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CONVERGENCE ANALYSIS OF THE SINC COLLOCATION METHOD FOR INTEGRO-DIFFERENTIAL EQUATIONS SYSTEM

MOHAMMAD ZAREBNIA

Abstract. In this paper, a numerical solution for a system of linear Fredholm integro-differential equations by means of the sinc method is considered. This approximation reduces the system of integro-differential equations to an explicit system of algebraic equations. The exponential convergence rate $O(e^{-k\sqrt{N}})$ of the method is proved. The analytical results are illustrated with numerical examples that exhibit the exponential convergence rate.

1. Introduction

In this paper, we extend the applications of the sinc method to find a numerical solution for the system of Fredholm integro-differential equations. The sinc methods which proposed in [4] and [10], are effectively and easily used to solve some problems in applied physics and engineering [3, 5, 6, 9].

Many mathematical formulations of physical phenomena contain integro differential equation. These equations arise in fluid dynamics, biological models and chemical phenomena. Integro-differential equations and systems are usually difficult to solve analytically so, it is required to obtain an efficient approximate solution. Therefore, several researchers have considered the numerical solution of these equations with different methods. In literatures, there exist numerical techniques such as differential transform method (DTM) [2], Chebyshev collocation method [1], Lagrange interpolation [11], time-stepping schemes [7], finite difference

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method and semi analytical-numerical techniques such as He's homotopy perturbation (HPM) method [12].

The system of linear Fredholm integro-differential equation has the form:

$$\mathbf{U}'(x) = \int_\Gamma \mathbf{K}(x,t)\mathbf{U}(t)dt + \widetilde{\mu}(x)\mathbf{U}(x) + \mathbf{F}(x), \quad x \in \Gamma = [a,b], \tag{1.1}$$

with boundary condition

$$\mathbf{U}(a) = \mathbf{U}_a,$$

where,

$$\mathbf{U}(x) = [u_1(x), u_2(x), \ldots, u_n(x)]^T,$$

$$\mathbf{U}'(x) = [u'_1(x), u'_2(x), \ldots, u'_n(x)]^T,$$

$$\widetilde{\mu}(x) = [\mu_{ij}(x,t)], \qquad i,j = 1,2,\ldots,n,$$

$$\mathbf{F}(x) = [f_1(x), f_2(x), \ldots, f_n(x)]^T,$$

$$\mathbf{K}(x,t) = [K_{ij}(x,t)], \qquad i,j = 1,2,\ldots,n,$$

$$\mathbf{U}(a) = [u_1(a), u_2(a), \ldots, u_n(a)]^T.$$

In system (1.1) the kernel $\mathbf{K}(x,t)$, functions $\mathbf{F}(x)$ and $\widetilde{\mu}(x)$ are given, and $\mathbf{U}(x)$ is the solution to be determined.

The sections of this paper are organized as follows. Section 2 is devoted to the basic formulation of the sinc function required for our subsequent development. In Section 3, we illustrate how the sinc method may be used to replace system (1.1) by an explicit system of linear algebraic equations. In Section 4, the convergence analysis of the method has been discussed. It is shown that the sinc procedure converges to the solution as an exponential rate. Finally, we report our numerical results and demonstrate the efficiency and accuracy of the proposed numerical scheme by considering some numerical examples in Section 5.

2. Preliminaries

For any $h > 0$, the sinc basis functions in [4] and [10] are given by

$$S(j,h)(z) = \mathrm{Sinc}\left(\frac{z - jh}{h}\right), \quad j = 0, \pm 1, \pm 2, \ldots, \tag{2.1}$$

where,

$$\mathrm{Sinc}(z) = \begin{cases} \frac{\sin(\pi z)}{\pi z}, & z \neq 0, \\ 1, & z = 0. \end{cases} \tag{2.2}$$

The sinc function for the interpolating points $z_k = kh$ is given by

$$S(j,h)(kh) = \delta_{jk}^{(0)} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \tag{2.3}$$

Let D be a simply-connected domain in the complex plane with boundary ∂D. Let a and b denote two distinct points of ∂D and ϕ denotes a conformal map of D onto D_d, where D_d denotes the region

$$\{w \in \mathbb{C} : |Iw| < d, d > 0\},$$

such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. Assume $\psi = \phi^{-1}$ and let Γ be defined by

$$\Gamma = \{z \in \mathbb{C} : z = \psi(u), u \in \mathbb{R}\}.$$

Given ϕ, ψ and a positive number h, let us set

$$z_k = z_k(h) = \psi(kh), \quad k = 0, \pm 1, \pm 2, \ldots.$$

A function u is said to decay exponentially with respect to the conformal map ϕ if there exist positive constants α and C such that

$$|u(x)| \leq C e^{(-\alpha|\phi(x)|)}. \tag{2.4}$$

Now, we introduce the space $L_\alpha(D)$ as it is defined in [4] and [10]. A function $u(z)$ is in the space $L_\alpha(D)$ if and only if $u(z)$ is analytic in D and there exists a constant $C > 0$, such that

$$|u(z)| \leq C \frac{|\rho(z)|^\alpha}{[1 + |\rho(z)|]^{2\alpha}}, \quad z \in D, \quad 0 < \alpha \leq 1, \tag{2.5}$$

where $\rho(z) = e^{\phi(z)}$. To construct an approximation in the interval (a, b), we consider the conformal map

$$\phi(z) = \ln\left(\frac{z-a}{b-z}\right). \tag{2.6}$$

The sinc grid points $z_k \in (a, b)$ in D will be denoted by x_k because they are real. For the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line, the image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{a + be^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{2.7}$$

Let N be a positive integer, $\frac{u}{\phi'} \in L_\alpha(D)$ and the mesh size h is chosen by the formula $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$. The definite integration of a function u is defined as it is defined in [10]:

$$\int_\Gamma u(z)dz = h \sum_{j=-N}^{N} \frac{u(z_j)}{\phi'(z_j)} + O\left(e^{-(\pi d\alpha N)^{\frac{1}{2}}}\right). \tag{2.8}$$

Moreover, for the function u, we have (see[9]):

$$\int_a^{z_k} u(z)dt = h \sum_{j=-N}^{N} \delta_{kj}^{(-1)} \frac{u(z_j)}{\phi'(z_j)} + O\Big(e^{-(\pi d\alpha N)^{\frac{1}{2}}}\Big), \tag{2.9}$$

where

$$\delta_{kj}^{(-1)} = \frac{1}{2} + \int_0^{k-j} \frac{\sin(\pi t)}{\pi t} dt. \tag{2.10}$$

An $n \times n$ matrix with (k, j)th entry $\delta_{kj}^{(-1)}$ is shown by $I^{(-1)}$.

3. Description of the proposed numerical method

Let us consider the system of linear integro-differential equations (1.1). For convenience, we consider the ith equation of (1.1)

$$u_i'(x) = \sum_{j=1}^{n} \int_\Gamma K_{ij}(x,t)u_j(t)dt + \sum_{j=1}^{n} \mu_{ij}(x)u_j(x) + f_i(x), \quad i = 1, 2, \ldots, n. \tag{3.1}$$

Integrating from a to x of (3.1) obtained

$$\begin{aligned} u_i(x) = \int_a^x \left\{ \sum_{j=1}^{n} \int_a^b K_{ij}(\xi,t)u_j(t)dt + \sum_{j=1}^{n} \mu_{ij}(\xi)u_j(\xi) + f_i(\xi) \right\} d\xi \\ + u_i(a), \quad i = 1, 2, \ldots, n, \quad x \in \Gamma = [a, b]. \end{aligned} \tag{3.2}$$

For simplicity, we write

$$H_{ij}(\xi) = \int_a^b K_{ij}(\xi,t)u_j(t)dt, \tag{3.3}$$

and suppose that

$$\frac{H_{ij}}{\phi'} \in L_\alpha(D), \qquad \frac{\mu_{ij}}{\phi'}u_j \in L_\alpha(D), \qquad \frac{f_i}{\phi'} \in L_\alpha(D).$$

We apply the collocation method by setting $x = x_k,\ k = -N, \dots, N$, in (13) which x_k are sinc grid points

$$\begin{aligned} x_k &= \psi(kh) \\ &= \phi^{-1}(kh) \\ &= \frac{a + be^{kh}}{1 + e^{kh}}, \end{aligned}$$

and by using Eq. (2.9) for the right-hand side of (3.2), and applying the collocation to it, we obtain the following equation;

(3.4)

$$\begin{aligned} u_i(x_k) \simeq h &\sum_{j=1}^{n} \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{H_{ij}(x_l)}{\phi'(x_l)} \\ &+ h \sum_{j=1}^{n} \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\mu_{ij}(x_l)}{\phi'(x_l)} u_j(x_l) \\ &+ h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f_i(x_l)}{\phi'(x_l)} + u_i(a), \quad i = 1, 2, \dots, n, \quad k = -N, \dots, N. \end{aligned}$$

For the first term on the right-hand side of (3.4), by considering (2.8), we obtain

(3.5)

$$h \sum_{j=1}^{n} \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{H_{ij}(x_l)}{\phi'(x_l)} \simeq h^2 \sum_{j=1}^{n} \sum_{l=-N}^{N} \sum_{l'=-N}^{N} \delta_{k,l}^{(-1)} \frac{K_{ij}(x_l, t_{l'})}{\phi'(x_l)\phi'(t_{l'})} u_j(t_{l'}).$$

Then, substituting the first term on the right-hand side of (3.4) with (3.5), yields

(3.6)

$$\begin{aligned} u_i(x_k \simeq h^2 &\sum_{j=1}^{n} \sum_{l=-N}^{N} \sum_{l'=-N}^{N} \delta_{k,l}^{(-1)} \frac{K_{ij}(x_l, t_{l'})}{\phi'(x_l)\phi'(t_{l'})} u_j(t_{l'}) \\ &+ h \sum_{j=1}^{n} \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\mu_{ij}(x_l)}{\phi'(x_l)} u_j(x_l) \\ &+ h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f_i(x_l)}{\phi'(x_l)} + u_i(a), \quad i = 1, 2, \dots, n, \quad k = -N, \dots, N, \end{aligned}$$

where

$$\phi(x) = \ln\left(\frac{x-a}{b-x}\right), \qquad \phi(a) = -\infty, \qquad \phi(b) = +\infty,$$

$$\phi'(x) = \frac{a-b}{(a-x)(b-x)}.$$

We rewrite the above equation and obtain the following system of $n \times (2N+1)$ linear equations with $n \times (2N+1)$ unknowns

$$u_{ik}, \quad k = -N, -N+1, \ldots, N-1, N, \quad i = 1, \ldots, n,$$

(3.7)

$$\begin{aligned} &u_{ik} - h^2 \sum_{j=1}^{n} \left[\sum_{l=-N}^{N} \sum_{l'=-N}^{N} \delta_{k,l}^{(-1)} \frac{K_{ij}(x_l, t_{l'})}{\phi'(x_l)\phi'(t_{l'})} u_{jl'} \right] \\ &\quad - h \sum_{j=1}^{n} \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\mu_{ij}(x_l)}{\phi'(x_l)} u_{jl} \\ &\simeq h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f_i(x_l)}{\phi'(x_l)} + u_i(a), \quad i = 1, 2, \ldots, n, \quad k = -N, \ldots, N, \end{aligned}$$

where u_{ik} denotes an approximate value of $u_i(x_k)$. We denote

$$\begin{aligned} &I^{(m)} = \left[\delta_{kj}^{(m)}\right], \quad m = -1, 0, \\ &D\left(\mu_{ij}/\phi'\right) = \operatorname{diag}\left(\mu_{ij}(x_{-N})/\phi'(x_{-N}), \ldots, \mu_{ij}(x_N)/\phi'(x_N)\right), \\ &\widetilde{K}_{ij} = \left[\frac{K_{ij}(x_l, t_{l'})}{\phi'(x_l)\phi'(t_{l'})}\right], \end{aligned}$$

and

$$A_{ij} = \begin{cases} I^{(0)} - h^2 I^{(-1)} \widetilde{K}_{ij} - h I^{(-1)} D\big(\mu_{ij}/\phi'\big), & i = j, \\ -h^2 I^{(-1)} \widetilde{K}_{ij} - h I^{(-1)} D\big(\mu_{ij}/\phi'\big), & i \neq j, \end{cases}$$

which are the square matrices of order $(2N+1) \times (2N+1)$, then the system of linear equations (3.7) for $n \times (2N+1)$ unknown coefficients $u_{jl},\ j = 1, \ldots, n,\ l = -N, \ldots, N$ can be expressed in the matrix form

$$\mathbf{A}\widetilde{\mathbf{U}} = \widetilde{\mathbf{F}}, \tag{3.8}$$

where

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{2n} & \cdots & A_{nn} \end{pmatrix},$$

$$\widetilde{\mathbf{F}} = \left[G_{1,-N}, \ldots, G_{1,N}, \ldots, G_{n,-N}, \ldots, G_{n,N}\right]^T,$$

$$G_{i,k} = h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f_i(x_l)}{\phi'(x_l)} + u_i(a),$$

$$\widetilde{\mathbf{U}} = \left[u_{1l}, \ldots, u_{nl}\right]^T, \quad l = -N, \ldots, N.$$

Existence of a solution of the system (3.8) is discussed in [8]. We use Gaussian elimination method for solving the system (3.8) and obtain an approximate solution

$$u_{jl}, \quad j = 1, 2, \ldots, n, \quad l = -N, -N+1, \ldots, N. \tag{3.9}$$

Having used the approximate solution (3.9) we can obtain an approximation to the solution of system (1.1) as:

(3.10)

$$\mathbf{U}_N(x) = h^2 \sum_{l=-N}^{N} \sum_{j=-N}^{N} \frac{\mathbf{K}(\xi_l, t_j)}{\phi'(\xi_l)\phi'(t_j)} \Omega_{h,l}(x) \widetilde{\mathbf{U}}_j + h \sum_{l=-N}^{N} \frac{\widetilde{\mu}(\xi_l)}{\phi'(\xi_l)} \Omega_{h,l}(x) \widetilde{\mathbf{U}}_l + h \sum_{l=-N}^{N} \frac{\mathbf{F}(\xi_l)}{\phi'(\xi_l)} \Omega_{h,l}(x) + \mathbf{U}(a),$$

where

$$\widetilde{\mathbf{U}}_l = \left[u_{1l}, u_{2l}, \ldots, u_{nl}\right]^T,$$

$$\mathbf{K}(\xi_l, t_j) = \left[K_{ij'}(\xi_l, t_j)\right], \qquad i, j' = 1, 2, \ldots, n,$$

$$\widetilde{\mu}(\xi_l) = [\mu_{ij}(\xi_l)], \qquad i, j = 1, 2, \ldots, n,$$

$$\mathbf{F}(\xi_l) = [f_1(\xi_l), f_2(\xi_l), \ldots, f_n(\xi_l)]^T,$$

$$\mathbf{U}(a) = [u_1(a), u_2(a), \ldots, u_n(a)]^T,$$

$$\Omega_{h,l}(x) = \tfrac{1}{2} + \textstyle\int_a^x S(l,h) \circ \phi(t) dt.$$

This procedure is the sinc-Nyström method.

4. Convergence Analysis

Now, we discuss the convergence of the proposed method for the system of linear Fredholm integro-differential equation (1.1). For each N, we can find

$$u_{jl}, \quad j = 1, \ldots, n, \quad l = -N, \ldots, N,$$

which is the solution of the linear system (3.8), also by using u_{jl} we obtain the approximate solution $\mathbf{U}_N(x)$ as (3.10).

Lemma 4.1. *Let $\boldsymbol{U}(x)$ be the exact solution of the given Eq.*(1.1) *and let $h = \left(\frac{\pi d}{\alpha N}\right)^{\frac{1}{2}}$, $\frac{K_{ij}}{\phi'} u_j \in L_\alpha(D)$, $\frac{\mu_{ij}}{\phi'} u_j \in L_\alpha(D)$ and $\frac{f_i}{\phi'} \in L_\alpha(D)$ for all $x \in \Gamma$. Then there exists a constant C_2 independent of N such that*

$$\|\mathbf{A}\,\boldsymbol{U} - \widetilde{\boldsymbol{F}}\|_2 \leq C_2 N^{\frac{1}{2}} \exp\left\{-(\pi d \alpha N)^{\frac{1}{2}}\right\}, \tag{4.1}$$

Proof. Consider the kth component of the vector $\nu = \mathbf{AU} - \widetilde{\mathbf{F}}$. Using the previous section, we get the following bound on v_k:

$$\begin{aligned}
|v_k| &= |(\mathbf{AU} - \widetilde{\mathbf{F}})_k| \\
&= \Big|\mathbf{U}(x_k) - h^2 \sum_{l=-N}^{N} \sum_{l'=-N}^{N} \delta_{k,l}^{(-1)} \frac{\mathbf{K}(x_l, t_{l'})}{\phi'(x_l)\phi'(t_{l'})} \mathbf{U}(t_{l'}) \\
&\quad - h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\widetilde{\mu}(x_l)}{\phi'(x_l)} \mathbf{U}(x_l) - h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\mathbf{F}(x_l)}{\phi'(x_l)} - \mathbf{U}(a)\Big| \\
&= E.
\end{aligned}$$

Now, we consider the ith equation of the above relation as:

$$\begin{aligned}
E_i &= \Big|u_i(x_k) - h^2 \sum_{j=1}^{n} \sum_{l=-N}^{N} \sum_{l'=-N}^{N} \delta_{k,l}^{(-1)} \frac{K_{ij}(x_l, t_{l'})}{\phi'(x_l)\phi'(t_{l'})} u_j(t_{l'}) \\
&\quad - h \sum_{j=1}^{n} \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{\mu_{ij}(x_l)}{\phi'(x_l)} u_j(x_l) - h \sum_{l=-N}^{N} \delta_{k,l}^{(-1)} \frac{f_i(x_l)}{\phi'(x_l)} - u_i(a)\Big| \\
&\leq c_i \exp\{-(\pi d \alpha N)^{\frac{1}{2}}\}, \quad i = 1, 2, \ldots, n.
\end{aligned}$$

Therefore we can write

$$E \leq C_1 \exp\left\{-(\pi d \alpha N)^{\frac{1}{2}}\right\},$$

and it thus follows that

$$\begin{aligned}\|\mathbf{A}\mathbf{U}-\widetilde{\mathbf{F}}\| &= \left(\sum_{k=-N}^{N}|v_k|^2\right)^{\frac{1}{2}}\\ &\le C_2 N^{\frac{1}{2}} e^{\left\{-(\pi d\alpha N)^{\frac{1}{2}}\right\}}.\end{aligned}$$

□

Now, we state and proof the following theorem which shows that the error in the approximate solution for the system of Fredholm integro-differential equations (1.1) is $O\big(\exp(-kN^{1/2})\big)$, where $k > 0$.

Theorem 4.2. *Let us consider all assumptions of Lemma 4.1. Let* $\boldsymbol{U}_N(x)$ *be the approximate solution of integro-differential equation* (1.1) *given by* (3.10). *Then there exists a constant* C_3 *independent of* N *such that*

$$\sup_{x\in(a,b)}\big|\boldsymbol{U}(x)-\boldsymbol{U}_N(x)\big| \le C_3 N^{\frac{1}{2}} e^{\left\{-(\pi d\alpha N)^{\frac{1}{2}}\right\}}. \tag{4.2}$$

Proof. By considering the given Eq. (1.1) and all assumptions, we obtain

$$\begin{aligned}\big|\mathbf{U}(x)-\mathbf{U}_N(x)\big| &= \Big|\int_a^x\int_\Gamma \mathbf{K}(\xi,t)\mathbf{U}(t)dtd\xi+\int_a^x\widetilde{\mu}(\xi)\mathbf{U}(\xi)d\xi\\ &\quad+\int_a^x\mathbf{F}(\xi)d\xi-h^2\sum_{l=-N}^{N}\sum_{j=-N}^{N}\frac{\mathbf{K}(\xi_l,t_j)}{\phi'(\xi_l)\phi'(t_j)}\Omega_{h,l}(x)\widetilde{\mathbf{U}}_j\\ &\quad-h\sum_{l=-N}^{N}\frac{\widetilde{\mu}(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)\widetilde{\mathbf{U}}_l-h\sum_{l=-N}^{N}\frac{\mathbf{F}(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)\Big|\\ &\le h^2\sum_{l=-N}^{N}\Big|\sum_{j=-N}^{N}\frac{\mathbf{K}(\xi_l,t_j)}{\phi'(\xi_l)\phi'(t_j)}\Omega_{h,l}(x)\Big|\Big|\mathbf{U}(t_j)-\widetilde{\mathbf{U}}_j\Big|\\ &\quad+h\sum_{l=-N}^{N}\Big|\frac{\widetilde{\mu}(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)\Big|\Big|\mathbf{U}(\xi_l)-\widetilde{\mathbf{U}}_l\Big| = E_N(Say).\end{aligned}$$

Note that, there exist constants S_1 and S_2 such that

$$h^2\left(\sum_{l=-N}^{N}\Big|\sum_{j=-N}^{N}\frac{\mathbf{K}(\xi_l,t_j)}{\phi'(\xi_l)\phi'(t_j)}\Omega_{h,l}(x)\Big|^2\right)^{\frac{1}{2}} \le S_1,$$

$$h\left(\sum_{l=-N}^{N}\Big|\frac{\widetilde{\mu}(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)\Big|^2\right)^{\frac{1}{2}} \le S_2,$$

hold for $x \in [a,b]$, then using the Schwarz inequality, we get

$$\begin{aligned} E_N &\le h^2\left(\sum_{l=-N}^{N}\Big|\sum_{j=-N}^{N}\frac{\mathbf{K}(\xi_l,t_j)}{\phi'(\xi_l)\phi'(t_j)}\Omega_{h,l}(x)\Big|^2\right)^{\frac{1}{2}}\left(\sum_{l=-N}^{N}\Big|\mathbf{U}(t_l)-\widetilde{\mathbf{U}}_l\Big|^2\right)^{\frac{1}{2}} \\ &\quad + h\left(\sum_{l=-N}^{N}\Big|\frac{\widetilde{\mu}(\xi_l)}{\phi'(\xi_l)}\Omega_{h,l}(x)\Big|^2\right)^{\frac{1}{2}}\left(\sum_{l=-N}^{N}\Big|\mathbf{U}(\xi_l)-\widetilde{\mathbf{U}}_l\Big|^2\right)^{\frac{1}{2}} \\ &\le S\|\mathbf{U}-\widetilde{\mathbf{U}}\|, \end{aligned}$$

where $S = S_1 + S_2$. Since from (3.8)

$$\widetilde{\mathbf{U}} = \mathbf{A}^{-1}\widetilde{\mathbf{F}},$$

then

$$\begin{aligned} \|\mathbf{U}-\widetilde{\mathbf{U}}\| &= \|\mathbf{U}-\mathbf{A}^{-1}\widetilde{\mathbf{F}}\| \\ &\le \|\mathbf{A}^{-1}\|\|\mathbf{A}\mathbf{U}-\widetilde{\mathbf{F}}\|, \end{aligned} \tag{4.3}$$

holds, we have from Lemma 4.1

$$\begin{aligned} E_N &\le S\|\mathbf{A}^{-1}\|\|\mathbf{A}\mathbf{U}-\widetilde{\mathbf{F}}\| \\ &\le SC_2\|\mathbf{A}^{-1}\|N^{\frac{1}{2}}\exp\left\{-(\pi d\alpha N)^{\frac{1}{2}}\right\}. \end{aligned} \tag{4.4}$$

Therefore from the above relation we conclude that

$$\sup_{x\in(a,b)}\big|\mathbf{U}(x)-\mathbf{U}_N(x)\big| \le C_3 N^{\frac{1}{2}}\exp\left\{-(\pi d\alpha N)^{\frac{1}{2}}\right\}.$$

□

5. Numerical examples

In order to illustrate the performance of the sinc method for solving linear system of Fredholm integral equations and justify the accuracy and efficiency of the method, we consider the following examples. The examples have been solved by presented method with different values of N. In all examples we take $\alpha = 1$ and $d = \frac{\pi}{2}$, which yields $h = \pi(\frac{1}{N})^{\frac{1}{2}}$. The errors are reported on the set of sinc grid points

$$S = \{x_{-N},\ldots,x_0,\ldots,x_N\},$$

$$x_k = \frac{e^{kh}}{1+e^{kh}}, \quad k = -N, \dots, N. \tag{5.1}$$

The maximum absolute error on the sinc grid points is

$$\|E_F^S(h)\|_\infty = \max_{-N \le j \le N} \left| \mathbf{U}(x_j) - \mathbf{U}_N(x_j) \right|. \tag{5.2}$$

Example 5.1. Consider the following Fredholm integro-differential equations system with exact solution $(u_1(x), u_2(x)) = (\sin x, \cos x)$.

(5.3)
$$\begin{cases} u_1'(x) = -\frac{1}{2}\int_0^\pi \cos(x+t)u_1(t)dt - \frac{1}{2}\int_0^\pi \cos(x-t)u_2(t)dt \\ \qquad\qquad + \frac{\pi}{4}u_1(x) + \frac{1}{2}u_2(x) + \frac{1}{2}(1+\frac{\pi}{2})\cos x - \frac{\pi}{2}\sin x, \\ u_2'(x) = -\int_0^\pi e^x u_1(t)dt + \int_0^\pi e^x u_2(t)dt + u_1(x) + 2e^x - 2\sin x, \\ u_1(0) = 0, \qquad u_2(0) = 1. \end{cases}$$

Example 5.1 is solved for different values of N. The maximum of absolute errors on the sinc grid S are tabulated in Table 1.

TABLE 1. Results for Example 5.1.

N	h	$\|E_{u_1}^S(h)\|_\infty$	$\|E_{u_2}^S(h)\|_\infty$
5	1.404963	2.38057×10^{-2}	2.51081×10^{-2}
10	0.993459	2.74166×10^{-3}	2.09660×10^{-3}
20	0.702481	1.15103×10^{-4}	7.38931×10^{-5}
30	0.573574	9.33684×10^{-6}	5.68854×10^{-6}
40	0.496729	1.08330×10^{-6}	6.39170×10^{-7}
50	0.444288	1.58842×10^{-7}	9.17031×10^{-8}
60	0.405578	2.75744×10^{-8}	1.56828×10^{-8}
70	0.375492	5.44808×10^{-9}	3.06922×10^{-9}
80	0.351241	1.1936×10^{-9}	6.65056×10^{-10}
90	0.331153	2.84773×10^{-10}	1.57662×10^{-10}
100	0.314159	7.29926×10^{-11}	4.01618×10^{-11}

Example 5.2. We consider the system of integro-differential equations

(5.4)
$$\begin{cases} u_1'(x) = \int_0^1 \frac{(x-t)}{x} u_1(t)dt + \int_0^1 (x-t)^2 u_2(t)dt + \frac{1}{x}u_1(x) + u_2(x) \\ \qquad\qquad + \frac{1}{60}(7 - \frac{5}{x} + 10x + 50x^2), \\ u_2'(x) = \int_0^1 (x-t)^2 u_1(t)dt + \int_0^1 (x-t) u_2(t)dt + x u_1(x) + \frac{1}{x-1}u_2(x) \\ \qquad\qquad + \frac{17}{15} - \frac{1}{6}x(8 + x(6x-7)), \\ u_1(0) = 0, \qquad u_2(0) = 0, \end{cases}$$

with exact solution $(u_1(x), u_2(x)) = \big(x^2 - x, x - x^2\big)$.

The approximate solutions are calculated for different values of N and the optimal sinc mesh size $h = \pi(\frac{1}{N})^{\frac{1}{2}}$. In this problem the kernel $K_{1,1}(x,t)$, $\mu_{1,1}(x)$ and $\mu_{2,2}(x)$ are singular at $x = 0$ and $x = 1$. Table 2 exhibits the absolute errors.

TABLE 2. Results for Example 5.2.

N	h	$\Vert E^S_{u_1}(h)\Vert_\infty$	$\Vert E^S_{u_2}(h)\Vert_\infty$
5	1.404963	2.78724×10^{-3}	2.73763×10^{-3}
10	0.993459	2.71654×10^{-4}	2.03569×10^{-4}
20	0.702481	5.70513×10^{-6}	4.71787×10^{-6}
30	0.573574	2.85487×10^{-7}	2.44982×10^{-7}
40	0.496729	2.24748×10^{-8}	1.96874×10^{-8}
50	0.444288	2.36912×10^{-9}	2.10371×10^{-9}
60	0.405578	3.07697×10^{-10}	2.75940×10^{-10}
70	0.375492	4.68438×10^{-11}	4.23366×10^{-11}
80	0.351241	8.09139×10^{-12}	7.36045×10^{-12}
90	0.331153	1.55559×10^{-12}	1.41712×10^{-12}
100	0.314159	3.46889×10^{-13}	2.97484×10^{-13}

Example 5.3. Consider the following Fredholm integro-differential equations system

(5.5)
$$\begin{cases} u_1'(x) = -\int_0^1 e^{(x-t)} u_1(t)dt + 8\int_0^1 e^{-xt} u_2(t)dt + x u_1(x) + \frac{1}{5(1-x)}u_2(x) \\ \qquad\qquad + f_1(x), \\ u_2'(x) = -\int_0^1 e^{xt} u_1(t)dt - \int_0^1 e^{(x+t)} u_2(t)dt + \frac{1}{3x}u_1(x) + u_2(x) + f_2(x), \end{cases}$$

subject to initial conditions

$$u_1(0) = 1, \qquad u_2(0) = 1,$$

where $f_1(x)$ and $f_2(x)$ are chosen such that the exact solution is $(u_1(x), u_2(x)) = (e^x, e^{-x})$. The approximate solution is calculated for different values of N. The functions $\mu_{1,2}$ and $\mu_{2,1}$ are singular at $x = 0$ and $x = 1$. The maximum absolute errors in computed solutions are tabulated in Table 3.

Table 3. Results for Example 5.3.

N	h	$\|E^S_{u_1}(h)\|_\infty$	$\|E^S_{u_2}(h)\|_\infty$
5	1.404963	2.94914×10^{-3}	2.27788×10^{-3}
10	0.993459	2.22128×10^{-4}	1.60401×10^{-4}
20	0.702481	6.20098×10^{-6}	3.54061×10^{-6}
30	0.573574	3.69349×10^{-7}	1.84917×10^{-7}
40	0.496729	3.15091×10^{-8}	1.49841×10^{-8}
50	0.444288	3.54608×10^{-9}	1.61426×10^{-9}
60	0.405578	4.75880×10^{-10}	2.13328×10^{-10}
70	0.375492	8.02687×10^{-11}	3.29481×10^{-11}
80	0.351241	1.51705×10^{-11}	5.76789×10^{-12}
90	0.331153	3.14238×10^{-12}	1.11966×10^{-12}
100	0.314159	7.07878×10^{-13}	2.41085×10^{-13}

Conclusion

The sinc method is used to solve the linear integro-differential equations system with initial conditions of the Fredholm type. The numerical examples show that the accuracy improves with an increasing number of sinc grid points N.

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[1] Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili,m, P.O.Box 56199-11367, Ardabil, Iran.

E-mail address: `zarebnia@uma.ac.ir`