Sahand Communications in Mathematical Analysis (SCMA) Vol. 5 No. 1 (2017), 1-7
http://scma.maragheh.ac.ir

THE FEKETE-SZEGÖ PROBLEM FOR A GENERAL CLASS OF BI-UNIVALENT FUNCTIONS SATISFYING SUBORDINATE CONDITIONS

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Abstract. In this work, we obtain the Fekete-Szegö inequalities for the class $P_{\Sigma}(\lambda, \phi)$ of bi-univalent functions. The results presented in this paper improve the recent work of Prema and Keerthi [11].

1. Introduction and Definitions

Let A denotes the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\},$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

and let S be the class of all functions from A which are univalent in U.

The Koebe one-quarter theorem [5] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U),$$

and

$$f\left(f^{-1}(w)\right) = w, \qquad \left(|w| < r_0(f), \quad r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

2010 *Mathematics Subject Classification.* 30C45, 30C50.

Key words and phrases. Bi-univalent functions, Convex functions with respect to symmetric points, Subordination, Fekete-Szegö inequality.

Received: 27 February 2016, Accepted: 30 June 2016.

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A function $f \in A$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denotes the class of bi-univalent functions defined in the unit disk U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \quad (z \in U),$$

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1, \quad (z \in U),$$

such that

$$f(z) = g(w(z)), \quad (z \in U).$$

Brannan and Taha [2] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $S^{\star}(\alpha)$ and $K(\alpha)$ of starlike and convex function of order $\alpha\,(0 < \alpha \leq 1)$, respectively. The classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding to the function classes $S^{\star}(\alpha)$ and $K(\alpha)$, were also introduced analogously. For each of the function classes $S^{\star}_{\Sigma}(\alpha)$ and $K_{\Sigma}(\alpha)$, they found non-sharp estimates on the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in [1, 4, 6, 9, 10, 13, 14].

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, there are only a few works determining the general coefficient bounds on $|a_n|$ for the analytic bi-univalent functions [3, 7, 8]. The coefficient estimate problem for each of $|a_n|$, $n \in \mathbb{N} \diagdown \{1, 2\}$; $\mathbb{N} = \{1, 2, \ldots\}$ is still an open problem.

In this paper, motivated by the earlier work of Zaprawa [15], we obtain the Fekete-Szegö inequalities for the class $P_{\Sigma}(\lambda, \phi)$. These inequalities will result in bounds of the third coefficient which are, in some cases, better than these obtained in [11].

In order to derive our main results, we require the following lemma.

Lemma 1.1. [12] *If $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$ is an analytic function in U with positive real part, then*

$$|p_n| \leq 2, \quad (n \in \mathbb{N} = \{1, 2, \ldots\}),$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2 - \frac{|p_1|^2}{2}. \tag{1.2}$$

2. Coefficient Estimates

In the following, let ϕ be an analytic function with positive real part in U, with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(U)$ be starlike with respect to 1 and symmetric with respect to the real axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \quad (B_1 > 0). \tag{2.1}$$

Definition 2.1. [11] A function $f(z)$ given by (1.1) is said to be in the class $P_\Sigma(\alpha,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \left|\arg\left(\frac{z^{1-\lambda} f'(z)}{f(z)^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leq 1, \lambda \geq 0, z \in U),$$

and

$$\left|\arg\left(\frac{w^{1-\lambda} g'(w)}{g(w)^{1-\lambda}}\right)\right| < \frac{\alpha\pi}{2}, \quad (0 < \alpha \leq 1, \lambda \geq 0, w \in U),$$

where the function g is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$

We note that for $\lambda = 0$, the class $P_\Sigma(\alpha,\lambda)$ reduces to the class $S^\star_\Sigma(\alpha)$ which was given by Brannan and Taha [2].

Definition 2.2. A function $f \in \Sigma$ is said to be in the class $P_\Sigma(\lambda,\phi)$, $0 < \phi \leq 1$ and $\lambda \geq 0$, if the following subordination holds

$$\left[\frac{z^{1-\lambda} f'(z)}{f(z)^{1-\lambda}}\right] \prec \phi(z),$$

and

$$\left[\frac{w^{1-\lambda} g'(w)}{g(w)^{1-\lambda}}\right] \prec \phi(w),$$

where $g(w) = f^{-1}(w)$.

Theorem 2.3. *Let f given by (1.1) be in the class $\mathrm{P}_\Sigma(\lambda,\phi)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{B_1}{2+\lambda}, \\ \qquad \textit{for } |\mu - 1| \leq \frac{1}{2+\lambda}\left|1 + \frac{3\lambda}{2} + \frac{\lambda^2}{2} + (1+\lambda)^2 \frac{(B_1 - B_2)}{B_1^2}\right|; \\ \\ \frac{|1-\mu| B_1^3}{\left|\left(1+\frac{3\lambda}{2}+\frac{\lambda^2}{2}\right)B_1^2 + (1+\lambda)^2 (B_1 - B_2)\right|}, \\ \qquad \textit{for } |\mu - 1| \geq \frac{1}{2+\lambda}\left|1 + \frac{3\lambda}{2} + \frac{\lambda^2}{2} + (1+\lambda)^2 \frac{(B_1 - B_2)}{B_1^2}\right|. \end{cases}$$

Proof. Let $f \in P_{\Sigma}(\lambda, \phi)$ and g be the analytic extension of f^{-1} to U. Then there exist two functions u and v, analytic in U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$ and $z, w \in U$ such that

$$\left[\frac{z^{1-\lambda} f'(z)}{f(z)^{1-\lambda}}\right] = \phi(u(z)), \tag{2.2}$$

and

$$\left[\frac{w^{1-\lambda} g'(w)}{g(w)^{1-\lambda}}\right] = \phi(v(w)), \tag{2.3}$$

where $g(w) = f^{-1}(w)$.

Next, define the functions p and q by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + \cdots \tag{2.4}$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + \cdots. \tag{2.5}$$

Clearly, $\Re(p(z)) > 0$ and $\Re(q(w)) > 0$. From (2.4) and (2.5) one can derive

$$u(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2} p_1 z + \frac{1}{2}\left(p_2 - \frac{1}{2} p_1^2\right) z^2 + \cdots, \tag{2.6}$$

and

$$v(w) = \frac{q(w)-1}{q(w)+1} = \frac{1}{2} q_1 w + \frac{1}{2}\left(q_2 - \frac{1}{2} q_1^2\right) w^2 + \cdots. \tag{2.7}$$

Combining (2.1), (2.2), (2.3), (2.6) and (2.7),

$$\frac{z^{1-\lambda} f'(z)}{f(z)^{1-\lambda}} = 1 + \frac{1}{2} B_1 p_1 z + \left(\frac{1}{4} B_2 p_1^2 + \frac{1}{2} B_1 \left(p_2 - \frac{1}{2} p_1^2\right)\right) z^2 + \cdots, \tag{2.8}$$

and

$$\frac{w^{1-\lambda} g'(w)}{g(w)^{1-\lambda}} = 1 + \frac{1}{2} B_1 q_1 w + \left(\frac{1}{4} B_2 q_1^2 + \frac{1}{2} B_1 \left(q_2 - \frac{1}{2} q_1^2\right)\right) w^2 + \cdots. \tag{2.9}$$

From (2.8) and (2.9), we deduce

$$(1+\lambda) a_2 = \frac{1}{2} B_1 p_1, \tag{2.10}$$

$$(2+\lambda) a_3 + \left(\frac{\lambda^2}{2} + \frac{\lambda}{2} - 1\right) a_2^2 = \frac{1}{4} B_2 p_1^2 + \frac{1}{2} B_1 \left(p_2 - \frac{1}{2} p_1^2\right), \tag{2.11}$$

and

$$-\left(1+\lambda\right)a_2=\frac{1}{2}B_1q_1, \tag{2.12}$$

and

$$\left(\frac{\lambda^2}{2}+\frac{5\lambda}{2}+3\right)a_2^2-\left(2+\lambda\right)a_3=\frac{1}{4}B_2q_1^2+\frac{1}{2}B_1\left(q_2-\frac{1}{2}q_1^2\right). \tag{2.13}$$

From (2.10) and (2.12), we obtain

$$p_1=-q_1. \tag{2.14}$$

Subtracting (2.11) from (2.13) and applying (2.14), we have

$$a_3=a_2^2+\frac{1}{4\left(2+\lambda\right)}B_1\left(p_2-q_2\right). \tag{2.15}$$

By adding (2.11) to (2.13), we get

$$\left(\lambda^2+3\lambda+2\right)a_2^2=\frac{1}{2}B_1\left(p_2+q_2\right)-\frac{1}{4}\left(B_1-B_2\right)\left(p_1^2+q_1^2\right).$$

Combining this with (2.10) and (2.12) leads to

$$a_2^2=\frac{B_1^3\left(p_2+q_2\right)}{2\left[\left(\lambda^2+3\lambda+2\right)B_1^2+2\left(1+\lambda\right)^2\left(B_1-B_2\right)\right]}. \tag{2.16}$$

From (2.15) and (2.16) it follows that

$$a_3-\mu a_2^2=B_1\left[\left(h\left(\mu\right)+\frac{1}{4\left(2+\lambda\right)}\right)p_2+\left(h\left(\mu\right)-\frac{1}{4\left(2+\lambda\right)}\right)q_2\right],$$

where

$$h\left(\mu\right)=\frac{B_1^2\left(1-\mu\right)}{2\left[\left(\lambda^2+3\lambda+2\right)B_1^2+2\left(1+\lambda\right)^2\left(B_1-B_2\right)\right]}.$$

Then, in view of (1.2) and (2.1), we conclude that

$$\left|a_3-\mu a_2^2\right|\leq\begin{cases}\frac{B_1}{2+\lambda}, & 0\leq\left|h\left(\mu\right)\right|\leq\frac{1}{4\left(2+\lambda\right)};\\ 4B_1\left|h\left(\mu\right)\right|, & \left|h\left(\mu\right)\right|\geq\frac{1}{4(2+\lambda)}.\end{cases}$$

□

Taking $\mu=1$ or $\mu=0$, we get

Corollary 2.4. *If* $f\in\mathrm{P}_{\Sigma}\left(\lambda,\phi\right)$ *then*

$$\left|a_3-a_2^2\right|\leq\frac{B_1}{2+\lambda}. \tag{2.17}$$

Corollary 2.5. *If* $f \in \mathrm{P}_{\Sigma}(\lambda,\phi)$ *then*

(2.18)
$$|a_3| \leq \begin{cases} \frac{B_1}{2+\lambda}, \\ \qquad for\ \frac{B_1-B_2}{B_1^2} \in \left(-\infty, -\frac{6+5\lambda+\lambda^2}{2(1+\lambda)^2}\right] \cup \left[\frac{2-\lambda-\lambda^2}{2(1+\lambda)^2}, \infty\right); \\ \frac{B_1^3}{\left|\left(1+\frac{3\lambda}{2}+\frac{\lambda^2}{2}\right)B_1^2+(1+\lambda)^2(B_1-B_2)\right|}, \\ \qquad for\ \frac{B_1-B_2}{B_1^2} \in \left[-\frac{6+5\lambda+\lambda^2}{2(1+\lambda)^2}, -\frac{2+3\lambda+\lambda^2}{2(1+\lambda)^2}\right) \cup \left(-\frac{2+3\lambda+\lambda^2}{2(1+\lambda)^2}, \frac{2-\lambda-\lambda^2}{2(1+\lambda)^2}\right]. \end{cases}$$

Corollary 2.6. *Let*

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\beta} = 1 + 2\beta z + 2\beta^2 z^2 + \cdots, \quad (0 < \beta \leq 1),$$

then inequalities (2.17) and (2.18) become

$$\left|a_3 - a_2^2\right| \leq \frac{2\beta}{2+\lambda},$$

and

$$|a_3| \leq \begin{cases} \frac{2\beta}{2+\lambda}, & \beta \leq \frac{(1+\lambda)^2}{3+\lambda}; \\ \frac{4\beta^2}{(1+\lambda)\beta+(1+\lambda)^2}, & \beta \geq \frac{(1+\lambda)^2}{3+\lambda}. \end{cases}$$

Corollary 2.7. *Let*

$$\phi(z) = \frac{1+(1-2\beta)z}{1-z} = 1+2(1-\beta)z+2(1-\beta)z^2+\cdots, \quad (0 \leq \beta < 1),$$

then inequalities (2.17) and (2.18) become

$$\left|a_3 - a_2^2\right| \leq \frac{2(1-\beta)}{2+\lambda},$$

and

$$|a_3| \leq \frac{2(1-\beta)}{1+\frac{3\lambda}{2}+\frac{\lambda^2}{2}}.$$

Remark 2.8. Corollary 2.6 and Corollary 2.7 provide an improvement of the estimate $|a_3|$ obtained by Prema and Keerthi [11].

Acknowledgment. The authors thank the referee for his valuable suggestions which led to improvement of this study.

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