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SYMMETRIC MODULE AND CONNES AMENABILITY

MOHAMMAD HOSSEIN SATTARI[1*] AND HAMID SHAFIEASL[2]

Abstract. In this paper we introduce two symmetric variants of amenability, symmetric module amenability and symmetric Connes amenability. We determine symmetric module amenability and symmetric Connes amenability of some concrete Banach algebras. Indeed, it is shown that $\ell^1(S)$ is a symmetric $\ell^1(E)$-module amenable if and only if S is amenable, where S is an inverse semigroup with subsemigroup $E(S)$ of idempotents. In symmetric connes amenability, we have proved that $M(G)$ is symmetric connes amenable if and only if G is amenable.

1. Introduction

The origin of term 'amenable' for Banach algebra, was established by Johnson [4], where he proved that $L^1(G)$ is amenable if and only if G is amenable as locally compact group. The Johnson's theorem is not valid for semigroups, even discrete semigroups. In [1] Amini showed, for the new kind of amenability, module amenability, Johnson's theorem holds for inverse semigroup. Amini's work based on an extra $\mathfrak{A}$-bimodule structure on A. He also gave an equivalent conditions to module amenability in terms of a special net and an element, called module approximate diagonal and module virtual diagonal, respectively. On $M(G)$ being amenable is much stronger than just G having amenability property, as locally compact group. Indeed, $M(G)$ is amenable if and only if G is discrete and amenable [3]. There is a notation of amenability that is a much less restrictive demand, Connes amenability. This variant of amenability was introduced and investigated by Runde in [6].

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* Corresponding author.

Connes amenability takes the dual space structure into account where is defined. In [7] Runde proved $M(G)$ is connes amenable if and only if G is amenable.

The organization of the paper is as follows. In section 2 we give some basic definition related to symmetric version of module amenability and Connes amenability. Also an equivalent condition to symmetric amenability is given in terms of virtual diagonal. In Section 3 we show that the semigroup algebra $\ell^1(S)$ of an inverse semigroup S is symmetric $\ell^1(E)$-module amenable if and only if S is an amenable semigroup, where $\ell^1(S)$ is considered appropriately as a $\ell^1(E)$-bimodule. Symmetric variant of Connes amenability is investigated in Section 4 and a characterization of symmetric Connes amenability is obtained for measure algebra of a locally compact group.

2. Preliminaries

Definition 2.1. Let A be a Banach algebra, $A\hat{\otimes}A$ be the projective tensor product of A by itself and $t \in A\hat{\otimes}A$. Then there are two sequences $\{a_n\}, \{b_n\}$ in A such that $t = \sum_{n=1}^{\infty} a_n \otimes b_n$, and $\sum_{n=1}^{\infty} \|a_n\|\|b_n\| < \infty$. The operator $^\circ : A\hat{\otimes}A \to A\hat{\otimes}A$ is defined with the following rule:

$$t^\circ = \sum_{n=1}^{\infty} b_n \otimes a_n.$$

The element $t \in A\hat{\otimes}A$ is called symmetric if $t^\circ = t$.

Definition 2.2. The Banach algebras A is symmetrically amenable if it has a bounded approximate diagonal consisting of symmetric elements, i.e., a net $\{m_i\} \subseteq A\hat{\otimes}A$ with the following properties:

(i) $a\Delta_A(m_i) \to a$ for all $a \in A$;
(ii) $a \cdot m_i - m_i \cdot a \to 0$ for all $a \in A$;
(iii) $m_i^\circ = m_i$ for all i,

where the diagonal operator $\Delta_A : A\hat{\otimes}A \to A$ is defined by $\Delta_A(a \otimes b) = ab$, for all $a, b \in A$.

Let A be a Banach algera and define $a \circ b = ba$ and for all $a, b, c \in A$ define

$$a \circ (b \otimes c) = b \otimes (ac), \qquad (b \otimes c) \circ a = (ba) \otimes c.$$

Also define

$$\begin{cases} \Delta_o : A\hat{\otimes}A \to A, \\ \Delta_o(a \otimes b) = a \circ b = ba. \end{cases}$$

We use the symbol $\circ$ for dual module actions and second dual module actions of A on $(A\hat{\otimes}A)^*$ and $(A\hat{\otimes}A)^{**}$.

Theorem 2.3. *For a Banach algebra A, the following are equivalent:*

(i) *A is symmetric amenable;*
(ii) *There is a bounded net $\{m_i\} \subset A\hat{\otimes}A$ such that for all $a \in A$*

$$a \cdot m_i - m_i \cdot a \to 0, \quad a\Delta(m_i) \to a,$$
$$a \circ m_i - m_i \circ a \to 0, \quad a\Delta_o(m_i) \to a;$$

(iii) *There is an element $M \in (A\hat{\otimes}A)^{**}$ such that for all $a \in A$*

$$a \cdot M = M \cdot a, \quad a \cdot \Delta^{**}(M) = a,$$
$$a \circ M = M \circ a, \quad a \cdot \Delta_o^{**}(M) = a.$$

Proof. *(i)* $\Leftrightarrow$ *(ii)*: Proposition 2.2 of [5].
(ii) $\Rightarrow$ *(iii)*: Let M be a w*-accumulation point of $\{m_i\}$ in $(A\hat{\otimes}A)^{**}$. We can suppose that $m_i \to M$ in w*-topology. Then w*-continuity of both Δ^{**} and Δ_o^{**} and weak continuity of the following operators lead to the result;

$$\begin{aligned} b &\mapsto b \cdot (a \otimes c) : A \to A\hat{\otimes}A, \\ b &\mapsto (a \otimes c) \cdot b : A \to A\hat{\otimes}A, \\ b &\mapsto b \circ (a \otimes c) : A \to A\hat{\otimes}A, \\ b &\mapsto (a \otimes c) \circ b : A \to A\hat{\otimes}A, \qquad (a, b, c \in A). \end{aligned}$$

For instance we prove $a \circ M = M \circ a$. Let $J : A\hat{\otimes}A \to (A\hat{\otimes}A)^{**}$ be the natural embeding. We have

$$\begin{aligned} \langle f, a \circ M\rangle &= \langle f \circ a, M\rangle \\ &= \lim_i \langle f \circ a, J(m_i)\rangle \\ &= \lim_i \langle m_i, f \circ a\rangle \\ &= \lim_i \langle a \circ m_i, f\rangle \\ &= \lim_i \langle m_i \circ a, f\rangle \\ &= \lim_i \langle m_i, a \circ f\rangle \\ &= \lim_i \langle a \circ f, J(m_i)\rangle \\ &= \lim_i \langle a \circ f, M\rangle \\ &= \lim_i \langle f, M \circ a\rangle, \qquad (f \in (A\hat{\otimes}A)^*, a \in A). \end{aligned}$$

(iii) $\Rightarrow$ *(ii)*: Let M be an element of $(A\hat{\otimes}A)^{**}$ satisfying *(iii)*. By the Goldstein theorem there is a bounded net $\{m_i\} \subset A\hat{\otimes}A$ such that $m_i \to M$ in w*-topology of $(A\hat{\otimes}A)^{**}$. Since any module action on a dual

module is w*-continuous for a fixed element of algebra, we have

$$\begin{array}{lll} a\cdot\Delta^{**}(m_i)\to a\cdot\Delta^{**}(M)=a, & a\cdot\Delta_o^{**}(m_i)\to a\cdot\Delta_o^{**}(M)=a, & \\ a\cdot m_i\to a\cdot M, & a\circ m_i\to a\circ M, & \\ m_i\cdot a\to M\cdot a, & m_i\circ a\to M\circ a, & (a\in A); \end{array}$$

in the w^*-topology of $(A\hat{\otimes}A)^{**}$ and A^{**}, respectively. Therefore

$$\begin{array}{lll} a\cdot m_i-m_i\cdot a\to 0, & a\circ m_i-m_i\circ a\to 0, & (a\in A); \\ a\cdot\Delta^{**}(m_i)\to a, & a\cdot\Delta_o^{**}\to a, & \end{array}$$

in the w^*-topology of $(A\hat{\otimes}A)^{**}$ and A^{**}, respectively. For any Banach space E, the w^*-topology of E^{**} restricted to E is the weak topology. This fact together with restrcition of Δ^{**} and Δ_o^{**} enable us to get 2.3 in weak topology of $A\hat{\otimes}A$ and A, respectively. Hence there is a net of convex combinations of $\{m_i\}$ with the same bound such that limits of 2.3 holds on norm topology. □

Definition 2.4. Let $A,\mathfrak{A}$ be two Banach algebras, such that A is a Banach $\mathfrak{A}$-bimodule and

$$\alpha\cdot(ab)=(\alpha\cdot a)b,\qquad a(\alpha\cdot b)=(a\cdot\alpha)b,\qquad (\alpha\in\mathfrak{A},a,b\in A).$$

Suppose that X is a Banach A-bimodule and Banach $\mathfrak{A}$-bimodule such that

$$\alpha\cdot(a\cdot x)=(\alpha\cdot a)\cdot x,\quad a\cdot(\alpha\cdot x)=(a\cdot\alpha)\cdot x,\quad \alpha\cdot(x\cdot a)=(\alpha\cdot x)\cdot a;$$

where $a\in A,\alpha\in\mathfrak{A},x\in X$. In this case, X is called A-$\mathfrak{A}$-bimodule and if moreover $\alpha\cdot x=x\cdot\alpha$, then X is called commutative A-$\mathfrak{A}$-bimodule. A bounded map $D:A\to X$ is called module derivation if

$$\begin{array}{ll} D(a\pm b)=D(a)\pm D(b), & D(ab)=D(a)\cdot b+a\cdot D(b), \\ D(\alpha\cdot a)=\alpha\cdot D(a), & D(a\cdot\alpha)=D(a)\cdot\alpha, \end{array}$$

for all $a,b\in A,\alpha\in\mathfrak{A}$. Note that here we used term 'bounded' to mean there exists $M>0$ such that $\|D(a)\|\le M\|a\|$, for all $a\in A$. For a commutative A-$\mathfrak{A}$-bimodule X and $x\in X$, an inner module derivation is defined by

$$D_x(a)=a\cdot x-x\cdot a,\quad a\in A.$$

Definition 2.5. A is called $\mathfrak{A}$-module amenable, if for any commutative Banach A-$\mathfrak{A}$-bimodule X, each module derivation $D:A\to X^*$ is inner; where X^* has dual module structure as usual.

3. Symmetric module amenability

Let I be the closed ideal of $A\hat{\otimes}A$ generated by the elements of the form $\alpha \cdot a \otimes b - a \otimes b \cdot \alpha$ for $\alpha \in \mathfrak{A}, a, b \in A$. Set $A\hat{\otimes}_{\mathfrak{A}}A = \frac{A\hat{\otimes}A}{I}$ and let J be the closed ideal of A generated by $\Delta(I)$. Define

$$\begin{cases} \tilde{\Delta} : A\hat{\otimes}_{\mathfrak{A}}A \to \frac{A}{J}, \\ \tilde{\Delta}(a \otimes b + I) = ab + J. \end{cases}$$

Definition 3.1. An element $\tilde{M} \in (A\hat{\otimes}_{\mathfrak{A}}A)^{**} = \frac{A\hat{\otimes}A}{I^{\perp\perp}}$ is called a symmetric module virtual diagonal for A if

$$\begin{array}{ll} \tilde{\Delta}^{**}(\tilde{M}) \cdot a = \tilde{a}, & \tilde{M} \cdot a = a \cdot \tilde{M}, \\ \tilde{\Delta_o}^{**}(\tilde{M}) \cdot a = \tilde{a}, & \tilde{M} \circ a = a \circ \tilde{M}, \end{array} \qquad (a \in A) \tag{3.1}$$

where $\tilde{a} = a + J^{\perp\perp}$ and $\tilde{M} = M + I^{\perp\perp}$. In this case A is said to be symmetric module amenable.

Proposition 3.2. *The following are equivalent:*

(i) *A has a symmetric module virtual diagonal,*

(ii) *There is $M \in (A\hat{\otimes}A)^{**}$ such that*

$$\Delta^{**}(M) \cdot a - a \in J^{\perp\perp}, \qquad M \cdot a - a \cdot M \in I^{\perp\perp}, \qquad (a \in A);$$

and

$$\Delta_o^{**}(M) \cdot a - a \in J^{\perp\perp}, \qquad M \circ a - a \circ M \in I^{\perp\perp}, \qquad (a \in A).$$

Proof. $(i) \Rightarrow (ii)$: Suppose that $\tilde{M}$ is a symmetric virtual diagonal. Then $\tilde{M} = M + I^{\perp\perp}$ for some $M \in (A\hat{\otimes}A)^{**}$. Hence for all $a \in A$ we have

$$M \circ a - a \circ M + I^{\perp\perp} = \tilde{M} \circ a - a \circ \tilde{M} = \tilde{0},$$

and so, $M \circ a - a \circ M \in I^{\perp\perp}$. Also $\Delta_o^{**}(M) \cdot a - a + J^{\perp\perp} = \tilde{\Delta}_o^{**}(\tilde{M}) \cdot a - a = \tilde{0}$, consequently, $\Delta_o^{**}(M) \cdot a - a \in J^{\perp\perp}$. Similarly one can prove other parts of (ii).
$(ii) \Rightarrow (i)$: Let M be as in (ii) and define $\tilde{M} \in (A\hat{\otimes}_{\mathfrak{A}}A)^{**}$ with $\tilde{M} = M + I^{\perp\perp}$. For each $a \in A$ we have

$$\tilde{M} \circ a - a \circ \tilde{M} = M \circ a - a \circ M + I^{\perp\perp} = I^{\perp\perp},$$

and so, $\tilde{M} \circ a = a \circ \tilde{M}$. On the other hand, the equations

$$\tilde{\Delta_o}^{**}(\tilde{M}) = \tilde{\Delta_o}^{**}(M + I^{\perp\perp}) = \Delta_o^{**}(M) + J^{\perp\perp},$$

and $\Delta_o^{**}(M) \cdot a - a \in J^{\perp\perp}$ imply, $\tilde{\Delta_o}^{**}(\tilde{M}) \cdot a = a$. Use a similar way to get other conditions of (3.1) in Definition 3.1. □

Definition 3.3. A discrete semigroup S is called an inverse semigroup if for each $s \in S$ there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. Also define $E(S) = \{s \in S : \ s^2 = s^* = s\}$.

$E = E(S)$ is a subsemigroup of S and consequently $\ell^1(E)$ is a subalgebra of the convolution semigroup algebra $\ell^1(S)$. Consider the following module actions of $\ell^1(E)$ on $\ell^1(S)$

$$\delta_e \cdot \delta_s \mapsto \delta_s : \ell^1(E) \times \ell^1(S) \to \ell^1(S),$$
$$\delta_s \cdot \delta_e \mapsto \delta_{se} : \ell^1(S) \times \ell^1(E) \to \ell^1(S),$$

which are compatible.

Remark 3.4. In the forth comming theorem we use the following notations and module actions:

$$r \circ (\delta_s \otimes \delta_t) := (\delta_s \otimes \delta_{rt}) : \ell^1(S) \times \ell^1(S \times S) \to \ell^1(S \times S),$$
$$(\delta_s \otimes \delta_t) \circ r := (\delta_{sr} \otimes \delta_t) : \ell^1(S \times S) \times \ell^1(S) \to \ell^1(S \times S),$$
$$(r \circ f)(s,t) := f(sr,t) : \ell^1(S) \times \ell^\infty(S \times S) \to \ell^\infty(S \times S),$$
$$(f \circ r)(s,t) := f(s,rt) : \ell^\infty(S \times S) \times \ell^1(S) \to \ell^\infty(S \times S),$$
$$(r \circ M)(f) := M(f \circ r) : \ell^1(S) \times (\ell^\infty(S \times S))^* \to (\ell^\infty(S \times S))^*,$$
$$(M \circ r)(f) := M(r \circ f) : (\ell^\infty(S \times S))^* \times \ell^1(S) \to (\ell^\infty(S \times S))^*;$$

where $r, s, t \in S, f \in \ell^\infty(S \times S), M \in (\ell^\infty(S \times S))^*$ and r is used instead of $\delta_r \in \ell^1(S)$. Similarly $r \cdot (\delta_s \otimes \delta_t) := (\delta_{rs} \otimes \delta_t)$, $(\delta_s \otimes \delta_t) \cdot r := (\delta_s \otimes \delta_{tr})$, their dual actions $r \cdot f, f \cdot r$ and second dual actions $r \cdot M, M \cdot r$ are defined. Note that we use the standard identification $\ell^1(S)\hat{\otimes}\ell^1(S) = \ell^1(S \times S)$.

Definition 3.5. Let S be a discrete inverse semigroup and define

$$ft(s) = f(ts), \qquad tf(s) = f(st), \qquad (s,t \in S, f \in \ell^\infty(S)).$$

A continuous linear functional $\mu : \ell^\infty(S) \to \mathbb{C}$ is said to be left (right) invariant if, for all $f \in \ell^\infty(S), t \in S$,

$$\mu(ft) = \mu(f), \qquad (\mu(tf) = \mu(f)).$$

μ is called invariant if it is both left and right invariant. The semigroup S is called amenable if it has an invariant μ such that $\mu(\mathbf{1}) = \|\mu\| = 1$, where $\mathbf{1}$ is the constant unit function in $\ell^\infty(S)$, and μ is called an *invariant mean*, [2].

Theorem 3.6. *Let S be an amenable inverse semigroup. Then $\ell^1(S)$ is symmetric $\ell^1(E)$-module amenable where $E = E(S)$.*

Proof. Let μ be an invariant mean on S. If we define

$$\begin{cases} M : \ell^\infty(S \times S) \to \mathbb{C}, \\ M(f) = \displaystyle\int_S f(s^*, s) d\mu(s), \end{cases}$$

then M is a bounded linear functional and for the constant function 1, $M(1 \otimes 1) = \|\mu\| = 1$ holds. For each $s \in S$ and $f \in \ell^\infty(S \times S)$ we have

$$\begin{aligned}
s \circ M(f) &= M(f \circ s) \\
&= \int_S f(t^*, st) d\mu(t) \\
&= \int_S f((s^*st)^*, s(s^*st)) d\mu(t) \\
&= \int_S f((s^*st)^*, st) d\mu(t) \\
&= \int_S f((s^*t)^*, t) d\mu(t) \\
&= \int_S f(t^*s, t) d\mu(t) \\
&= M(s \circ f) = M \circ s(f).
\end{aligned}$$

Suppose that $s \in S$ and $g \in J^\perp \subseteq \ell^\infty(S)$. Then

$$\begin{aligned}
\tilde{\Delta}^{**}(M) \cdot s(g) &= \tilde{\Delta}^{**}(M)(g \cdot s) \\
&= M(\tilde{\Delta}^{**}(g \cdot s)) \\
&= \int_S \tilde{\Delta}^*(g \cdot s)(t^*, t) d\mu(t) \\
&= \int_S g \cdot s(tt^*) d\mu(t) \\
&= \int_S g(stt^*) d\mu(t) \\
&= \int_S g(s) d\mu(t) \\
&= g(s) \int_S d\mu(t) = g(s).
\end{aligned}$$

□

Corollary 3.7. *Let S be an inverse semigroup. Then S is amenable if and only if $\ell^1(S)$ is $\ell^1(E)$-module amenable if and only if $\ell^1(S)$ is symmetric $\ell^1(E)$-module amenable.*

Proof. Since symmetric module amenability implies module amenability the conclusion obtaines by Theorem 3.1 of [1] and the previous theorem. □

4. Symmetric Connes amenability

Dual Banach algebras form a special class of Banach algebras which includes the von Neumann algebras, and the measure algebra of a locally compact group. The concept of Connes amenability, defined by Runde in [7], is a version of amenability which is compatible with the structure of dual Banach algebras.
Let A be a Banach algebra. A is called a dual Banach algebra if there is a closed A-submodule A_* of A^* such that $A = (A_*)^*$. In this case, a dual Banach A-bimodule E is called *normal* if, for each $x \in E$, the maps

$$A \to E, \quad \begin{cases} a \mapsto x \cdot a, \\ a \mapsto a \cdot x, \end{cases}$$

are w*-continuous.
A dual Banach algebra A is *Connes-amenable* if, for every normal and dual Banach A-bimodule E, every w*-continuous derivation from A into E is inner.

Definition 4.1. Let B be a dual Banach algebra. Define $\sigma wc(B\hat{\otimes}B)$ to be the set of all $x \in B\hat{\otimes}B$ such that $b \mapsto x \cdot b$, $b \mapsto b \cdot x$, $b \mapsto x \circ b$ and $b \mapsto b \circ x$ be weak*-weak continuous from B into $B\hat{\otimes}B$. Similarly define $\sigma wc((B\hat{\otimes}B)^*)$ to be the set of all $f \in (B\hat{\otimes}B)^*$ such that $b \mapsto f \cdot b$, $b \mapsto b \cdot f$, $b \mapsto f \circ b$ and $b \mapsto b \circ f$ be weak*-weak continuous from B into $(B\hat{\otimes}B)^*$.

Remark 4.2. $\sigma wc(B\hat{\otimes}B)$ and $\sigma wc((B\hat{\otimes}B)^*)$ are respectively closed submodules of $B\hat{\otimes}B$ and $(B\hat{\otimes}B)^*$ with respect to both bimodule action. As before, let $\Delta : B\hat{\otimes}B \to B$ is defined by $\Delta(a \otimes b) = ab$, and let B_* be the predual of B, then $\Delta^*(B_*) \subset \sigma wc((B\hat{\otimes}B)^*)$ by Corollary 4.6 of [7]. Define $\Delta_{\sigma wc} = (\Delta^*|_{B_*})^* : \sigma wc((B\hat{\otimes}B)^*)^* \to B$. By considering $B\hat{\otimes}B \subset \sigma wc((B\hat{\otimes}B)^*)$ we can say that $\Delta_{\sigma wc}$ is an extension of Δ to $\sigma wc((B\hat{\otimes}B)^*)^*$. In an analogue way we can define $\Delta_{o\delta wc} = (\Delta_o)_{\delta wc}$.

Definition 4.3. The dual Banach algebra B is said to be *symmetrically Connes amenable* if there is $M \in \sigma wc((B\hat{\otimes}B)^*)^*$ such that for all $b \in B$ we have

$$\begin{aligned} b \cdot M = M \cdot b, &\quad b \cdot \Delta_{\sigma wc}(M) = b, \\ b \circ M = M \circ b, &\quad b \cdot \Delta_{o\sigma wc}(M) = b. \end{aligned}$$

Theorem 4.4. *Let A be a symmetric amenable Banach algebra, B be a dual Banach algebra and $\theta : A \to B$ be a continuous homomorphism such that $\theta(A)$ is w*-dense in B. Then B is symmetrically Connes amenable.*

Proof. Since A is symmetrically amenable there is $M \in (A\hat{\otimes}A)^{**}$ such that for all $a \in A$,

$$a \cdot M = M \cdot a, \qquad a \cdot \Delta^{**}(M) = a,$$
$$a \circ M = M \circ a, \qquad a \cdot \Delta_o^{**}(M) = a.$$

Define $\lambda = \theta \otimes \theta : A\hat{\otimes}A \to B\hat{\otimes}B$ by $\theta \otimes \theta(a \otimes b) = \theta(a) \otimes \theta(b)$. Set $M_0 = \lambda^{**}(M)|_{\sigma wc(B\hat{\otimes}B)^*}$. Let $f \in \sigma wc(B\hat{\otimes}B)^*$ and $b \in B$. By assumption there is a net $\{a_\alpha\}$ in A such that $\theta(a_\alpha) \to b$ in the w*-topology of B. First we show that $\lambda^*(f \cdot \theta(a)) = \lambda^*(f) \cdot a$ $(a \in A)$. For, let $x, y \in A$, then

$$\begin{aligned}
\langle x \otimes y, \lambda^*(f \cdot \theta(a))\rangle &= \langle \lambda(x \otimes y), f \cdot \theta(a)\rangle \\
&= \langle \theta(x) \otimes \theta(y), f \cdot \theta(a)\rangle \\
&= \langle \theta(a)\theta(x) \otimes \theta(y), f\rangle \\
&= \langle \theta(ax) \otimes \theta(y), f\rangle \\
&= \langle \lambda(ax \otimes y), f\rangle \\
&= \langle ax \otimes y, \lambda^*(f)\rangle \\
&= \langle x \otimes y, \lambda^*(f) \cdot a\rangle.
\end{aligned}$$

We have

$$\begin{aligned}
\langle f, b \cdot M_0\rangle &= \langle f \cdot b, M_0\rangle \\
&= \lim_\alpha \langle f \cdot \theta(a_\alpha), M_0\rangle \\
&= \lim_\alpha \langle f \cdot \theta(a_\alpha), \lambda^{**}(M)\rangle \\
&= \lim_\alpha \langle \lambda^*(f \cdot \theta(a_\alpha)), M\rangle \\
&= \lim_\alpha \langle \lambda^*(f) \cdot a_\alpha, M\rangle \\
&= \lim_\alpha \langle \lambda^*(f), a_\alpha \cdot M\rangle \\
&= \lim_\alpha \langle \lambda^*(f), M \cdot a_\alpha\rangle \\
&= \langle f, M_0 \cdot b\rangle,
\end{aligned}$$

thus $b \cdot M_0 = M_0 \cdot b$. If also $u \in B_* \subseteq B^*$ then

$$\begin{aligned}
\langle u, b \cdot \Delta_{\sigma wc}(M_0)\rangle &= \langle u \cdot b, \Delta^{**}(M_0)\rangle \\
&= \langle \Delta^*(u \cdot b), M_0\rangle \\
&= \langle \Delta^*(u) \cdot b, M_0\rangle \\
&= \lim_\alpha \langle \Delta^*(u) \cdot \theta(a_\alpha), M_0\rangle \\
&= \lim_\alpha \langle \Delta^*(u) \cdot \theta(a_\alpha), \lambda^{**}(M)\rangle \\
&= \lim_\alpha \langle \lambda^*(\Delta^*(u) \cdot \theta(a_\alpha)), M\rangle \\
&= \lim_\alpha \langle \lambda^*(\Delta^*(u)) \cdot a_\alpha, M\rangle \\
&= \lim_\alpha \langle \lambda^*(\Delta^*(u)), a_\alpha \cdot M\rangle \\
&= \lim_\alpha \langle \Delta_A^*(\theta^*(u)), a_\alpha \cdot M\rangle \\
&= \lim_\alpha \langle \theta^*(u), \Delta_A^{**}(a_\alpha \cdot M)\rangle \\
&= \lim_\alpha \langle \theta^*(u), a_\alpha \cdot \Delta_A^{**}(M)\rangle \\
&= \lim_\alpha \langle \theta^*(u), a_\alpha\rangle \\
&= \lim_\alpha \langle u, \theta(a_\alpha)\rangle \\
&= \langle u, b\rangle,
\end{aligned}$$

thus $b \cdot \Delta_{\sigma wc}(M_0) = b$. Similarly $b \circ M_0 = M_0 \circ b$ and $b \cdot \Delta_{o\sigma wc}(M_0) = b$. □

Theorem 4.5. *Let G be a locally compact group. Then $M(G)$ is symmetrically Connes amenable if and only if G is amenable.*

Proof. It is proved in Theorem 4.1 of [5], that G is amenable if and only if $L^1(G)$ is symmetrically amenable. Also we know that $L^1(G)$ is $\sigma(M(G), C_0(G))$-dense in $M(G)$. Hence from Theorem 4.4 we get the result. □

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[1] Department of Mathematics, Faculty of Science, Azarbaijan Shahid Madani University, P.O.Box 53751-71379, Tabriz, Iran.

E-mail address: sattari@azaruniv.ac.ir

[2] Department of Mathematics, Faculty of Science, Azarbaijan Shahid Madani University, P.O.Box 53751-71379, Tabriz, Iran.

E-mail address: h.shafieasl@azaruniv.ac.ir