SKCPs . To do this, we consider an approximation of the fractional state rate $D_t^\alpha x(t)$ in the dynamical system (1.2) as:

$$D_t^\alpha x(t) \simeq \sum_{i=0}^{N} c_i \psi_i(t) = C^T \psi(t), \tag{3.1}$$

where the elements c_i of the vector C are unknown and $\psi(t)$ is given by (2.6). Considering equations (1.3), (2.2) and (3.1) and using the operational matrix of fractional integration in equation (2.7), we have

$$x(t) \simeq C^T P^{(\alpha)} \psi(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!}. \tag{3.2}$$

Now, we can approximate $u(t)$ using the dynamical system in equation (1.2) as:

$$u(t) \simeq \frac{1}{b(t)}\left[C^T\psi(t) - g\left(t, C^T P^{(\alpha)}\psi(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!}\right)\right]. \tag{3.3}$$

Substituting (3.2) and (3.3) into (1.1), we obtain

$$J[C] = \int_0^1 f\left(t, C^T P^{(\alpha)}\psi(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!},\right.$$
$$\left.\frac{1}{b(t)}\left[C^T\psi(t) - g\left(t, C^T P^{(\alpha)}\psi(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!}\right)\right]\right) dt. \tag{3.4}$$

Let us introduce

$$Q\,(C,t) = f\left(t, C^T P^{(\alpha)}\psi(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!},\right.$$
$$\left.\frac{1}{b(t)}\left[C^T\psi(t) - g\left(t, C^T P^{(\alpha)}\psi(t) + \sum_{i=0}^{n-1} x_i \frac{t^i}{i!}\right)\right]\right) dt.$$

So, using Legendre-Gauss quadrature formula we have

$$J[C] \simeq \frac{1}{2}\sum_{k=1}^{m} w_k Q\left(\frac{s_k+1}{2}, C\right),$$

where s_k, $k = 1, 2, \ldots, m$ are m zeros of Legendre polynomial of degree m and w_k are the corresponding weights [9]. Finally, the necessary conditions for the optimality of the performance index imply:

$$\frac{\partial J}{\partial c_j}[C] = 0, \qquad j = 0, 1, \ldots, N. \tag{3.5}$$

Equation (3.5) forms a nonlinear system of algebraic equations in terms of the unknown elements of the vector C. In our implementation, we have solved this system using the Mathematica function FindRoot, which uses the Newton's method as the default method. After solving this system the numerical results for $x(t)$, $u(t)$ and optimum value of J are given using equations (3.2), (3.3) and (3.4), respectively.

4. Numerical examples

In this section, some examples are given to demonstrate the applicability and accuracy of our method. The codes were written in Mathematica software. In all the examples in this section we have used $m = 10$ for Legendre-Gauss quadrature formula.

Example 4.1. Consider a minimization problem as follows [19]:

$$\min J = \int_0^1 \left[\left(x(t) - t^2\right)^2 + \left(u(t) + t^4 - \frac{20t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})}\right)^2 \right] dt,$$

subject to:

$$D_t^{1.1} x(t) = t^2 x(t) + u(t),$$

$$x(0) = x'(0) = 0.$$

The functions $x(t) = t^2$ and $u(t) = -t^4 + \frac{20t^{\frac{9}{10}}}{9\Gamma(\frac{9}{10})}$ minimize the performance index J and the minimum value is $J = 0$. We have solved this problem with different values for N. For instance, with $N = 3$ the operational matrix of fractional integration is obtained as:

$$P^{(1.1)} = \begin{bmatrix} 0.452542 & 0.242828 & 0.00714199 & -0.00140498 \\ -0.35846 & -0.0354935 & 0.107481 & 0.00509663 \\ 0.166877 & -0.0600908 & -0.0166159 & 0.0679421 \\ -0.120681 & -0.00390251 & -0.0478958 & -0.0109875 \end{bmatrix},$$

and the unknown vector C is calculated by solving the final system in equation (3.5) with initial approximation $C = [0, 0, 0, 0]^T$ as:

$$C = \begin{bmatrix} 1.09993 \\ 0.507813 \\ -0.0163147 \\ 0.0042012 \end{bmatrix}.$$

Numerical results for the minimum of J with different values of N together with the results obtained in [19] using Legendre functions are shown in Table 1.

TABLE 1. Numerical results for Example 4.1

Present method		Method of [19]	
N	J	m	J
3	5.82938×10^{-6}	3	6.0753×10^{-6}
4	1.46226×10^{-6}	4	1.67255×10^{-6}
5	4.3337×10^{-7}	5	5.91532×10^{-7}
7	3.5498×10^{-8}	7	1.21966×10^{-7}
8	6.00953×10^{-9}	8	7.03371×10^{-8}

Example 4.2. Consider the following minimization problem [19]

$$\min J = \int_0^1 \Big[e^t \left(x(t) - t^4 + t - 1 \right)^2$$
$$+ (t^2+1) \left(u(t) + 1 - t + t^4 - \frac{8000 t^{\frac{21}{10}}}{77 \Gamma\left(\frac{1}{10}\right)} \right)^2 \Bigg] dt,$$

subject to:

$$D_t^{1.9} x(t) = x(t) + u(t),$$
$$x(0) = 1, \quad x'(0) = -1.$$

In this problem the performance index J takes its minimum value when $x(t) = 1 - t + t^4$ and the minimum value is $J = 0$. By choosing $N = 3$, the operational matrix of fractional integration is as follows:

$$P^{(1.9)} = \begin{bmatrix} 0.178143 & 0.138152 & 0.031611 & -0.00061084 \\ -0.18579 & -0.108697 & 0.00600659 & 0.0117374 \\ 0.113265 & 0.0388703 & -0.0197332 & 0.00231601 \\ -0.0726456 & -0.0254721 & -0.000646971 & -0.0104707 \end{bmatrix},$$

and the unknown vector C is obtained with initial approximation $C = [0, 0, 0, 0]^T$ as:

$$C = \begin{bmatrix} 3.27996 \\ 2.70114 \\ 0.728038 \\ 0.0128955 \end{bmatrix}.$$

Table 2 presents the numerical results for the minimum of performance index J with different values of N obtained by the presented method in this paper and the results obtained in [19] using Legendre functions.

TABLE 2. Numerical results for Example 4.2

Present method		Method of [19]	
N	J	m	J
3	1.16577×10^{-4}	3	8.93768×10^{-6}
4	5.32891×10^{-7}	4	5.42028×10^{-7}
5	6.50811×10^{-8}	5	6.77757×10^{-8}
7	1.94565×10^{-9}	7	2.84624×10^{-9}
8	2.76563×10^{-10}	8	8.22283×10^{-10}

Example 4.3. Consider the following minimization problem [19]

$$\min J = \int_0^1 \Big[x^2(t) - 2t^{\frac{3}{2}}x(t) + u^2(t) - \frac{3\sqrt{\pi}}{4}e^{-t}u(t) + e^{-t+t^{\frac{3}{2}}}u(t) + t^3 + \frac{9\pi}{64}e^{-2t} - \frac{3\sqrt{\pi}}{8}e^{-2t+t^{\frac{3}{2}}} + \frac{1}{4}e^{-2t+2t^{\frac{3}{2}}} + e^{2t}\Big]\,dt,$$

subject to:

$$D_t^{1.5}x(t) = e^{x(t)} + 2e^t u(t),$$
$$x(0) = x'(0) = 0.$$

In this example the state function $x(t) = \sqrt{t^3}$ and the control function $u(t) = \frac{1}{2}e^{-t}\left(-e^{t^{\frac{3}{2}}} + \frac{3\sqrt{\pi}}{4}\right)$ minimize the performance index J and the minimum value is $J = 3.19452805$. We applied the proposed method in this paper with $N = 2$ such that for this choice we have

$$P^{(1.5)} = \begin{bmatrix} 0.2919 & 0.1946 & 0.0265364 \\ -0.27244 & -0.106146 & 0.0449077 \\ 0.147846 & 0.00898155 & -0.0336128 \end{bmatrix},$$

and the unknown vector C is obtained with initial approximation $C = [0, 0, 0]^T$ as:

$$C = \begin{bmatrix} 1.32797 \\ 0.00085962 \\ -0.00105561 \end{bmatrix}.$$

Finally, the performance index with $N = 2$ is gained $J = 3.19455842$. Also, Table 3 gives the numerical results for the minimum of performance index J with different values of N.

TABLE 3. Numerical results for Example 4.3

$N=3$	$N=5$	$N=7$	Exact solution
3.19453043	3.19452812	3.19452805	3.19452805

Example 4.4. As the final example, consider the following problem [19]:

$$\min J = \frac{1}{2}\int_0^1 \left[x^2(t) + u^2(t)\right] dt,$$

subject to:

$$D_t^{\alpha}x(t) = -x(t) + u(t), \quad 0 < \alpha \le 1,$$
$$x(0) = 1.$$

The exact solution of this problem with $\alpha = 1$ was given by [8] as:

$$x(t) = \beta \sinh\left(\sqrt{2}t\right) + \cosh\left(\sqrt{2}t\right),$$

$$u(t) = \left(\beta + \sqrt{2}\right) \sinh\left(\sqrt{2}t\right) + \left(\sqrt{2}\beta + 1\right) \cosh\left(\sqrt{2}t\right),$$

where

$$\beta = -\frac{\sqrt{2}\sinh\left(\sqrt{2}\right) + \cosh\left(\sqrt{2}\right)}{\sinh\left(\sqrt{2}\right) + \sqrt{2}\cosh\left(\sqrt{2}\right)}.$$

Numerical results for this problem are shown in Table 4 and Figures 1 and 2. In Table 4, minimum value for J when $\alpha = 1$ is presented with different values of N and Figures 1 and 2 display the approximate solutions for $x(t)$ and $u(t)$, respectively, with $\alpha = 0.8, 0.9, 1$ and $N = 2$ together with their exact solutions.

TABLE 4. Numerical results for Example 4.4

$N = 2$	$N = 3$	$N = 4$	$N = 6$	Exact solution
0.192756	0.192867	3.192908	0.192909	0.192909

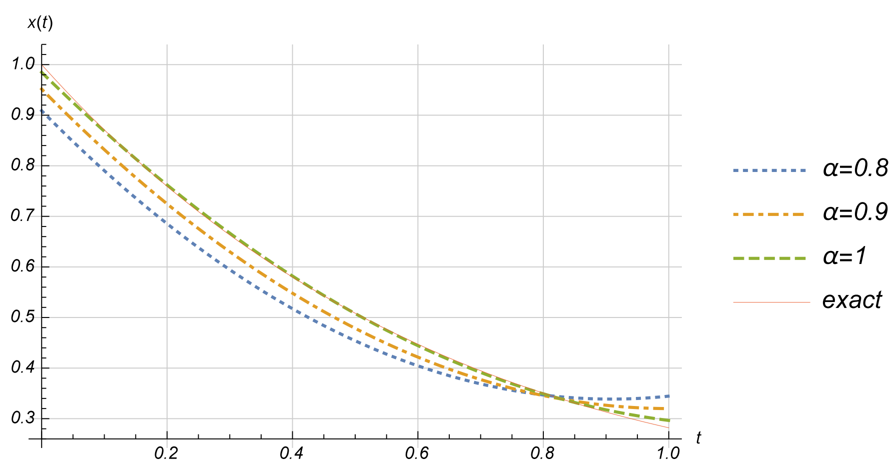

FIGURE 1. Comparison of $x(t)$ with $N = 2$ for Example 4.4

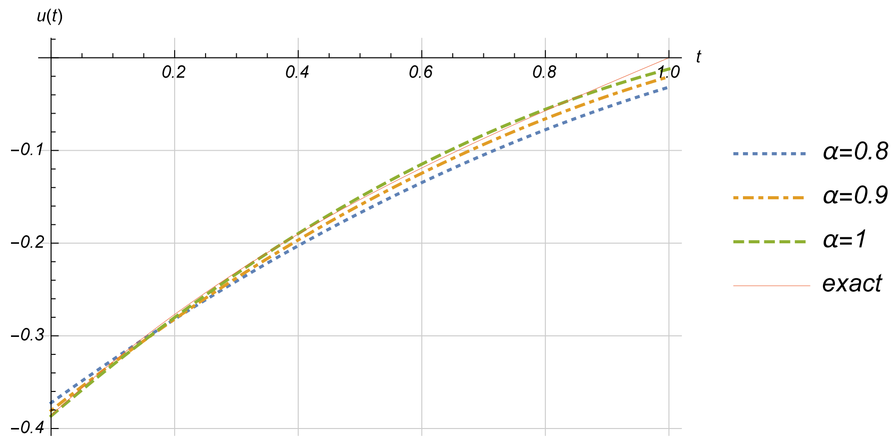

FIGURE 2. Comparison of $u(t)$ with $N = 2$ for Example 4.4

5. Conclusions

In this paper, a numerical method has been proposed for numerical solution of the FOCPs. By using the definition of Riemann-Liouville fractional integral operator and properties of the shifted SKCPs, the operational matrix of fractional integration has been introduced. This matrix and Legendre-Gauss quadrature formula have been employed to reduce the considered problem into a system of nonlinear algebraic equations. The method was tested on some examples and the numerical results obtained by the proposed method in this paper have been compared with the results achieved using the numerical technique discussed in [19]. Also, the results approved that the proposed method is efficient and has high accuracy.

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