Sahand Communications in Mathematical Analysis (SCMA) Vol. 8 No. 1 (2017), 55-71
http://scma.maragheh.ac.ir

Contra β^*-continuous and almost contra β^*-continuous functions

Appachi Vadivel[1]*, Radhakrishnan Ramesh[2], and Duraisamy Sivakumar[3]

Abstract. The notion of contra continuous functions was introduced and investigated by Dontchev. In this paper, we apply the notion of β^*-closed sets in topological space to present and study a new class of functions called contra β^*-continuous and almost contra β^*-continuous functions as a new generalization of contra continuity.

1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologist worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc, by utilizing generalized closed sets. Recently, as generalization of closed sets, the notion of β^*-closed sets were introduced and studied by [15].

Dontchev [4] introduced the notions of contra continuity and strong S-closedness in topological spaces. He defined a function $f : X \to Y$ is contra continuous if the preimage of every open set of Y is closed in X. A new weaker form of this class of functions called contra semicontinuous function is introduced and investigated by Dontchev and Noiri [5]. Caldas and Jafari [2] have introduced and studied contra β-continuous function. Jafari and Noiri [8, 9] introduced and investigated the notions of contra super continuous, contra precontinuous and contra α-continuous functions. Almost contra precontinuous functions were introduced by [6] and recently have been investigated further by Noiri and Popa [14]. Nasef [13] has introduced and studied contra γ-continuous function. In

2010 *Mathematics Subject Classification.* 54C08, 54C10, 54D10.

Key words and phrases. β^*-closed sets, Contra β^*-continuous, Almost contra β^*-continuous functions.

Received: 5 March 2016, Accepted: 7 August 2016.

* Corresponding author.

this direction, we will introduce the concept of contra β^*-continuous and almost contra β^*-continuous functions via the notion of β^*-open set and study some properties of contra β^*-continuous and almost contra β^*-continuous functions.

All through this paper, (X, τ) and (Y, σ) stand for topological space with no separation axioms assumed, unless otherwise stated. Let $A \subseteq X$, the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(B)$, respectively.

Let A be a subset of a space $(X,\ T)$. The set $\cap\{U : U \in T$ and $A \subset U\}$ is called *kernal of* A and is denoted by $ker(A)$ [12].

Lemma 1.1 ([8]). *The following properties hold for subsets* A, B *of a space* X*:*

(1) $x \in \ker(A)$ *if and only if* $A \cap F \neq \phi$ *for any* $F \in C(X, x)$,
(2) $A \subseteq \ker(A)$ *and* $A = \ker(A)$ *if* A *is open in* X,
(3) *if* $A \subseteq B$, *then* $\ker(A) \subseteq \ker(B)$.

2. Contra β^*-continuous functions

Definition 2.1. A subset A of a topological space (X, τ) is called a

(i) generalized closed (briefly g-closed) [11] if $Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ).
(ii) β^*-closed [15] if $Cl(\text{Int}(A)) \subseteq U$ whenever $A \subseteq U$ and U is g-open in (X, τ).

Example 2.2. If we give $\mathbb{R}$ the topology having as basis all the intervals $[a, b)$, then with this topology $\mathbb{R}$ is β^*-closed (resp., g-closed) because the intervals $[a, b)$ are closed (resp., g-closed) as well as open, since their complements $(-\infty, a) \cup [b, \infty)$ are open (resp., g-open), being the union of the basis intervals $[a - n, a)$ and $[b, b + n)$ for $n = 1, 2, \ldots$.

Definition 2.3. A function $f : X \to Y$ is called β^*-continuous [7] if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq V$.

Definition 2.4. A bijection $f : (X, \tau) \to (Y, \sigma)$ is called β^*-homeomorph ism [18] if f and f^{-1} are both β^*-continuous and β^*-closed.

Example 2.5. Let $(-1, 1)$ has the standard topology, and define a function $f : \mathbb{R} \to (-1, 1)$ by $f(x) = \frac{x}{1+|x|}$. It is apparent from the graph that f is bijection between $\mathbb{R}$ and $(-1, 1)$. It is also apparent that preimages of open intervals are open intervals for both f and f^{-1}. Therfore, f and f^{-1} are both β^*-continuous, and it follows that f is a homeomorphism between $\mathbb{R}$ and $(-1, 1)$.

Definition 2.6. A function $f : X \to Y$ is called contra-β^*-continuous (resp., contra-continuous [4]) if $f^{-1}(V)$ is β^*-closed (resp., closed) in X for each open set V of Y.

Example 2.7. Let $X = \mathcal{R}$ with the usual topology τ and $Y = \{a, b, c, d\}$ with the topology $\sigma = \{\phi, Y, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}\}$. Then the function $f : (X, \tau) \to (Y, \sigma)$ defined by

$$f(x) = \begin{cases} c, & \text{if } x = Q, \\ a, & \text{if } x \neq Q, \end{cases}$$

is contra β^*-continuous.

Definition 2.8. A function $f : X \to Y$ is said to be almost-β^*-continuous (resp., almost continuous [14]) if $f^{-1}(V)$ is β^*-open (resp., open) in X for each regular open set V of Y.

Remark 2.9. Every contra continuous function is contra β^*-continuous but the converse is not true, as shown in the following example.

Example 2.10. Let $X = Y = \{a, b\}$ with $\tau = \sigma = \{X, \phi, \{a\}\}$. Let $f : (X, \tau) \to (Y, \sigma)$ be the identity map. Then f is contra β^*-continuous but not contra continuous, where $\{a\}$ is open in Y but it is not closed in X.

Theorem 2.11. *The following are equivalent for a function $f : X \to Y$:*

(1) *f is contra-β^*-continuous,*
(2) *for every closed subset F of Y, $f^{-1}(F) \in \beta^*O(X)$,*
(3) *for each $x \in X$ and $F \in C(Y, f(x))$, there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq F$,*
(4) *$f(\beta^*Cl(A)) \subseteq \ker(f(A))$ for every subset A of X,*
(5) *$\beta^*Cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$ for every subset B of Y.*

Proof. The implications (1)⇔(2) and (2)⇒(3) are obvious.

(3)⇒(2) Let F be any closed subset of Y and $x \in f^{-1}(F)$. Then $f(x) \in F$ and there exists $U_x \in \beta^*O(X, x)$ such that $f(U_x) \subseteq F$. Therefore, we obtain $f^{-1}(F) = \bigcup\{U_x | x \in f^{-1}(F)\}$ and $f^{-1}(F)$ is β^*-open, since τ_{β^*} is a topological space.

(2)⇒(4) Let A be a subset of X. Suppose that $y \notin \ker(f(A))$. Then by Lemma 1.1, there exists $F \in C(Y, f(x))$ such that $f(A) \cap F = \phi$. Thus, we have $A \cap f^{-1}(F) = \phi$ and since $f^{-1}(F)$ is β^*-open then we have $\beta^*Cl(A) \cap f^{-1}(F) = \phi$. Therefore, we have obtain $f(\beta^*Cl(A) \cap F) = \phi$ and $y \notin f(\beta^*Cl(A))$. This implies that

$$f(\beta^*Cl(A) \subseteq \ker(f(A)).$$

(4)⇒(5) Let B be any subset of Y. By (4) and Lemma 1.1, we have
$$f(\beta^*Cl(f^{-1}(B))) \subseteq \ker(f(f^{-1}(B))) \subseteq \ker(B)$$
thus $\beta^*Cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B))$.

(5)⇒(1) Let V be any open set of Y. Then by Lemma 1.1, we have $\beta^*Cl(f^{-1}(V)) \subseteq f^{-1}(\ker(V)) = f^{-1}(V)$ and $\beta^*Cl(f^{-1}(V)) = f^{-1}(V)$. This show that $f^{-1}(V)$ is β^*-closed in X.

□

Theorem 2.12. *If a function $f : X \to Y$ is contra-β^*-continuous and Y is regular, then f is β^*-continuous.*

Proof. Let x be an arbitrary point of X and let V be an open set of Y containing $f(x)$, since Y is regular, there exists an open set W in Y containing $f(x)$ such that $Cl(W) \subseteq V$. Since f is contra β^*-continuous, so by Theorem 2.11(3) there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq Cl(W)$. Then $f(U) \subseteq Cl(W) \subseteq V$. Hence f is β^*-continuous. □

Definition 2.13. A space (X, τ) is said to be β^*-space (resp., locally β^*-indiscrete) if every β^*-open set is open (resp., closed) in X.

For any space (X, τ), we have $\tau \subseteq \tau_{\beta^*}$. So the following results follows immediatly.

Theorem 2.14. *A function $f : (X, \tau) \to (Y, \sigma)$ is contra-β^*-continuous if and only if $f : (X, \tau_{\beta^*}) \to (Y, \sigma)$ is contra-continuous.*

Theorem 2.15. *If a function $f : X \to Y$ is contra-β^*-continuous and X is β^*-space, then f is contra-continuous.*

Theorem 2.16. *Let X be locally β^*-indiscrete. If a function $f : X \to Y$ is contra β^*-continuous, then f is continuous.*

Definition 2.17. A function $f : X \to Y$ is called almost-β^*-continuous if for each $x \in X$ and each open set V of Y containing $f(x)$, there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq \beta^*Int(Cl(V))$.

Definition 2.18. A function $f : X \to Y$ is said to be pre-β^*-open if the image of each β^*-open set is β^*-open.

Theorem 2.19. *If a function $f : X \to Y$ is pre-β^*-open and contra-β^*-continuous function, then f is almost-β^*-continuous.*

Proof. Let x be any arbitrary point of X and V be an open set containing $f(x)$. Since f is contra-β^*-continuous, then by Theorem 2.11 (3) there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq Cl(V)$. Since f is pre-β^*-open, $f(U)$ is β^*-open in Y. Therefore,
$$f(U) = \beta^*Intf(U) \subseteq \beta^*Int(Cl(f(U))) \subseteq \beta^*Int(Cl(V)).$$
This shows that f is almost-β^*-continuous. □

Definition 2.20. A function $f : X \to Y$ is said to be almost weakly β^*-continuous if for each $x \in X$ and each open neighborhood V of $f(x)$ there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq Cl(V)$.

Theorem 2.21. *If a function $f : X \to Y$ is contra-β^*-continuous, then f is almost weakly β^*-continuous.*

Proof. Let V be any open set of Y. Since $Cl(V)$ is closed in Y, by Theorem 2.11 (3), $f^{-1}(Cl(V))$ is β^*-open in X and set $U = f^{-1}(Cl(V))$, then we have $f(U) \subseteq Cl(V)$. This shows that f is almost weakly β^*-continuous. □

Since the family of all β^*-open subsets of a space (X, τ), denoted by τ_{β^*}, forms a topology on X finer than τ, then the β^*-frontier of A, where $A \subseteq X$, is defined by $\beta^*Fr(A) = \beta^*Cl(A) \cap \beta^*Cl(X - A)$.

Theorem 2.22. *The set of all points x of X for which $f : X \to Y$ is not contra-β^*-continuous is identical with the union of the β^*-frontier of the inverse images of closed sets of Y containing $f(x)$.*

Proof. Suppose f is not contra-β^*-continuous at $x \in X$. There exists $F \in C(Y, f(x))$ such that $f(U) \cap (Y - F) \neq \phi$ for every $U \in \beta^*O(X, x)$ by Theorem 2.11 This implies that $U \cap f^{-1}(Y - F) \neq \phi$. Therefore, we have

$$\begin{aligned} x \in f^{-1}(F) &\subseteq \beta^*Cl(f^{-1}(Y - F)) \\ &= \beta^*Cl(X - f^{-1}(F)). \end{aligned}$$

However, since $x \in f^{-1}(F) \subseteq \beta^*Cl(f^{-1}(F))$, thus

$$x \in \beta^*Cl(f^{-1}(F)) \cap \beta^*Cl(f^{-1}(Y - F)).$$

Therefore, we obtain $x \in \beta^*Fr(f^{-1}(F))$. Suppose that

$$x \in \beta^{-1}Frf(f^{-1}(F)),$$

for some $F \in C(Y, f(x))$, and f is contra-β^*-continuous at x, then there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq F$. Therefore, we have $x \in U \subseteq f^{-1}(F)$ and hence

$$x \in Int(f^{-1}(F)) \subseteq X - \beta^*Fr(f^{-1}(F)).$$

This is a contradiction. This mean that f is not contra-β^*-continuous. □

Theorem 2.23. *Let $f : X \to Y$ be a function and let $g : X \to X \times Y$ be the graph function of f defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra-β^*-continuous, then f is contra-β^*-continuous.*

Proof. Let U be an open set in Y, then $X \times U$ is an open set in $X \times Y$. Since g is contra-β^*-continuous. It follows that $f^{-1}(U) = g^{-1}(X \times U)$ is an β^*-closed in X. Thus, f is contra-β^*-continuous. □

Theorem 2.24. *If $f : X \to Y$ and $g : X \to Y$ are contra-β^*-continuous, and Y is Urysohn, then $E = \{x \in X : f(x) = g(x)\}$ is β^*-closed in X.*

Proof. Let $x \in X - E$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $g(x) \in W$, and $Cl(V) \cap Cl(W) \neq \phi$. Since f and g are contra-β^*-continuous, then $f^{-1}(Cl(V))$ and $g^{-1}(Cl(W))$ are β^*open sets in X. Let $U = f^{-1}(Cl(V))$ and $G = g^{-1}(Cl(W))$. Then U and V are β^*-open sets containing x. Set $A = U \cap G$, thus A is β^*-open in X. Hence,

$$\begin{aligned} f(A) \cap g(A) &= f(U \cap G) \cap g(U \cap G) \\ &\subseteq f(U) \cap g(G) \\ &= Cl(V) \cap Cl(W) \\ &= \phi. \end{aligned}$$

Therefore, $A \cap E = \phi$ and $x \notin \beta^*Cl(E)$. Hence, E is β^*-closed in X. □

A subset A of a topological space X is said to be β^*-dense in X if $\beta^*Cl(A) = X$.

Theorem 2.25. *Let $f : X \to Y$ and $g : X \to Y$ be functions. If Y is Urysohn, f and g are contra-β^*-continuous and $f = g$ on β^*-dense set $A \subseteq X$, then $f = g$ on X.*

Proof. Since f and g are contra-β^*-continuous and Y is Urysohn, by the previous Theorem 2.24, $E = \{x \in X : f(x) = g(x)\}$ is β^*-closed in X. By assumption, we have $f = g$ on β^*-dense set $A \subseteq X$. Since $A \subseteq E$ and A is β^*-dense set in X, then $X = \beta^*Cl(A) \subseteq \beta^*Cl(E) = E$. Hence, $f = g$ on X. □

Definition 2.26. A space X is called β^*-connected [16] provided that X is not the union of two disjoint nonempty β^*-open sets.

Example 2.27. The real line R with the usual topology is a β^*-connected space since R and ϕ are the only subsets of R which are both β^*-open and β^*-closed.

Example 2.28. If we give R the topology having as basis all the intervals $[a, b)$, then with this topology R is not β^*-connected, because the intervals $[a, b)$ are closed (resp., g-closed) as well as open (resp., g-open) since this complements $(-\infty, a] \cup [b, \infty)$ are open (resp., g-open), being the union of the basis intervals $[a - n, a)$ and $[b, b + n)$ for $n = 1, 2, \ldots$.

Theorem 2.29. *If $f : X \to Y$ is a contra-β^*-continuous function from an β^*-connected space X onto any space Y, then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper nonempty open and closed subset of Y. Then $f^{-1}(A)$ is a proper nonempty β^*-clopen subset of X, which is a contradiction to the fact that X is β^*-connected. □

Theorem 2.30. *If $X \to Y$ is contra-β^*-continuous surjection and X is β^*-connected, then Y is connected.*

Proof. Suppose that Y is not connected space. Then there exists two nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen in Y. Since f is contra-β^*-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are β^*-open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not β^*-connected. This is a contradiction. This means that Y is connected. □

Theorem 2.31. *A space X is β^*-connected, if every contra-β^*-continuous function from a space X in to any T_0-space Y is constant.*

Proof. Suppose that X is not β^*-connected and every contra-β^*-continuous function from X in to Y is constant. Since X is not β^*-connected, there exists a proper nonempty β^*-clopen such that A of X. Let $Y = \{a, b\}$, and $\tau = \{Y, \phi, \{a\}, \{b\}\}$ be a topology for Y. Let $f : X \to Y$ be a function subset $f(A) = \{a\}$ and $f(X - A) = \{b\}$. Then f is nonconstant and contra-β^*-continuous such that Y is T_0 which is a contradiction. Hence, X must be β^*-connected. □

Definition 2.32. A space X is said to be β^*-T_1 if each pair of distinct points x and y of X, there exists β^*-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$.

Theorem 2.33. *A topological space X is a β^*-T_1-space iff every singleton subsets $\{p\}$ of X are β^*-closed.*

Since finite unions of β^*-closed sets are β^*-closed, the above Theorem implies:

Corollary 2.34. *(X, T) is a β^*-T_1-space if and only if T contains the cofinite topology on X.*

Example 2.35. The cofinite topology on X is the coarsest toplogy on X for which (X, T) is a β^*-T_1-space by corollary 2.34. Hence the cofinite topology is also called the β^*-T_1-topology.

Definition 2.36. A space X is said to be β^*-T_2 if for each pair of distinct points x and y in X, there exist $U \in \beta^*O(X, x)$ and $V \in \beta^*O(X, y)$ such that $U \cap V = \phi$.

Example 2.37. Let Z be the topology on the real line R generated by the open-closed intervals $(a, b]$. Then (R, Z) is β^*-T_2-space.

Example 2.38. Let

$$Y = [-2, 0] \cup \left\{ \frac{1}{n+1} | \, n = 1, 2, \dots \right\} \cup [1, 2],$$

write the topology induced from

$$X = (-\infty, 0] \cup \left\{ \frac{1}{n+1} | \, n = 1, 2, \dots \right\} \cup [1, \infty).$$

Take the topology on X defined by neighborhood system:

(1) neighborhoods of any $x \in (-\infty, 0) \cup (1, \infty)$ are the sets $(x - \epsilon, \ x + \epsilon)$, where $\epsilon > 0$,
(2) The neighborhoods of 0 are the sets $(-\epsilon, 0] \cup \left\{ \frac{1}{n+1} | \, n \in N - F \right\}$, where $\epsilon > 0$ and F is finite.
(3) The neighborhoods of 1 are the sets $[1, \epsilon) \cup \left\{ \frac{1}{n+1} | \, n \in N - F \right\}$, where $\epsilon > 0$ and F is finite.

The neighborhood of $\frac{1}{n+1}$, where $n \in N$, is the set $\left\{ \frac{1}{n+1} \right\}$. Then 0 and 1 have no disjoint neighborhoods. Therefore X is not β^*-T_2. Then Y is a β^* compact closed subspace of X which is not β^*-T_2.

Definition 2.39 ([17])**.** A topological space (X, τ) is said to be β^*-regular if for every β^*-closed set F and a point $x \notin F$, there exist disjoint open sets U and V such that $F \subseteq U$ and $x \in V$.

Definition 2.40. A topological space (X, τ) is said to be β^*-T_3 if it is β^*-regular with β^*-T_1.

Definition 2.41 ([17])**.** A topological space (X, τ) is said to be β^*-normal if for each pair of disjoint β^*-closed sets A and B, there exist a pair of disjoint open sets U and V in X such that $A \subseteq U$ and $B \subseteq V$.

Definition 2.42. A topological space (X, τ) is said to be β^*-T_4 if it is β^*-normal with β^*-T_1.

Example 2.43. Every indiscrete space is trivially a β^*-regular space and every discrete space is β^*-regular.

Example 2.44. A β^*-space need not be β^*-regular and hence it is not β^*-regular. Consider the set of all real numbers. Let $B_x = \{x\} \cup I_x$, where I_x contains only the rationals in an open intervals around x. Then $\mathscr{B} = \{B_x : x \in R\}$ is a base for a topology τ. Thus (R, τ) is a topological space which is β^*-T_2 also. But it is not a β^*-regular space, since the irrationals in R form a β^*-closed set and cannot be sequenced from a rational number by β^*-open sets.

Example 2.45. Any indiscrete space X is vacuously β^*-normal, if it is not a β^*-T_4-space unless X is a singleton set.

Example 2.46. Every discrete space is a β^*-T_4-space.

Theorem 2.47. *Let X and Y be topological spaces. If*

(1) *for each pair of distinct points x and y in X there exists a function f of X into Y such that $f(x) \neq f(y)$,*
(2) *Y is an Urysohn space,*
(3) *f is contra-β^*-continuous at x and y, then X is β^*-T_2.*

Proof. Let x and y be any distinct points in X. Then, there exists an Urysohn space Y and a function $f : X \to Y$ such that $f(x) \neq f(y)$ and f is contra-β^*-continuous at x and y. Let $a = f(x)$ and $b = f(y)$. Then $a \neq b$. Since Y is an Urysohn space, there exist open sets V and W containing a and b, respectively, such that $Cl(V) \cap Cl(W) = \phi$. Since f is contra-β^*-continuous at x and y, then there exist β^*-open sets A and B containing a and b, respectively, such that $f(A) \subseteq Cl(V)$ and $f(B) \subseteq Cl(W)$. Then $f(A) \cap f(B) = \phi$, so $A \cap B = \phi$. Hence, X is β^*-T_2. □

Corollary 2.48. *Let $f : X \to Y$ be contra-β^*-continuous injection. If Y is an Urysohn space, then X is β^*-T_2.*

3. Almost contra-β^*-continuous functions

In this section, we introduce a new type of continuity called almost contra-β^*-continuous which is weaker than contra-β^*-continuous.

Definition 3.1. A function $f : X \to Y$ is said to be almost contra-β^*-continuous (resp., almost contra-precontinuous [6]) if $f^{-1}(V) \in \beta^*C(X)$ (resp., $f^{-1}(V) \in PC(X)$) for every $V \in RO(X)$.

Theorem 3.2. *The following statements are equivalents for a function $f : X \to Y$:*

(1) *f is almost contra-β^*-continuous,*
(2) *$f^{-1}(F) \in \beta^*O(X, x)$ for every $F \in RC(Y)$,*
(3) *for each $x \in X$ and each regular closed set F in Y containing $f(x)$, there exists a β^*-open set U in X containing x such that $f(U) \subseteq F$,*
(4) *for each $x \in X$ and each regular open set V in Y noncontaining $f(x)$, there exists a β^*-closed set K in X noncontaining x such that $f^{-1}(V) \subseteq K$.*

Proof. (1)⇔(2) Let F be any regular closed set of Y. Then $Y - F$ is regular open. By (1), $f^{-1}(Y - F) = X - f^{-1}(F) \in \beta^*C(X)$. We have $f^{-1}(F) \in \beta^*O(X)$. The converse is obvious.

(2)⇒(3) Let F be any regular closed set in Y containing $f(x)$. Then by (2), $f^{-1}(F) \in \beta^*O(X)$ and $x \in f^{-1}(F)$. Take $U = f^{-1}(F)$. Then $f(U) \in F$.

(3)⇒(2) Let F be any regular closed set in Y and $x \in f^{-1}(F)$. From (3), there exists a β^*-open $U_x \in X$ containing x such that $f(U_x) \subseteq F$, thus $U_x \subseteq f^{-1}(F)$. We have $f^{-1}(F) \subseteq U_{x \in f^{-1}(F)} U_x$. This implies that $f^{-1}(F)$ is β^*-open.

(3)⇔(4) Let V be any regular open set in Y noncontaining $f(x)$. Then $Y - V$ is a regular closed set containing $f(x)$. By (3), there exists a β^*-open set U in X containing x such that $f(U) \subseteq Y - V$. Hence, $U \subseteq f^{-1}(Y - V) \subseteq X - f^{-1}(V)$ and then $f^{-1}(V) \subseteq X - U$. Take $H = X - U$. We obtain that H is an β^*-closed set in X noncontaining x. The converse is obvious.

□

A space (X, τ) is anti-locally countable [1] if all non-empty open subsets are uncountable. Note that R with usual topology is anti-locally countable space.

Lemma 3.3 ([1])**.** *If (X, τ) is an anti-locally countable space, then $\beta^*Cl(A) = Cl(A)$ for every β^*-open subset of X and $Int(A) = \beta^*Int(A)$ for every β^*-closed subset of X.*

Definition 3.4 ([1])**.** A space (X, τ) is called locally countable, if each point $x \in X$ has a countable open neighborhood.

Lemma 3.5 ([1])**.** *If (X, τ) is a locally countable space, then τ_{β^*} is the discrete topology on X.*

Definition 3.6. A function $f : X \to Y$ is said to be regular set-connected if $f^{-1}(V)$ is clopen in X for each regular open set V of Y.

Theorem 3.7. *Let (X, τ) be an anti-locally countable space, if a function $f : X \to Y$ is almost contra-β^*-continuous and almost continuous, then f is regular set-connected.*

Proof. Let V be any regular open set in Y. Since f is almost contra-β^*-continuous and contra continuous, $f^{-1}(V)$ is β^*-closed and open. Thus $\beta^*Cl(f^{-1}(V)) = f^{-1}(V)$, since (X, τ) is an anti-locally countable space, then by Lemma 3.3, we have $\beta^*Cl(f^{-1}(V)) = Cl(f^{-1}(V))$. Hence $f^{-1}(V)$ is clopen. We obtain that f is regular set-connected. □

Definition 3.8 ([20])**.** A space X is said to be weakly Hausdorff if each element of X is an intersection of regular closed sets.

Theorem 3.9. *If $f : X \to Y$ is an almost contra-β^*-continuous injection and Y is weakly Hausdorff, then X is β^*-T_1.*

Proof. Suppose that Y is weakly Hausdorff. For any distinct points x and y in X, there exist V, W which are regular closed in Y such that $f(x) \in V$, $f(y) \notin V$, $f(x) \notin W$, and $f(y) \in W$. Since f is almost contra-β^*-continuous, then $f^{-1}(V)$ and $f^{-1}(W)$ are β^*-open subsets of X such that $x \in f^{-1}(V)$, $y \notin f^{-1}(V)$, $x \notin f^{-1}(W)$ and $y \in f^{-1}(W)$. This shows that X is β^*-T_1. □

Corollary 3.10. *If $f : X \to Y$ is a contra-β^*-continuous injection and Y is weakly Hausdorff, then X is β^*-T_1.*

Theorem 3.11. *If $f : X \to Y$ is almost contra-β^*-continuous surjection and X is β^*-connected, then Y is connected.*

Proof. Suppose that Y is not connected space. There exist nonempty disjoint open sets V_1 and V_2 such that $Y = V_1 \cup V_2$. Therefore, V_1 and V_2 are clopen sets. Thus they are regular open in Y. Since f is almost contra-β^*-continuous, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are β^*-open in X. Moreover, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are nonempty disjoint and $X = f^{-1}(V_1) \cup f^{-1}(V_2)$. This shows that X is not β^*-connected. This is a contradiction. This means that Y is connected. □

Definition 3.12. A space X is said to be

(1) β^*-compact if every β^*-open cover of X has a finite subcover,
(2) countably β^*-compact if every countable cover of X by β^*-open sets has a finite subcover,
(3) β^*-Lindelöf if every β^*-open cover of X has a countable subcover,
(4) S-Lindelöf [6] if every cover of X by regular closed sets has a countable subcover,
(5) countably S-closed [3] if countable cover of X by regular closed sets has a finite subcover,
(6) S-closed [10] if regular closed cover of X has a finite subcover.

Example 3.13. In Example 2.38, The subspace

$$Z = \{0, 1\} \cup \left\{ \frac{1}{n+1} | \, n = 1, 2, \dots \right\}$$

of X is β^*-closed and β^*-compact.

Theorem 3.14. *Let $f : X \to Y$ be an almost contra-β^*-continuous surjection. The following statements hold:*

(1) *if X is β^*-compact, then Y is S-closed,*
(2) *if X is β^*-Lindelöf, then Y is S-Lindelöf,*
(3) *if X is countably β^*-compact, then Y is countably S-closed.*

Proof. We prove only (1), let $\{V_\alpha : \alpha \in I\}$ be any regular closed cover of Y. Since f is almost contra-β^*-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an β^*-open cover of X and hence there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$ therefore we have $Y = \cup\{V_\alpha : \alpha \in I_0\}$ and Y is S-closed. □

Definition 3.15. A space X is said to be

(1) β^*-closed compact if every β^*-closed cover of X has a finite subcover,
(2) countably β^*-closed compact if every countable cover of X by β^*-closed sets has a finite subcover,
(3) β^*-closed-Lindelöf if every cover of X by β^*-closed sets has a countable subcover,
(4) nearly compact [19] if every regular open cover of X has a finite subcover,
(5) nearly countably compact [19] if every countably cover of X by regular open sets has a finite subcover,
(6) nearly Lindelöf [19] if every cover of X by regular open sets has a countably subcover.

Theorem 3.16. *Let $f : X \to Y$ be an almost contra-β^*-continuous surjection. The following statements hold:*

(1) *if X is β^*-closed compact, then Y is nearly compact,*
(2) *if X is β^*-closed-Lindelöf, then Y is nearly-Lindelöf,*
(3) *if X is countably β^*-closed compact, then Y is nearly countably compact.*

Proof. We prove only (1), let $\{V_\alpha : \alpha \in I\}$ be any regular open cover of Y. Since f is almost contra-β^*-continuous, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an β^*-closed cover of X. Since X is β^*-closed compact, there exists a finite subset I_0 of I such that $X = \bigcup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Thus, we have $Y = \bigcup\{V_\alpha : \alpha \in I_0\}$ and Y is nearly compact. □

Definition 3.17 ([20])**.** A space X is said to be mildly compact (resp., mildly countably compact, mildly Lindelöf) if every clopen cover (resp., clopen countably cover, clopen cover) of X has a finite (resp., a finite, a countable) subcover.

Theorem 3.18. *Let (X, τ) be an anti-locally countable space, if $f : X \to Y$ be an almost contra-β^*-continuous and almost continuous surjection and X is mildly compact (resp., mildly countably compact, mildly Lindelöf), then Y is nearly compact (resp., nearly countably compact, nearly Lindelöf) and S-closed (resp., countably S-closed, S-Lindelöf).*

Proof. Let V be any regular closed set on Y. Then since f is almost contra-β^*-continuous and almost continuous, then $f^{-1}(V)$ is β^*-open and closed in X. By Lemma 3.3, we have $Int(f^{-1}(V)) = \beta^* Int(f^{-1}(V)) = f^{-1}(V)$. Hence, $f^{-1}(V)$ is clopen. Let $\{V_\alpha : \alpha \in I\}$ be any regular closed (resp., regular open) cover of Y. Then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a clopen cover of X and since X is mildly compact, there exists a finite subset I_0 of I such that $X = \cup\{f^{-1}(V_\alpha) : \alpha \in I_0\}$. Since f is surjection, we obtain $Y = \cup\{V_\alpha : \alpha \in I_0\}$. This shows that Y is S-closed (resp., nearly compact). The other proofs are similar. □

Theorem 3.19. *If $f : X \to Y$ is contra-β^*-continuous and A is β^*-compact relative to X, then $f(A)$ is strongly S-closed in Y.*

Proof. Let $\{V_i : i \in I\}$ be any cover of $f(A)$, by closed sets of the subspace $f(A)$. For $i \in I$, there exists a closed set A_i of Y such that $V_i = A_i \cap f(A)$. For each $x \in A$, there exists $i(x) \in I$ such that $f(x) \in A_{i(x)}$ and by Theorem 2.11, there exists $U_x \in \beta^* O(X, x)$ such that $f(U_x) \subseteq A_{i(x)}$. Since the family $\{U_x : x \in A\}$ is a cover of A by β^*-open sets of X, there exists a finite subset A_0 of A such that $A \subseteq \bigcup\{U_x : x \in A_0\}$. Therefore, we obtain $f(A) \subseteq \bigcup\{f(U_x) : x \in A_0\}$. Which is a subset of $\bigcup\{A_{i(x)} : x \in A_0\}$. Thus $f(A) = \bigcup\{V_{i(x)} : x \in A_0\}$ and hence $f(A)$ is strongly S-closed. □

Corollary 3.20. *If $f : X \to Y$ is contra-β^*-continuous surjection and X is β^*-compact, then Y is strongly S-closed.*

4. Contra-closed graphs

Recall that for a function $f : X \to Y$, the subset $\{(x, f(x)) : x \in X\} \subseteq X \times Y$ is called the graph of f and is denoted by $G(f)$.

Definition 4.1. The graph $G(f)$ of a function $f : X \to Y$ is said to be contra-β^*-closed if for each $(x, y) \in (X, Y) - G(f)$, there exist $U \in \beta^* O(X, x)$ and $V \in C(Y, y)$ such that $(U \times V) \cap G(f) = \phi$.

The following results can be easily verified.

Lemma 4.2 ([6]). *Let $G(f)$ be the graph of f, for any subset $A \subseteq X$ and $B \subseteq Y$, we have $f(A) \cap B = \phi$ if and only if $(A \times B) \cap G(f) = \phi$.*

Lemma 4.3. *The graph $G(f)$ of $f : X \to Y$ is contra-β^*-closed in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \beta^* O(X, x)$ and $V \in C(Y, y)$ such that $f(U) \cap V = \phi$.*

Theorem 4.4. *If $f : X \to Y$ is contra-β^*-continuous and Y is Urysohn, then $G(f)$ is contra-β^*-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exist open sets V, W such that $f(x) \in V$, $y \in W$, and $Cl(V) \cap Cl(W) = \phi$. Since f is contra-β^*-continuous, there exists $U \in \beta^* O(X, x)$ such that

$f(U) \subseteq Cl(V)$. Therefore, we obtain $f(U) \cap Cl(W) = \phi$. This shows that $G(f)$ is contra-β^*-closed. □

Theorem 4.5. *If $f : X \to Y$ is β^*-continuous and Y is T_1, then $G(f)$ is contra-β^*-closed in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and there exists open set V of Y, such that $f(x) \in V$, $y \notin V$. Since f is β^*-continuous, there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq V$. Therefore, $f(U) \cap (Y - V) = \phi$ and $Y - V \in C(Y, y)$. This shows that $G(f)$ is contra-β^*-closed in $X \times Y$. □

Definition 4.6. The graph $G(f)$ of a function $f : X \to Y$ is said to be strongly contra-β^*-closed if for each $(x, y) \in (X, Y) - G(f)$, there exist $U \in \beta^*O(X, x)$ and $V \in RC(Y, y)$ such that $(U \times V) \cap G(f) = \phi$.

Lemma 4.7. *The graph $G(f)$ of $f : X \to Y$ is strongly contra-β^*-closed graph in $X \times Y$ if and only if for each $(x, y) \in (X \times Y) - G(f)$, there exist $U \in \beta^*O(X, x)$ and $V \in RC(Y, y)$ such that $f(U) \cap V = \phi$.*

Theorem 4.8. *If $f : X \to Y$ is almost weakly-β^*-continuous and Y is Urysohn, then $G(f)$ is strongly contra-β^*-closed in $X \times Y$.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Urysohn, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $Cl(V) \cap Cl(W) = \phi$. Since f is almost weakly-β^*-continuous, by Definition 2.20 there exists $U \in \beta^*(X, x)$ such that $f(U) \subseteq Cl(W)$. This shows that $f(U) \cap Cl(V) = f(U) \cap Cl(Int(V)) = \phi$, where $Cl(Int(V)) \in RC(Y)$ and hence by Lemma 4.7 we have $G(f)$ is strongly contra-β^*-closed. □

Theorem 4.9. *If $f : X \to Y$ is almost contra-β^*-continuous, then f is almost weakly-β^*-continuous.*

Proof. Let $x \in X$ and V be any open set of Y containing $f(x)$. Then $Cl(V)$ is a regular closed set of Y containing $f(x)$. Since f is almost contra-β^*-continuous, by Theorem 3.2, there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq Cl(V)$. By Definition 2.20 f is almost weakly-β^*-continuous. □

Corollary 4.10. *If $f : X \to Y$ is almost contra-β^*-continuous and Y is Urysohn, then $G(f)$ is strongly contra-β^*-closed.*

The following results can be easily verified.

Lemma 4.11. *A function $f : X \to Y$ is almost β^*-continuous, if and only if for each $x \in X$ and each regular open set V of Y containing $f(x)$, there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq V$.*

Theorem 4.12. *If $f : X \to Y$ is almost β^*-continuous and Y is Hausdorff, then $G(f)$ is strongly contra-β^*-closed.*

Proof. Suppose that $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$. Since Y is Hausdorff, there exist open sets V and W in Y containing y and $f(x)$, respectively, such that $V \cap W = \phi$, hence $Cl(V) \cap Int(Cl(W)) = \phi$. Since f is almost β^*-continuous, and W is regular open by Lemma 4.11, there exists $U \in \beta^*O(X, x)$ such that $f(U) = W \subseteq Int(Cl(W))$. This shows that $f(U) \cap Cl(V) = \phi$ and hence by Lemma 4.7, we have $G(f)$ is strongly contra-β^*-closed. □

We recall that a topological space (X, τ) is said to be extremely disconnected (E.D) if the closure of every open set of X is open in X.

Theorem 4.13. *Let Y be E.D. Then a function $f : X \to Y$ is almost contra-β^*-continuous if and only if it is almost β^*-continuous.*

Proof. Let $x \in X$ and V be any regular open set of Y containing $f(x)$. Since Y is E.D then V is clopen and hence V is regular closed. By Theorem 3.2, there exists $U \in \beta^*O(X, x)$ such that $f(U) \subseteq V$. Then Lemma 4.11 implies that f is almost β^*-continuous. Conversely, let F be any regular closed set of Y. Since Y is E.D, F is also regular open and $f^{-1}(F)$ is β^*-open in X. This shows that f is almost contra-β^*-continuous. □

Theorem 4.14. *If $f : X \to Y$ is an injective almost contra-β^*-continuous function with the strongly contra-β^*-closed graph, then (X, τ) is β^*-T_2.*

Proof. Let x and y be distinct points of X. Then, since f is injective, we have $f(x) \neq f(y)$. Then we have $(x, f(y)) \in (X \times Y) - G(f)$. Since $G(f)$ is strongly contra-β^*-closed, by Lemma 4.7 there exist $U \in \beta^*O(X, x)$ and a regular closed set V containing $f(y)$ such that $f(U) \cap V = \phi$. Since f is almost contra-β^*-continuous, by Theorem 3.2, there exists $G \in \beta^*O(X, y)$ such that $f(G) \subseteq V$. Therefore, we have $f(U) \cap f(G) = \phi$; hence, $U \cap G = \phi$. This shows that (X, τ) is β^*-T_2. □

Acknowledgment. The authors would like to thank the referees for their valuable comments and suggestions.

References

1. K. Al-Zoubi and B. Al-Nashef, *The topology of ω-open subsets*, Al-Manarah J., 9 (2003), pp. 169-179.
2. M. Caldas and S. Jafari, *Some properties of contra-β-continuous functions*, Mem. Fac. Sci. Kochi Univ. Series A. Math., 22 (2001), pp. 19-28.

3. K. Dlaska, N. Ergun, and M. Ganster, *Countably S-closed spaces*, Math. Slovaca, 44 (1994), pp. 337-348.
4. K. Dontchev, *Contra-continuous functions and strongly S-closed spaces*, Int. J. Math. Math. Sci., 19 (1996), pp. 303-310.
5. J. Dontchev and T. Noiri, *Contra-semicontinuous functions*, Math. Pannon., 10 (1999), pp. 159-168.
6. E. Ekici, *Almost contra-precontinuous functions*, Bull. Malays. Math. Sci. Soc. , 27 (2004), pp. 53-65.
7. H.Z. Hdeib, *ω-closed mappings*, Rev. Colombiana Mat., 16 (1982), pp. 65-78.
8. S. Jafari and T. Noiri, *On contra-precontinuous functions*, Bull. Malays. Math. Sci. Soc. , 25 (2002), pp. 115-128.
9. S. Jafari and T. Noiri, *Contra-α-continuous functions between topological spaces*, Iran. Int. J. Sci., 2 (2001), pp. 153-167.
10. J.E. Joseph and M.H. Kwack, *On S-closed spaces*, Proc. Amer. Math. Soc. , 80 (1980), pp. 341-348.
11. N. Levine, *Generalized closed sets in topology*, Rend. Circ. Math. Palermo, 19 (1920), pp. 89-96.
12. M. Mrsevic, *On pairwise R and pairwise R_1 bitopological spaces*, Bull Math Soc Sci Math RS Roumanie, 30 (1986), pp. 141-148.
13. A.A. Nasef, *Some properties of contra-γ-continuous functions*, Chaos Solitons Fractals, 24 (2005), pp. 471-477.
14. T. Noiri and V. Popa, *Some properties of almost contra-precontinuous functions*, Bull. Malays. Math. Sci. Soc., 28 (2005), pp. 107-116.
15. P.G. Palanimani and R. Parimelazhagan, *β^*-closed sets in topological spaces*, ROSR Journal of Mathematics, 5 (2013), pp. 47-50.
16. R. Ramesh, A. Vadivel, and D. Sivakumar, *On β^*-connectedness and β^*-disconnectedness and thier applications*, J. Adv. Res. Sci. Comput., 7(2015), pp. 10-18.
17. R. Ramesh, A. Vadivel, and D. Sivakumar, *β^*-regular and β^*-normal spaces*, Int. J. of Pure and Engg. Mathematics, 2 (2014), pp. 78-80.
18. R. Ramesh, A. Vadivel, and D. Sivakumar, *Properties of β^*-homeomorphisms in topological spaces*, Gen. Math. Notes, 26 (2015), pp. 1-7.
19. M.K. Singal and A. Mathur,*On nearly-compact spaces*, Boll. Un. Mat. Ital., 2 (1969), pp. 702-710.
20. T. Soundararajan, *Weakly Hausdorff spaces and the cardinality of topological spaces in General Topology and Its Relations to Modern Analysis and Algebra*, III (Proc. Conf., Kanpur, 1968), Academia, Prague, 1971, 301-306.

[1] Department of Mathematics, Annamalai University, Annamalai Nagar-608 002, Tamil Nadu, India.
E-mail address: avmaths@gmail.com

[2] Department of Mathematics, Pope John Paul II College of Education, Reddiar Palayam, Puducherry-605010, India.
E-mail address: rameshroshitha@gmail.com

[3] Department of Mathematics (DDE), Annamalai University, Annamalai Nagar-608 002, Tamil Nadu, India.
E-mail address: sivakumardmaths@yahoo.com