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SOLUTION OF NONLINEAR VOLTERRA-HAMMERSTEIN INTEGRAL EQUATIONS USING ALTERNATIVE LEGENDRE COLLOCATION METHOD

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Abstract. Alternative Legendre polynomials (ALPs) are used to approximate the solution of a class of nonlinear Volterra-Hammerstein integral equations. For this purpose, the operational matrices of integration and the product for ALPs are derived. Then, using the collocation method, the considered problem is reduced into a set of nonlinear algebraic equations. The error analysis of the method is given and the efficiency and accuracy are illustrated by applying the method to some examples.

1. Introduction

In this study, we consider the nonlinear Volterra-Hammerstein integral equations of the form

$$u(x) = f(x) + \int_0^x k(x,t) N\big(u(t)\big) dt, \quad x \in I := [0,1], \tag{1.1}$$

where $u(x)$ is an unknown real valued function and $f(x)$ and $k(x,t)$ are given continuous functions defined, respectively on I and $I \times I$, and $N\big(u(x)\big)$ is a polynomials of $u(x)$ with constant coefficients. The existence and uniqueness of the solution of Eq. (1.1) are discussed in [4, 18].

Integral equations of type (1.1) arise in the mathematical modelling of parabolic boundary value problems and in various areas of physics, engineering and biological sciences [2, 8, 9, 21].

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To obtain the numerical solution of integral equation (1.1), several approaches have been proposed. For example, two methods based on the use of Haar wavelets are presented in [1, 19]. Chebyshev collocation method is applied in [22] to get the solution in the domain $[-1, 1]$. Two methods based on the operational matrices of Bernstein and Bernoulli polynomials, respectively proposed in [15] and [3]. Also, a collocation method based on the Legendre polynomials is proposed in [16]. In this paper, we use the alternative Legendre polynomials to approximate the solution of (1.1). These polynomials have recently created by Vladimir Chelyshkov [6, 7], which are orthogonal over the interval $[0, 1]$ with respect to the weight function $w(x) = 1$.

For convenience, we assume that

$$N(u(x)) = u^m(x), \tag{1.2}$$

where m is a positive integer, but the method can be easily extended and applied to any nonlinear VHIE of the form (1.1), where $N\big(u(x)\big)$ is a polynomial of $u(x)$ with constant coefficients.

The reminder of this paper is organized as follows. A review on ALPs and their properties is given in Section 2.1. Some bounds on the error of approximation of smooth functions by ALPs in L_2 norm is provided in Section 2.2. In Section 3, the operational matrices of integration and the product for ALPs is derived. In Section 4, the derived operational matrices together with the collocation method are applied to reduce the solution of integral equation (1.1) with assumption (1.2) to the solution of a nonlinear system of algebraic equations. We also show in this section that the ALP coefficient vector of function $u^m(x)$ can be computed in terms of the ALP coefficient vector of $u(x)$. A number of numerical examples are presented in Section 5, to demonstrate the efficiency and accuracy of the proposed method. Finally, conclusions are given in Section 6.

2. Alternative legendre polynomials

In this section, we briefly introduce ALPs and give a number of theirs properties. For more details, the reader is referred to [6, 7].

2.1. Definition and properties.

Let n be a fixed integer number. The set $\boldsymbol{\mathcal{P}}_n = \{\mathcal{P}_{nl}(x)\}_{l=0}^{n}$ of alternative Legendre polynomials were explicitly introduced [6] as

$$\mathcal{P}_{nl}(x) = \sum_{j=0}^{n-l} (-1)^j \binom{n-l}{j} \binom{n+l+j+1}{n-l} x^{l+j}, \quad l = 0, 1, \ldots, n. \tag{2.1}$$

It is easy to see that, in contrast to common sets of orthogonal polynomials, every member in $bm\mathcal{P}_n$ has degree n.

Relation (2.1) yields Rodrigues's type representation,

(2.2)
$$\mathcal{P}_{nl}(x) = \frac{1}{(n-l)!}\frac{1}{x^{l+1}}\frac{d^{n-l}}{dx^{n-l}}\left(x^{n+l+1}(1-x)^{n-l}\right), \quad l = 0, 1, \ldots, n.$$

It follows from (2.2) that

$$\int_0^1 \mathcal{P}_{nl}(x)dx = \frac{1}{n+1}, \quad l = 0, 1, \ldots, n. \tag{2.3}$$

The set $\boldsymbol{\mathcal{P}}_n = \{\mathcal{P}_{nl}(x)\}_{l=0}^n$ of alternative Legendre polynomials are orthogonal over the interval $[0, 1]$ with respect to the weight function $w(x) = 1$, i.e. they satisfy the orthogonality relationships

$$\int_0^1 \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)dx = \frac{1}{i+j+1}\delta_{ij}, \quad i, j = 0, 1, \ldots, n,$$

where δ_{ij} denotes the Kronecker delta.

Let T denotes the transpose of a matrix or vector. By using Eq. (2.1), the ALP vector

$$\Phi(x) = [\mathcal{P}_{n0}(x), \mathcal{P}_{n1}(x), \ldots, \mathcal{P}_{nn}(x)]^T, \tag{2.4}$$

can be written in the form

$$\Phi(x) = QX(x),$$

where

$$X(x) = [1, x, x^2, \ldots, x^n]^T,$$

and Q is the upper triangular matrix defined by

$$Q = [q_{kj}]_{k,j=0}^n, \qquad q_{kj} = \begin{cases} 0, & 0 \le j < k, \\ (-1)^{j-k}\binom{n-k}{j-k}\binom{n+j+1}{n-k}, & k \le j \le n. \end{cases}$$

The dual matrix of $\Phi(x)$ is a diagonal matrix defined by

$$\begin{aligned} D &= \int_0^1 \Phi(x)\Phi^T(x)dx \\ &= \operatorname{diag}\left\{\frac{1}{2i+1}\right\}_{i=0}^n. \end{aligned} \tag{2.5}$$

The ALPs also satisfy the three-term recurrence relation [6]

$$\mathcal{P}_{nn}(x) = x^n,$$
$$\mathcal{P}_{n,n-1}(x) = 2nx^{n-1} - (2n+1)x^n,$$
$$a_{nk}\mathcal{P}_{n,k-1}(x) = (b_{nk}x^{-1} - c_{nk})\mathcal{P}_{nk}(x) - d_{nk}\mathcal{P}_{n,k+1}(x),$$

where

$$a_{nk} = (k+1)(n-k+1)(n+k+1),$$
$$b_{nk} = k(2k+1)(2k+2),$$
$$c_{nk} = (2k+1)((n+1)^2 + k^2 + k),$$
$$d_{nk} = k(n-k)(n+k+2).$$

The ALPs have properties, which are analogues to the properties of common orthogonal polynomials. These polynomials are not solutions of the equation of the hypergeometric type, but they can be expressed in terms of the Jacobi polynomials $P_i^{(\alpha,\beta)}(x)$ by the following relation

$$\mathcal{P}_{nl}(x) = (-1)^{n-l}x^l P_{n-l}^{(0,2l+1)}(2x-1), \quad l = 0, 1, \dots, n. \tag{2.6}$$

Hence, the ALPs are related to different families of the Jacobi polynomials and keep distinctively all the attributes of regular orthogonal polynomials.

Properties of the zeros of the Jacobi polynomials [20] and relation (2.6) give the following result.

Corollary 2.1 ([6])**.** *Polynomials $\mathcal{P}_{nl}(x)$ have k multiple zeros $x = 0$ and $n - l$ distinct real zeros in the interval* $[0, 1]$.

By the above corollary, $\mathcal{P}_{n0}(x)$ has exactly n simple roots in $[0, 1]$.

2.2. **Function approximations and error estimates.** Let $H = L^2(I)$ be the space of square integrable functions with respect to Lebesgue measure on the closed interval $I = [0, 1]$. The inner product in this space is defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx,$$

and the norm is as follows

$$\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}} = \Big(\int_0^1 f^2(x)dx\Big)^{\frac{1}{2}}.$$

Let

$$H_n = \text{span}\{\mathcal{P}_{n0}(x), \mathcal{P}_{n1}(x), \dots, \mathcal{P}_{nn}(x)\}.$$

Since H_n is a finite dimensional subspace of H, then it is closed [13, Theorem 2.4-3] and for every given $f \in H$ there exist a unique best approximation $\bar{f} \in H_n$ [13, Theorem 6.2-5] such that

$$\|f - \bar{f}\|_2 \le \|f - g\|_2, \quad \forall g \in H_n.$$

Moreover, we have [10, Theorem 4.14]

$$\bar{f}(x) = \sum_{k=0}^{n} f_k \mathcal{P}_{nk}(x),$$

where

$$\begin{aligned} f_k &= \frac{\langle f, \mathcal{P}_{nk}\rangle}{\langle \mathcal{P}_{nk}, \mathcal{P}_{nk}\rangle} \\ &= (2k+1)\langle f, \mathcal{P}_{nk}\rangle, \quad k = 0, 1, \dots, n. \end{aligned} \tag{2.7}$$

Therefore, any function $f(x) \in H$ may be approximated in terms of ALP basis as

$$f(x) \simeq \bar{f}(x) = \Phi^T(x)F, \tag{2.8}$$

where

$$F = [f_0, f_1, \dots, f_n]^T,$$

is the ALP coefficient vector of f. Also, any function $k(x,t) \in L^2(I \times I)$ can be similarly expanded in terms of ALP basis as

$$k(x,t) \simeq \bar{k}(x,t) = \Phi^T(x)K\Phi(t), \tag{2.9}$$

where $K = [k_{ij}]_{i,j=0}^{n}$ is a $(n+1) \times (n+1)$ matrix in which the ALP coefficients k_{ij} are given by

$$\begin{aligned} k_{ij} &= \frac{\langle\langle k, \mathcal{P}_{ni}\rangle, \mathcal{P}_{nj}\rangle}{\langle \mathcal{P}_{ni}, \mathcal{P}_{ni}\rangle\langle \mathcal{P}_{nj}, \mathcal{P}_{nj}\rangle} \\ &= (2i+1)(2j+1)\langle\langle k, \mathcal{P}_{ni}\rangle, \mathcal{P}_{nj}\rangle, \quad i, j = 0, 1, \dots, n. \end{aligned} \tag{2.10}$$

By the same argument as done for univariate functions, $\bar{k}(x,t)$ is the unique best approximation to $k(x,t)$ in the space

$$H_n^2 = \text{span}\{\mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x), i, j = 0, 1, \dots, n\}.$$

Now, we obtain an estimation of the error norm of the best approximation of smooth functions of one and two variables, respectively by some polynomials in H_n and H_n^2. For this purpose, let us denote by $\mathcal{C}^m(I)$ the space of functions $f : I \to \mathbb{R}$ with continuous derivatives

$$f^{(i)}(x) = \frac{d^i}{dx^i} f(x), \quad x \in I,$$

for all i such that $0 \le i \le m$ and by $\mathcal{C}^{m,n}(I \times I)$ the space of functions $f : I \times I \to \mathbb{R}$ with continuous partial derivatives

$$f^{(i,j)}(x,t) = \frac{\partial^{i+j}}{\partial x^i \partial t^j} f(x,t), \quad (x,t) \in I \times I,$$

for all (i,j) such that $0 \le i \le m, 0 \le j \le n$.

Theorem 2.2. *Suppose that $f(x) \in \mathcal{C}^n(I)$ and $\bar{f}(x) = \Phi^T(x)F$ be its expansion in terms of ALPs, as described by Eq.* (2.8)*. Then*

$$\|f - \bar{f}\|_2 \le \frac{c}{(n+1)!2^{2n+1}}, \tag{2.11}$$

where c is a constant such that

$$|f^{(n+1)}(x)| \le c, \quad x \in I.$$

Proof. Assume that $p_n(x)$ is the interpolating polynomial to f at points x_l, where $x_l, l = 0, 1, \dots, n$ are the roots of the $(n+1)$-degree shifted Chebyshev polynomial in I. Then, for any $x \in I$, we have [12]

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \prod_{l=0}^{n} (x - x_l), \quad \xi_x \in I.$$

By taking into account the estimates for Chebyshev interpolation nodes [12], we obtain

$$|f(x) - p_n(x)| \le \frac{c}{(n+1)!2^{2n+1}}, \quad \forall x \in I.$$

As shown in Section 2.2, $\bar{f}$ is the unique best approximation of f in H_n, then we have

$$\begin{aligned} \|f - \bar{f}\|_2^2 \le \|f - p_n\|_2^2 &= \int_0^1 |f(x) - p_n(x)|^2 dx \\ &\le \int_0^1 \left(\frac{c}{(n+1)!2^{2n+1}} \right)^2 dx \\ &= \left(\frac{c}{(n+1)!2^{2n+1}} \right)^2. \end{aligned} \tag{2.12}$$

Taking the square root of both sides of (2.12) gives the result. □

Theorem 2.3. *Suppose that $k(x,t) \in \mathcal{C}^{n,n}(I \times I)$ and $\bar{k}(x,t) = \Phi^T(x) K \Phi(t)$ be its expansion in terms of ALPs, as described by Eq.* (2.9)*. Then*

$$\|k - \bar{k}\|_2 \le \frac{1}{(n+1)!2^{2n+1}} \left(c_1 + c_2 + \frac{c_3}{(n+1)!2^{2n+1}} \right),$$

where c_1, c_2, c_3 *are constants such that*

$$\max_{(x,t)\in I\times I}\left|\frac{\partial^{n+1}k(x,t)}{\partial x^{n+1}}\right| \le c_1,$$
$$\max_{(x,t)\in I\times I}\left|\frac{\partial^{n+1}k(x,t)}{\partial t^{n+1}}\right| \le c_2,$$
$$\max_{(x,t)\in I\times I}\left|\frac{\partial^{2n+2}k(x,t)}{\partial x^{n+1}\partial t^{n+1}}\right| \le c_3.$$

Proof. Assume that $p_{n,n}(x,t)$ is the interpolating polynomial to $k(x,t)$ at points (x_l, t_m), where $x_l = t_l, l = 0, 1, \dots, n$ are the roots of the $(n+1)$-degree shifted Chebyshev polynomial in I. Then, for any $(x,t) \in I \times I$, we have [11]

$$\begin{aligned} k(x,t) - p_{n,n}(x,t) &= \frac{\partial^{n+1}k(\xi,t)}{\partial x^{n+1}(n+1)!}\prod_{l=0}^{n}(x-x_l) \\ &+ \frac{\partial^{n+1}k(x,\eta)}{\partial t^{n+1}(n+1)!}\prod_{m=0}^{n}(t-t_m) \\ &- \frac{\partial^{2n+2}k(\xi',\eta')}{\partial x^{n+1}\partial t^{n+1}(n+1)!^2}\prod_{l=0}^{n}(x-x_l)\prod_{m=0}^{n}(t-t_m), \end{aligned}$$

where $\xi, \eta, \xi', \eta' \in [0,1]$. Therefore, by taking into account the estimates for Chebyshev interpolation nodes [12], we obtain

$$|k(x,t) - p_{n,n}(x,t)| \le \frac{1}{(n+1)!2^{2n+1}}\left(c_1 + c_2 + \frac{c_3}{(n+1)!2^{2n+1}}\right),$$

The rest of the proof proceeds in a similar fashion to that of Theorem 2.2 □

Later, in Section 5, we will need the following theorem to evaluate the distance between the best approximation u_n of the exact solution u in H_n and the computed solution $\bar{u}_n$.

Theorem 2.4. *Let*

$$g(x) = \sum_{l=0}^{n} g_l \mathcal{P}_{nl}(x),$$

and

$$h(x) = \sum_{l=0}^{n} h_l \mathcal{P}_{nl}(x),$$

be two functions of space H_n. *Then*

$$\|g-h\|_2 \le \alpha_n \|G-H\|_2, \qquad \alpha_n = \left(\sum_{l=0}^{n} \frac{1}{2l+1}\right)^{\frac{1}{2}}, \tag{2.13}$$

in which G *and* H *are* $(n+1)$*-vectors with elements* g_l *and* h_l*, respectively. The norm on the right hand side is the usual Euclidean norm for vectors.*

Proof. We can write

$$\begin{aligned} \|g-h\|_2^2 &= \int_0^1 |g(x)-h(x)|^2\,dx \qquad (2.14)\\ &= \int_0^1 \Big|\sum_{l=0}^{n}(g_l-h_l)\mathcal{P}_{nl}(x)\Big|^2 dx \\ &\le \int_0^1 \left(\sum_{l=0}^{n}|g_l-h_l|^2\right)\left(\sum_{l=0}^{n}|\mathcal{P}_{nl}(x)|^2\right)dx \\ &= \left(\sum_{l=0}^{n}|g_l-h_l|^2\right)\left(\sum_{l=0}^{n}\int_0^1 |\mathcal{P}_{nl}(x)|^2 dx\right) \\ &= \|G-H\|_2^2 \left(\sum_{l=0}^{n}\frac{1}{2l+1}\right). \qquad (2.15) \end{aligned}$$

Taking the square root of both sides of (2.14) gives the result. □

The previous theorem can be extended to the 2-dimensional case as follows.

Theorem 2.5. *Let*

$$k(x,t) = \sum_{i,j=0}^{n} k_{ij}\mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x),$$

and

$$s(x,t) = \sum_{i,j=0}^{n} s_{ij}\mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x),$$

be two functions of space H_n^2. *Then*

$$\|k-s\|_2 \le \alpha_n^2 \|K-S\|_F,$$

where K *and* S *are* $(n+1)\times(n+1)$ *matrices with entries* k_{ij} *and* s_{ij}*, respectively and* α_n *is the value defined by* (2.13)*. The norm on the right hand side is the usual Frobenius norm for matrices.*

Proof. The proof proceeds in a similar fashion to that of Theorem 2.4. □

3. ALP operational matrices of integration and the product

In this section, we derive the ALP operational matrices . We need the following two lemmas, which we state without proof since the proofs are easy.

Lemma 3.1. *Let*

$$\mathcal{P}_{nk}(x) = \sum_{r=0}^{n} p_r^{(k)} x^r,$$

and

$$\mathcal{P}_{nj}(x) = \sum_{r=0}^{n} p_r^{(j)} x^r,$$

are kth and jth ALP polynomials, respectively. Then the product of $\mathcal{P}_{nk}(x)$ and $\mathcal{P}_{nj}(x)$ is a polynomial of degree at most $2n$ that can be written as

$$\mathcal{Q}_{2n}^{(k,j)} = \mathcal{P}_{nk}(x)\mathcal{P}_{nj}(x) = \sum_{r=0}^{2n} q_r^{(k,j)} x^r,$$

where

$$q_r^{(k,j)} = \begin{cases} \displaystyle\sum_{l=0}^{r} p_l^{(k)} p_{r-l}^{(j)}, & r \leq n, \\ \displaystyle\sum_{l=r-n}^{n} p_l^{(k)} p_{r-l}^{(j)}, & r > n. \end{cases}$$

Lemma 3.2. *Let r be a nonnegative integer. Then*

$$\int_0^1 x^r \mathcal{P}_{nk}(x)dx = \sum_{l=0}^{n-k} \frac{(-1)^l \binom{n-k}{l}\binom{n+k+l+1}{n-k}}{k+l+r+1}, \quad k = 0, 1, \ldots, n.$$

By combining Lemmas 3.1 and 3.2 we can easily obtain the following result.

Lemma 3.3. *Let $\mathcal{P}_{ni}(x)$, $\mathcal{P}_{nj}(x)$ and $\mathcal{P}_{nk}(x)$ are ith, jth and kth ALP polynomials, respectively. If, according to Lemma 3.1, we define*

$$\mathcal{P}_{nk}(x)\mathcal{P}_{nj}(x) = \sum_{r=0}^{2n} q_r^{(k,j)} x^r,$$

then

$$\int_0^1 \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)\mathcal{P}_{nk}(x)dx = \sum_{r=0}^{2n} q_r^{(k,j)} \sum_{l=0}^{n-i} \frac{(-1)^l \binom{n-i}{l}\binom{n+i+l+1}{n-i}}{i+l+r+1}.$$

In the next theorem, the ALP operational matrix of integration is first derived.

Theorem 3.4. *Let* $\Phi(x)$ *be the ALP vector defined by* (2.4). *Then*

$$\int_0^x \Phi(t)dt \simeq P\Phi(x), \tag{3.1}$$

where $P = [p_{kr}]_{k,r=0}^n$ *is the ALP operational matrix of integration of order* $(n+1)\times(n+1)$, *in which*

$$p_{kr} = (2r+1)\sum_{j=0}^{n-k}\frac{(-1)^j\binom{n-k}{j}\binom{n+k+j+1}{n-k}}{k+j+1}\sum_{l=0}^{n-r}\frac{(-1)^l\binom{n-r}{l}\binom{n+r+l+1}{n-r}}{k+r+l+j+2}.$$

Proof. By integrating of $\mathcal{P}_{nk}(t)$, i.e. Eq. (2.1), from 0 to x we get

$$\int_0^x \mathcal{P}_{nk}(t)dt = \sum_{j=0}^{n-k}\frac{(-1)^j\binom{n-k}{j}\binom{n+k+j+1}{n-k}x^{k+j+1}}{k+j+1}.$$

Now, using Eq. (2.8), one can approximate x^{k+j+1} in terms of ALPs as

$$x^{k+j+1} \simeq \sum_{r=0}^{n} b_{kjr}\mathcal{P}_{nr}(x),$$

where, by Eq. (2.7) and Lemma 3.2, we have

$$\begin{aligned} b_{kjr} &= (2r+1)\int_0^1 x^{k+j+1}\mathcal{P}_{nr}(x)dx \\ &= (2r+1)\sum_{l=0}^{n-r}\frac{(-1)^l\binom{n-r}{l}\binom{n+r+l+1}{n-r}}{k+r+l+j+2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^x \mathcal{P}_{nk}(t)dt &\simeq \sum_{r=0}^{n}(2r+1)\Bigg[\sum_{j=0}^{n-k}\frac{(-1)^j\binom{n-k}{j}\binom{n+k+j+1}{n-k}}{k+j+1} \\ &\quad \sum_{l=0}^{n-r}\frac{(-1)^l\binom{n-r}{l}\binom{n+r+l+1}{n-r}}{k+r+l+j+2}\Bigg]\mathcal{P}_{nr}(x) \\ &= \sum_{r=0}^{n} p_{kr}\mathcal{P}_{nr}(x). \end{aligned}$$

Varying k from 0 to n gives the result. □

In the next theorem, we introduce the ALP operational matrix of the product. This matrix is of great importance in the next section where we reduce the solution of integral equation (1.1) to the solution of a set of algebraic equations.

Theorem 3.5. *Let $V = [v_0, v_1, \ldots, v_n]^T$ be an arbitrary vector in $\mathbb{R}^{n+1}$. Then*

$$\Phi(x)\Phi^T(x)V \simeq \hat{V}\Phi(x), \tag{3.2}$$

where $\hat{V} = [\hat{v}_{ik}]_{i,k=0}^{n}$ is the ALP operational matrix of the product in which

$$\hat{v}_{ik} = (2k+1)\sum_{j=0}^{n} v_j \varrho_{ijk}, \qquad \varrho_{ijk} = \int_0^1 \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)\mathcal{P}_{nk}(x)dx. \tag{3.3}$$

The values ϱ_{ijk} in (3.3) *are computed by means of Lemma 3.3.*

Proof. We have

$$\Phi(x)\Phi^T(x)V = \begin{bmatrix} \sum_{j=0}^{n} v_j \mathcal{P}_{n0}(x)\mathcal{P}_{nj}(x) \\ \sum_{j=0}^{n} v_j \mathcal{P}_{n1}(x)\mathcal{P}_{nj}(x) \\ \vdots \\ \sum_{j=0}^{n} v_j \mathcal{P}_{nn}(x)\mathcal{P}_{nj}(x) \end{bmatrix}. \tag{3.4}$$

Using Eq. (2.8), one can approximate $\mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)$, for $i, j = 0, 1, \ldots, n$, in terms of ALPs as

$$\mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x) \simeq \sum_{k=0}^{n} a_{ijk}\mathcal{P}_{nk}(x),$$

where, by Eq. (2.7), we obtain

$$\begin{aligned} a_{ijk} &= (2k+1)\int_0^1 \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x)\mathcal{P}_{nk}(x)dx \\ &= (2k+1)\varrho_{ijk}. \end{aligned}$$

So, for every $i = 0, 1, \dots, n$, we have

$$\begin{aligned} \sum_{j=0}^{n} v_j \mathcal{P}_{ni}(x)\mathcal{P}_{nj}(x) &\simeq \sum_{j=0}^{n} v_j \sum_{k=0}^{n} (2k+1)\varrho_{ijk}\mathcal{P}_{nk}(x) \\ &= \sum_{k=0}^{n} \left((2k+1) \sum_{j=0}^{n} v_j \varrho_{ijk} \right) \mathcal{P}_{nk}(x) \\ &= \sum_{k=0}^{n} \hat{v}_{ik} \mathcal{P}_{nk}(x). \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4) gives the result. □

4. Numerical solution of nonlinear VFH integral equations

In this section, we try to convert problem (1.1) with assumption (1.2) to a nonlinear set of algebraic equations. So, we consider the following integral equation

$$u(x) = f(x) + \int_0^x k(x,t) u^m(t) dt, \quad x \in I. \tag{4.1}$$

If we approximate functions $u(x)$, $u^m(x)$ and $k(x,t)$ in terms of ALPs, as described by equations (2.8) and (2.9), then we obtain

$$u(x) \simeq \Phi^T(x) U, \tag{4.2}$$

$$u^m(x) \simeq \Phi^T(x) U^{(m)}, \tag{4.3}$$

$$k(x,t) \simeq \Phi^T(x) K \Phi(t), \tag{4.4}$$

where the vectors U, $U^{(m)}$ and matrix K are ALP coefficients of $u(x)$, $u^m(x)$ and $k(x,t)$ respectively.

In the sequel, we need to express the components of vector $U^{(m)}$ as nonlinear functions of the elements of the vector U. We do this by the following lemma.

Lemma 4.1. *Let m be a positive integer and U and $U^{(m)}$ be the ALP coefficient vectors of functions $u(x)$ and $u^m(x)$, respectively. Then, we have*

$$U^{(m)} \simeq \frac{1}{n+1} D^{-1} \hat{U}^m \boldsymbol{1}, \tag{4.5}$$

where D is the diagonal matrix defined in (2.5) *and* **1** *denotes a* $(n+1)$-*vector of ones.*

Proof. We can write

$$\begin{aligned} DU^{(m)} &= \left(\int_0^1 \Phi(x)\Phi^T(x)dx\right) U^{(m)} \\ &= \int_0^1 \Phi(x)\Phi^T(x)U^{(m)}dx. \end{aligned}$$

Also, using relations (4.3), (4.2) and (3.2), we have

$$\begin{aligned} \int_0^1 \Phi(x)\Phi^T(x)U^{(m)}dx &\simeq \int_0^1 \Phi(x)u^m(x)dx \\ &\simeq \int_0^1 \Phi(x)\Big(\Phi^T(x)U\Big)^m dx \\ &= \int_0^1 \overbrace{\Big(\Phi(x)\Phi^T(x)U\Big)}^{\simeq \hat{U}\Phi(x)} \Big(\Phi^T(x)U\Big)^{m-1} dx \\ &\simeq \hat{U} \int_0^1 \Phi(x)\Big(\Phi^T(x)U\Big)^{m-1} dx \\ &\vdots \\ &\simeq \hat{U}^m \int_0^1 \Phi(x)dx. \end{aligned}$$

From the other side, by relation (2.2), we have

$$\int_0^1 \Phi(x)dx = \frac{1}{n+1}\mathbf{1}.$$

Therefore,

$$DU^{(m)} \simeq \frac{1}{n+1}\hat{U}^m\mathbf{1}.$$

Since D is invertible, we obtain (4.5). □

Using relations (4.3), (4.4), (3.1), (3.2) and (2.5), the Volterra part of Eq. (4.1) may be written as

$$\begin{aligned}\int_0^x k(x,t)u^m(t)dt &\simeq \int_0^x \Phi^T(x)K\Phi(t)\Phi^T(t)U^{(m)}dt\\ &= \Phi^T(x)K\int_0^x \Phi(t)\Phi^T(t)U^{(m)}dt\\ &\simeq \Phi^T(x)K\int_0^x \widehat{U^{(m)}}\Phi(t)dt\\ &= \Phi^T(x)K\widehat{U^{(m)}}\int_0^x \Phi(t)dt\\ &\simeq \Phi^T(x)K\widehat{U^{(m)}}P\Phi(x),\end{aligned} \tag{4.6}$$

where P is the ALP operational matrix of integration introduced in Theorem 3.4. With substituting approximations (4.2) and (4.6) into equation (4.1), we get

$$\Phi^T(x)U \simeq f(x) + \Phi^T(x)K\widehat{U^{(m)}}P\Phi(x). \tag{4.7}$$

Now, by Corollary 2.1, let

$$S_n = \{x_l|\mathcal{P}_{n+1,0}(x_l) = 0, l = 0, 1, \dots, n\}, \tag{4.8}$$

be the set of the Gauss-Chelyshkov (GC) collocation nodes. Collocating equation (4.7) at the GC nodes $x_l, l = 0, 1, \dots, n$, will result in

$$\Phi^T(x_l)U \simeq f(x_l) + \Phi^T(x_l)K\widehat{U^{(m)}}P\Phi(x_l), \quad l = 0, 1, \dots, n. \tag{4.9}$$

Since $U^{(m)}$ is a column vector whose elements are nonlinear functions of the elements of the unknown vector $U = [u_i]_{i=0}^n$, then equation (4.9) is a set of $(n+1)$ nonlinear algebraic equations with $(n+1)$ unknowns u_0, u_1, $\dots$, u_n. This nonlinear system of algebraic equations can be solved by numerical methods such as Newton's iterative method. If $\bar{U}$ be an approximate solution of this system, then $\bar{u}_n(x) = \Phi^T(x)\bar{U}$ is an approximate solution of equation (4.1).

5. Illustrative examples

In this section, we provide some numerical examples to illustrate the applicability and the accuracy of the proposed method compared with other existing methods. In order to analyze the error of the method we introduce notations

$$e_n(x) = u(x) - u_n(x),$$
$$\bar{e}_n(x) = u(x) - \bar{u}_n(x),$$
$$\gamma_n = \frac{1}{(n+1)!2^{2n+1}},$$
$$\beta_n = \|U_n - \bar{U}_n\|_2,$$

where $u(x)$, $u_n(x)$ and $\bar{u}_n(x)$ are the exact solution, its best approximation using ALPs and the computed solution by the presented method, respectively and U_n and $\bar{U}_n$ are the ALP coefficient vectors of $u_n(x)$ and $\bar{u}_n(x)$, respectively. We also define

$$\|\bar{e}_n\|_\infty = \max\{|\bar{e}_n(x_l)|, \quad l = 0, 1, \ldots, n\},$$

and the global error as [17]

$$\epsilon_n = \frac{1}{|u|_{\max}} \sqrt{\frac{1}{n} \sum_{l=0}^{n} [\bar{e}_n(x_l)]^2},$$

where x_l, $l = 0, 1, \ldots, n$, are the GC collocation nodes defined by (4.8) and $|u|_{\max}$ denotes the maximum absolute value of the exact solution u.

Experiments were performed on a personal computer using a 2.50 GHz processor and the codes were written in *Mathematica 9*.

Example 5.1. Consider the following linear Volterra integral equation

$$u(x) = f(x) + \int_0^x (x-t)u(t)dt, \quad x \in [0, 1]. \tag{5.1}$$

The function $f(x)$ was chosen so that the analytical solution of (5.1) is

$$u(x) = \gamma x e^{1-\gamma x},$$

with γ denoting a given (real) parameter. The results obtained for this example are given in Tables 1 and 2. Also, the error function $\bar{e}_n(x)$ is plotted for $n = 9$ in Fig. 1. We see from Table 1 that by decreasing γ, the total variation of the exact solution $u(x)$ (which is denoted by u_{tv}) increases and the method converges slowly. In the case of $\gamma = -1$, the results can be compared with those of Brunner and Yan [5] who used the collocation and iterated collocation methods for the numerical solution of this problem.

TABLE 1. Computed errors $\|\bar{e}_n\|_\infty$ for Example 5.1.

n	$\gamma = 1 (u_{tv} = 1)$	$\gamma = -1 (u_{tv} = e^2)$	$\gamma = -2 (u_{tv} = 2e^3)$	$\gamma = -3 (u_{tv} = 3e^4)$
1	3.723×10^{-2}	2.842×10^{-1}	$1.858 \times 10^{+0}$	$7.594 \times 10^{+0}$
3	1.961×10^{-3}	9.313×10^{-3}	8.949×10^{-2}	5.840×10^{-1}
5	1.639×10^{-5}	6.775×10^{-5}	2.256×10^{-3}	2.407×10^{-2}
7	4.673×10^{-8}	1.714×10^{-7}	2.142×10^{-5}	4.819×10^{-4}
9	6.407×10^{-11}	2.194×10^{-10}	1.058×10^{-7}	5.158×10^{-6}

TABLE 2. L_2 errors for Example 5.1 with $\gamma = -1$.

n	1	3	5	7
γ_n	6.25×10^{-2}	3.255×10^{-4}	6.782×10^{-7}	7.569×10^{-10}
$\|e_n\|_2$	4.341×10^{-1}	4.074×10^{-3}	0	0
α_n	1.1547	1.29468	1.37048	1.4219
β_n	4.374×10^{-1}	1.451×10^{-2}	9.275×10^{-5}	2.860×10^{-7}
$\|\bar{e}_n\|_2$	5.578×10^{-1}	1.390×10^{-2}	8.847×10^{-5}	6.744×10^{-7}

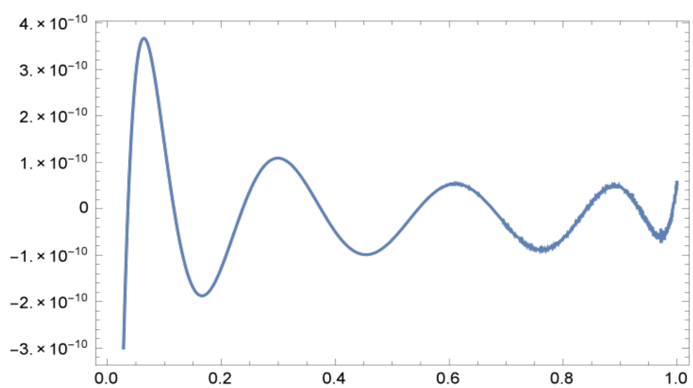

FIGURE 1. Graph of $\bar{e}_n(x)$ for Example 5.1 with $\gamma = -1$ and $n = 9$.

Example 5.2 ([1, 14, 15])**.** Consider the following nonlinear Volterra integral equation

$$u(x) = \frac{3}{2} - \frac{1}{2}e^{-2x} - \int_0^x \Big(u^2(t) + u(t)\Big)dt, \quad x \in [0,1]. \tag{5.2}$$

The exact solution of this problem is $u(x) = e^{-x}$. Tables 3 and 4 illustrate the numerical results obtained by the present method for this example. Also, Fig. 2 shows the graph of the error function $\bar{e}_n(x)$ for $n = 7$. In Table 5, we have compared the L_2 errors of the present method with those of the triangular function (TF) method and the Bernstein polynomial (BP) method. The table shows that better accuracy is obtained with the present method using a very fewer number of the basis functions and collocation nodes.

TABLE 3. L_∞ errors for Example 5.2.

n	$\lVert\bar{e}_n\rVert_\infty$
1	6.282×10^{-2}
3	1.001×10^{-3}
5	8.294×10^{-6}
7	3.913×10^{-8}
9	1.163×10^{-10}

TABLE 4. L_2 errors for Example 5.2.

n	1	3	5	7
γ_n	6.25×10^{-2}	3.255×10^{-4}	6.782×10^{-7}	7.569×10^{-10}
$\lVert e_n\rVert_2$	2.309×10^{-2}	1.218×10^{-4}	0	0
α_n	1.1547	1.29468	1.37048	
β_n	5.093×10^{-2}	1.075×10^{-3}	1.009×10^{-5}	4.986×10^{-8}
$\lVert\bar{e}_n\rVert_2$	4.444×10^{-2}	9.431×10^{-4}	9.212×10^{-6}	4.947×10^{-8}

TABLE 5. Comparison of L_2 errors for Example 5.2.

Present method		n	E_2	
n	$\lVert\bar{e}_n\rVert_2$		TF method [14]	BP method [15]
1	4.444×10^{-2}	4	0.003738014268	0.000068193353
3	9.431×10^{-4}	8	0.000937018240	0.000000084293
5	9.212×10^{-6}	16	0.000234412613	0.000000000000
7	4.947×10^{-8}	32	0.002374324588	0.000000000000
9	0	–	–	–

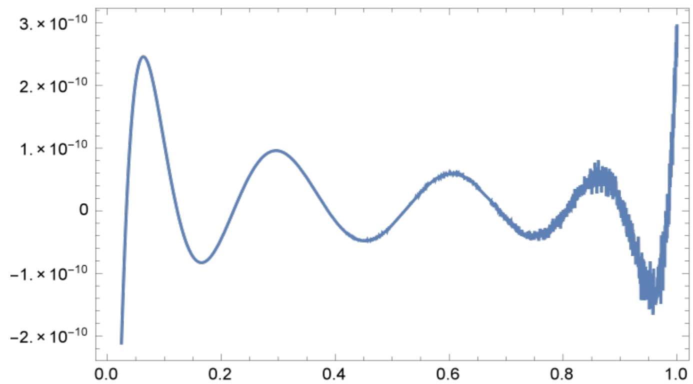

FIGURE 2. Graph of $\bar{e}_n(x)$ for Example 5.2 with $n = 9$.

Example 5.3. As the final example, we consider the nonlinear Volterra integral equation

$$u(x) = f(x) + \int_0^x (x-t)u^2(t)dt, \quad x \in [0,1]. \tag{5.3}$$

The function $f(x)$ was chosen so that the analytical solution of (5.3) is $u(x) = \ln(x+1)$. The results obtained for this example are given in Tables 6 and 7. Also, the error function $\bar{e}_n(x)$ is plotted for $n = 9$ in Fig. 3.

TABLE 6. L_2 errors for Example 5.3.

n	1	3	5	7
γ_n	6.25×10^{-2}	3.255×10^{-4}	6.782×10^{-7}	7.569×10^{-10}
$\lVert e_n \rVert_2$	1.752×10^{-2}	2.638×10^{-4}	1.079×10^{-5}	0
α_n	1.1547	1.29468	1.37048	1.4219
β_n	1.897×10^{-2}	2.528×10^{-4}	6.448×10^{-6}	1.157×10^{-7}
$\lVert \bar{e}_n \rVert_2$	2.484×10^{-2}	3.219×10^{-4}	7.359×10^{-6}	1.399×10^{-7}

TABLE 7. L_∞ errors for Example 5.3.

n	$\|\bar{e}_n\|_\infty$
1	7.156×10^{-3}
3	3.674×10^{-4}
5	9.200×10^{-6}
7	1.957×10^{-7}
9	4.371×10^{-9}

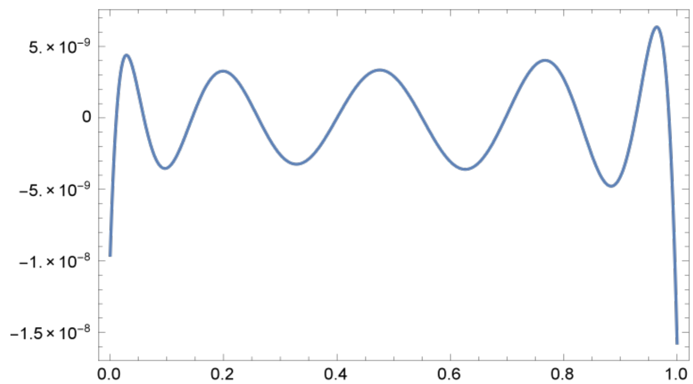

FIGURE 3. Graph of $\bar{e}_n(x)$ for Example 5.3 with $n = 9$.

6. Conclusion and comments

In the presented method, the ALP operational matrices of integration and the product together with the collocation method are used to get the solution of problem (1.1) with assumption (1.2). We transformed the considered problem to a nonlinear system of algebraic equations with unknown ALP coefficients of the exact solution. As it is illustrated by the numerical examples, high accuracy results can be achieved only using a small number of basis functions.

The numerical experiments have also shown that the convergence rate of the numerical solution is similar to the ones of the best approximation of the exact solution by a polynomial in H_n.

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