r-FUZZY REGULAR SEMI OPEN SETS IN SMOOTH TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce and study the concept of r-fuzzy regular semi open (closed) sets in smooth topological spaces. By using r-fuzzy regular semi open (closed) sets, we define a new fuzzy closure operator namely r-fuzzy regular semi interior (closure) operator. Also, we introduce fuzzy regular semi continuous and fuzzy regular semi irresolute mappings. Moreover, we investigate the relationship among fuzzy regular semi continuous and fuzzy regular semi irresolute mappings. Finally, we have given some counter examples to show that these types of mappings are not equivalent.

1. Introduction and Preliminaries

Šostak [11] introduced the fundamental concept of a fuzzy topological structure, as an extension of both crisp topology and Chang’s fuzzy topology [2]. Subsequently, Badard [1] introduced the concept of smooth topological space. Chattopadhyay et al. [4] have redefined the same concept under the name gradation of openness. Ramadan [9] introduced a similar definition, namely, smooth topological space for lattice $L = [0, 1]$. Thus, the term ‘fuzzy topology’ in Šostak sense, ‘gradation of openness’ and ‘smooth topology’ are essentially referring to the same concept. In our paper, we adopt the term smooth topology. It has been developed in many direction [3-5, 10]. Mashhour et. al., [8], Zahran [13] and Kerre et. al., [6] introduced the notion of fuzzy regular semi open and fuzzy regular semi closed sets and investigate the relationship among fuzzy regular semi continuity and fuzzy regular semi irresolute mappings.

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In this paper, we introduce and study the concept of \(r\)-fuzzy regular semi open (closed) sets in smooth topological spaces. By using \(r\)-fuzzy regular semi open (closed) sets, we define a new fuzzy closure operator namely \(r\)-fuzzy regular semi interior (closure) operator. Also, we generalize the notions of fuzzy regular semi continuous (open, closed) and fuzzy regular semi irresolute (irresolute open, irresolute closed) mappings in smooth topological spaces. We show that the fuzzy regular continuous mapping is fuzzy regular semi continuous, however, the converse is not true. Also, we show that the fuzzy regular semi irresolute mapping is fuzzy regular semi continuous, but the converse is not true. Therefore, we have given characterizations of all mentioned types of mappings. Finally, we establish some counter examples to show that these types are not equivalent.

Throughout this paper, let \(X\) be a non-empty set, \(I = [0, 1], I_0 = (0, 1]\). A fuzzy set \(\lambda\) of \(X\) is a mapping \(\lambda : X \to I\), and \(I^X\) be the family of all fuzzy sets on \(X\). The complement of a fuzzy set \(\lambda\) is denoted by \(\overline{\lambda}\). For \(\lambda \in I^X\), \(\overline{\lambda}(x) = \lambda\) for all \(x \in X\). For each \(x \in X\) and \(t \in I_0\), a fuzzy point \(x_t\) is defined by

\[x_t(y) = \begin{cases} t & \text{if } y = x; \\ 0 & \text{if } y \neq x. \end{cases} \]

Let \(Pt(X)\) be the family of all fuzzy points in \(X\). All other notations and definitions are standard in the fuzzy set theory.

**Definition 1.1** ([11]). A function \(\tau : I^X \to I\) is called a smooth topology on \(X\) if it satisfies the following conditions:

1. \(\tau(\emptyset) = \tau(\overline{I}) = 1,\)
2. \(\tau(\bigvee_{i \in J} \mu_i) \geq \bigwedge_{i \in J} \tau(\mu_i), \) for any \(\{\mu_i : i \in J\} \subseteq I^X,\)
3. \(\tau(\mu_1 \wedge \mu_2) \geq \tau(\mu_1) \wedge \tau(\mu_2), \) for all \(\mu_1, \mu_2 \in I^X.\)

The pair \((X, \tau)\) is called a smooth topological space. A fuzzy set \(\lambda\) is called an \(r\)-fuzzy open (\(r\)-fo, for short) if \(\tau(\lambda) \geq r\). A fuzzy set \(\lambda\) is called an \(r\)-fuzzy closed (\(r\)-fc, for short) set if \(\tau(\overline{\lambda}) \geq r\).

**Theorem 1.2** ([3]). Let \((X, \tau)\) be a smooth topological space. Then for each \(\lambda \in I^X\) and \(r \in I_0\), we define an operator \(C_\tau : I^X \times I_0 \to I^X\) as follows:

\[C_\tau(\lambda, r) = \bigwedge \{\mu \in I^X : \lambda \leq \mu, \ \tau(\overline{\lambda} - \mu) \geq r\}.\]

For \(\lambda, \mu \in I^X\) and \(r, s \in I_0\), the operator \(C_\tau\) satisfies the following statements:

(C1) \(C_\tau(\overline{\emptyset}, r) = \overline{\emptyset},\)
(C2) \(\lambda \leq C_\tau(\lambda, r),\)
\( C_\tau(\lambda, r) \lor C_\tau(\mu, r) = C_\tau(\lambda \lor \mu, r), \)
\( C_\tau(\lambda, r) \leq C_\tau(\lambda, s) \text{ if } r \leq s, \)
\( C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r). \)

**Theorem 1.3** ([3]). Let \((X, \tau)\) be a smooth topological space. Then for each \( \lambda \in I^X \) and \( r \in I_0 \), we define an operator \( I_\tau : I^X \times I_0 \to I^X \) as follows:

\[
I_\tau(\lambda, r) = \bigvee \{ \mu \in I^X : \mu \leq \lambda, \tau(\mu) \geq r \}.
\]

For \( \lambda, \mu \in I^X \) and \( r, s \in I_0 \), the operator \( I_\tau \) satisfies the following statements:

(1) \( I_\tau(\lambda, r) = \lambda \),
(2) \( I_\tau(\lambda, r) \leq \lambda \),
(3) \( I_\tau(\lambda, r) \cap I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r), \)
(4) \( I_\tau(\lambda, r) \leq I_\tau(\lambda, s) \text{ if } s \leq r, \)
(5) \( I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r), \)
(6) \( I_\tau(\lambda, r) = I_\tau(\lambda, r) \text{ and } C_\tau(I_\tau(\lambda, r), r) = C_\tau(\lambda, r) \text{ and } C_\tau(\lambda, r) = C_\tau(I_\tau(\lambda, r), r). \)

**Definition 1.4** ([7]). Let \((X, \tau)\) be a smooth topological space, \( \lambda \in I^X \) and \( r \in I_0 \). Then

(1) A fuzzy set \( \lambda \) is called \( r \)-fuzzy regular open (for short, \( r \)-fro) if \( \lambda = I_\tau(C_\tau(\lambda, r), r). \)
(2) A fuzzy set \( \lambda \) is called \( r \)-fuzzy regular closed (for short, \( r \)-frc) if \( \lambda = C_\tau(I_\tau(\lambda, r), r). \)

**Definition 1.5** ([12]). Let \((X, \tau)\) be a smooth topological space. For \( \lambda, \mu \in I^X \) and \( r \in I_0 \),

(1) The \( r \)-fuzzy regular closure of \( \lambda \), denoted by \( RC_\tau(\lambda, r) \), and is defined by

\[
RC_\tau(\lambda, r) = \bigwedge \{ \mu \in I^X : \mu \geq \lambda, \mu \text{ is } r \text{-frc} \}.
\]
(2) The \( r \)-fuzzy regular interior of \( \lambda \), denoted by \( RI_\tau(\lambda, r) \), and is defined by

\[
RI_\tau(\lambda, r) = \bigvee \{ \mu \in I^X : \mu \leq \lambda, \mu \text{ is } r \text{-fro} \}.
\]

**Definition 1.6** ([13]). Let \((X, \tau)\) be a smooth topological space and \( \lambda \in I^X \), \( r \in I_0 \). Then

(1) \( \lambda \) is called \( r \)-fuzzy semi open (for short, \( r \)-fso) if there exists \( \mu \in I^X \) with \( \tau(\mu) \geq r \) and \( \mu \leq \lambda \leq C_\tau(\mu, r). \)
(2) \( \lambda \) is called \( r \)-fuzzy semi closed (for short, \( r \)-fsc) if there exists \( \mu \in I^X \) with \( \tau(\lambda - \mu) \geq r \) and \( I_\tau(\mu, r) \leq \lambda \leq \mu. \)
2. \textit{r}-fuzzy regular semi open and \textit{r}-fuzzy regular semi closed sets

\textbf{Definition 2.1.} Let \((X, \tau)\) be a smooth topological space and \(\lambda \in I^X, \ r \in I_0\). Then

1. \(\lambda\) is called \(r\)-fuzzy regular semi open (for short, \(r\)-frso) if there exists \(r\)-fro set \(\mu \in I^X\) such that \(\mu \leq \lambda \leq C_\tau(\mu, r)\).
2. \(\lambda\) is called \(r\)-fuzzy regular semi closed (for short, \(r\)-frsc) if there exists \(r\)-frc set \(\mu \in I^X\) such that \(I_\tau(\mu, r) \leq \lambda \leq \mu\).
3. The \(r\)-fuzzy regular semi interior of \(\lambda\), denoted by \(RSI_\tau(\lambda, r)\), is defined by
   \[RSI_\tau(\lambda, r) = \bigvee \{\mu \in I^X \mid \mu \leq \lambda, \ \mu \ \text{is} \ r\text{-frso}\}\] 
4. The \(r\)-fuzzy regular semi closure of \(\lambda\), denoted by \(RSC_\tau(\lambda, r)\) is defined by
   \[RSC_\tau(\lambda, r) = \bigwedge \{\mu \in I^X \mid \mu \geq \lambda, \ \mu \ \text{is} \ r\text{-frsc}\}\]

\textbf{Theorem 2.2.} Let \((X, \tau)\) be a smooth topological space. For \(\lambda \in I^X, \ r \in I_0\), the following statements are equivalent:

1. \(\lambda\) is \(r\)-frso.
2. \(I_\tau(\lambda, r) = I_\tau(C_\tau(\lambda, r), r)\).
3. \(I - \lambda\) is \(r\)-frso.
4. \(C_\tau(I - \lambda, r) = C_\tau(I_\tau(I - \lambda, r), r)\).

\textbf{Proof.} (1)\(\Leftrightarrow\) (2) Let \(\lambda\) be an \(r\)-frso set. Then there exists an \(r\)-fro set \(\mu\) such that \(\mu \leq \lambda \leq C_\tau(\mu, r)\). Hence \(C_\tau(\lambda, r) = C_\tau(\mu, r)\). Since \(I_\tau(C_\tau(\lambda, r), r) = \mu\),

\[
I_\tau(C_\tau(\lambda, r), r) = \mu \leq I_\tau(\lambda, r)
\leq I_\tau(C_\tau(\mu, r), r)
= I_\tau(C_\tau(\lambda, r), r).
\]

Thus, \(I_\tau(\lambda, r) = I_\tau(C_\tau(\lambda, r), r)\).

(1)\(\Leftrightarrow\) (3) Let \(\lambda\) be an \(r\)-frso set. Then there exists an \(r\)-fro set \(\mu\) such that \(\mu \leq \lambda \leq C_\tau(\mu, r)\). Since \(I - \mu\) is an \(r\)-frc set, \(I_\tau(I - \mu, r)\) is an \(r\)-fro set such that

\[
I_\tau(I - \mu, r) \leq I - \lambda
\leq I - \mu
= C_\tau(I_\tau(I - \mu, r), r).
\]

Thus \(I - \lambda\) is \(r\)-frso.

The implication (1)\(\Leftrightarrow\) (4) follow immediately by taking the complement of the two sides. \(\square\)
Theorem 2.3. Let \((X, \tau)\) be a smooth topological space. Then

1. any union of \(r\)-frso sets is an \(r\)-frso set.
2. any intersection of \(r\)-frsc sets is an \(r\)-frsc set.

Proof. (1) Let \(\{\lambda_\alpha : \alpha \in \Gamma\}\) be a family of \(r\)-frso. Then for each \(\alpha \in \Gamma\) there exists \(r\)-fro set \(\nu_\alpha \in I^X\) such that

\[
\nu_\alpha \leq \lambda_\alpha \leq C_\tau(\nu_\alpha, r).
\]

Also, \(\nu_\alpha \leq \vee \nu_\alpha\) implies that \(C_\tau(\nu_\alpha, r) \leq C_\tau(\vee \nu_\alpha, r)\). Then, from (2.1) we have:

\[
\vee \nu_\alpha \leq \vee \lambda_\alpha \leq C_\tau(\nu_\alpha, r) \leq C_\tau(\vee \nu_\alpha, r).
\]

Since, \(\tau(\vee \nu_\alpha) \geq \wedge(\nu_\alpha)\) is \(r\)-fro set, then \(\vee \lambda_\alpha\) is \(r\)-frso.

(2) Similar to (1).

\(\square\)

Theorem 2.4. Let \((X, \tau)\) be an smooth topological space, \(\lambda, \mu \in I^X\) and \(r \in I_0\).

1. If \(\lambda\) is an \(r\)-fro set, then \(\lambda\) is an \(r\)-frso.
2. If \(\lambda\) is an \(r\)-frso and \(I_\tau(\lambda, r) \leq \mu \leq C_\tau(\lambda, r)\), then \(\mu\) is an \(r\)-frso.
3. If \(\lambda\) is an \(r\)-frsc and \(I_\tau(\lambda, r) \leq \mu \leq C_\tau(\lambda, r)\), then \(\mu\) is an \(r\)-frsc.

Proof. (1) Let \(\lambda\) is an \(r\)-fro, since \(\lambda \leq \lambda \leq C_\tau(\lambda, r)\). Then \(\lambda\) is \(r\)-frso.

(2) Let \(\lambda\) be \(r\)-frso such that \(I_\tau(\lambda, r) \leq \mu \leq C_\tau(\lambda, r)\). Then there exists \(r\)-fro set \(\nu \in I^X\) such that \(\nu \leq \lambda \leq C_\tau(\nu, r)\). It implies \(C_\tau(\lambda, r) \leq C_\tau(\nu, r)\) and therefore \(\mu \leq C_\tau(\nu, r)\). Also, \(\nu \leq I_\tau(\lambda, r) \leq \mu\). It follows that, \(\nu \leq \mu \leq C_\tau(\nu, r)\). So, \(\mu\) is \(r\)-frso.

(3) It is similar to part (2).

\(\square\)

Remark 2.5. (1) Every \(r\)-fro set is an \(r\)-frso set.

(2) Every \(r\)-frso set is an \(r\)-fso set.

The converse of the above Remark 2.5 is not true in general, as shown from the following examples.

Example 2.6. Let \(X = \{a, b, c\}\) and \(\lambda, \mu, \delta \in I^X\) are defined as

\[
\begin{align*}
\lambda(a) &= 0.5, & \lambda(b) &= 0.5, & \lambda(c) &= 0.6, \\
\mu(a) &= 0.4, & \mu(b) &= 0.5, & \mu(c) &= 0.6, \\
\delta(a) &= 0.4, & \delta(b) &= 0.5, & \delta(c) &= 0.4.
\end{align*}
\]

We define smooth topology \(\tau : I^X \to I\) as follows:
\[\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{0, 1\}, \\
\frac{1}{2} & \text{if } \lambda = \mu, \\
\frac{1}{4} & \text{if } \lambda = \delta, \\
0 & \text{otherwise.} \end{cases}\]

For \(r = \frac{1}{3}\), then the fuzzy set \(\lambda\) is an \(r\)-frso set, because there exists an \(r\)-fro set \(\mu \in I^X\) such that \(\mu \leq \lambda \leq C_r(\mu, r)\). But \(\lambda\) is not an \(r\)-frso set.

**Example 2.7.** Let \(X = \{a, b, c\}, \lambda, \mu, \delta \in I^X\) are defined as

\[
\begin{align*}
\lambda(a) &= 0.4, & \lambda(b) &= 0.5, & \lambda(c) &= 0.6, \\
\mu(a) &= 0.3, & \mu(b) &= 0.5, & \mu(c) &= 0.6, \\
\delta(a) &= 0.6, & \delta(b) &= 0.5, & \delta(c) &= 0.5.
\end{align*}
\]

We define smooth topology \(\tau : I^X \to I\) as follows:

\[\tau(\lambda) = \begin{cases} 1 & \text{if } \lambda \in \{0, 1\}, \\
\frac{1}{2} & \text{if } \lambda = \mu, \\
\frac{1}{4} & \text{if } \lambda = \delta, \\
\frac{1}{8} & \text{if } \lambda = \mu \lor \delta, \\
\frac{1}{8} & \text{if } \lambda = \mu \land \delta, \\
0 & \text{otherwise.} \end{cases}\]

For \(r = \frac{1}{2}\), then the fuzzy set \(\lambda\) is an \(r\)-fso set, because there is an \(r\)-fuzzy open set \(\mu \in I^X\) such that \(\mu \leq \lambda \leq C_r(\mu, r)\). But \(\lambda\) is not an \(r\)-frso set.

**Theorem 2.8.** Let \((X, \tau)\) be a smooth topological space. For \(\lambda, \mu \in I^X\) and \(r \in I_0\), it satisfies the following statements:

1. \(\lambda\) is \(r\)-frso \(\iff\) \(\lambda = RSI_r(\lambda, r)\).
2. \(\lambda\) is \(r\)-frsc \(\iff\) \(\lambda = RSC_r(\lambda, r)\).
3. \(RSC_r(\bar{0}, r) = \bar{0}\).
4. \(RI_r(\lambda, r) \leq RSI_r(\lambda, r) \leq \lambda \leq RSC_r(\lambda, r) \leq RC_r(\lambda, r)\).
5. \(RSC_r(\lambda, r) \lor RSC_r(\mu, r) \leq RSC_r(\lambda \lor \mu, r)\).
6. \(RC_r(RSC_r(\lambda, r), r) = RSC_r(\lambda, r), r) = RC_r(\lambda, r)\).

**Proof.**

1. Let \(\lambda\) be \(r\)-frso. Then

\[
RSI_r(\lambda, r) = \lor \{\rho \in I^X : \rho \leq \lambda, \text{ } \rho \text{ is } r \text{-frso}\} = \lambda.
\]

Conversely, let \(\lambda = RSI_r(\lambda, r)\). Since \(RSI_r(\lambda, r)\) is the arbitrary union of \(r\)-frso’s, then \(\lambda\) is \(r\)-frso.

2. It is similar to part (1).

3. It is easily obtained from Definition 2.1.
(4) Since
\[ R_{I}(\lambda, r) = \bigvee \{ \rho \in I^{X} : \rho \leq \lambda, \; \rho \text{ is } r \text{-fro} \} \]
\[ \leq \bigvee \{ \rho \in I^{X} : \rho \leq \lambda, \; \rho \text{ is } r \text{-frso} \} = RSI_{I}(\lambda, r), \]
it follows that, \( R_{I}(\lambda, r) \leq RSI_{I}(\lambda, r). \) Also,
\[ RSC_{\tau}(\lambda, r) = \bigwedge \{ \rho \in I^{X} : \rho \geq \lambda, \; \rho \text{ is } r \text{-frsc} \} \]
\[ \leq RC_{\tau}(\lambda, r). \]
Finally, we have
\[ R_{I}(\lambda, r) \leq RSI_{I}(\lambda, r) \leq \lambda \leq RSC_{\tau}(\lambda, r) \leq RC_{\tau}(\lambda, r). \]

(5) Since, \( \mu \leq \mu \lor \rho, \; \rho \leq \mu \lor \rho, \) then
\[ RSC_{\tau}(\mu, r) \leq RSC_{\tau}(\mu \lor \rho, r) \]
and
\[ RSC_{\tau}(\rho, r) \leq RSC_{\tau}(\mu \lor \rho, r). \]
Hence,
\[ RSC_{\tau}(\mu, r) \lor RSC_{\tau}(\rho, r) \leq RSC_{\tau}(\mu \lor \rho, r). \]

(6) Since \( RC_{\tau}(\lambda, r) \) is \( r \)-frsc set, then
\[ RSC_{\tau}(RC_{\tau}(\lambda, r), r) = RC_{\tau}(\lambda, r). \]

Now it remains to prove only the relation:
\[ RC_{\tau}(RSC_{\tau}(\lambda, r), r) = RC_{\tau}(\lambda, r). \]
Since, \( \lambda \leq RSC_{\tau}(\lambda, r), \) then \( RC_{\tau}(\lambda, r) \leq RC_{\tau}(RSC_{\tau}(\lambda, r)). \) It remains to prove:
\[ RC_{\tau}(RSC_{\tau}(\lambda, r), r) \leq RC_{\tau}(\lambda, r). \]
Let the contrary, that is,
\[ RC_{\tau}(RSC_{\tau}(\lambda, r), r) \not\leq RC_{\tau}(\lambda, r). \]
Then
\[ RC_{\tau}(RSC_{\tau}(\lambda, r), r) > RC_{\tau}(\lambda, r). \]
So, there exists an \( r \)-frc set \( \rho \in I^{X}, \; \rho \geq \lambda \) such that
\[ RC_{\tau}(\lambda, r)(x) < \rho(x) < RC_{\tau}(RSC_{\tau}(\lambda, r), r)(x). \]
Since,
\[ \lambda \leq \rho \quad \Rightarrow \quad RSC_{\tau}(\lambda, r) \leq RSC_{\tau}(\rho, r) \]
\[ = RSC_{\tau}(RC_{\tau}(\lambda, r), r) \]
\[ = RC_{\tau}(\rho, r). \]
Then, $RSC_\tau(\lambda, r) \leq RC_\tau(\rho, r)$ and this implies

$$RC_\tau(RSC_\tau(\lambda, r), r) \leq RC_\tau(\lambda, r),$$

which contradicts to the relation (2.3). Hence the result. \qed

### 3. Fuzzy Regular Semi Continuous Mappings

**Definition 3.1.** Let $(X, \tau)$ and $(Y, \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. Then $f$ is said to be:

1. fuzzy regular semi irresolute (resp. fuzzy regular semi continuous) if $f^{-1}(\mu)$ is $r$-frso for each $r$-frso set $\mu \in I^Y$ (resp. $\mu \in \bar{I}^Y$).
2. fuzzy regular semi irresolute open (resp. fuzzy regular semi open) if $f(\lambda)$ is $r$-frso in $Y$ for each $r$-frso set $\lambda \in I^X$ (resp. $\lambda \in \bar{I}^X$).
3. fuzzy regular semi irresolute closed (resp. fuzzy regular semi closed) if $\bar{f}(\lambda)$ is $r$-frsc in $Y$ for each $r$-frsc set $\lambda \in I^X$ (resp. $\lambda \in \bar{I}^X$).
4. fuzzy regular semi irresolute homeomorphism if $f$ is bijective, $f$ and $f^{-1}$ are fuzzy regular semi irresolute.

**Remark 3.2.** Every fuzzy regular continuous (resp. fuzzy regular semi irresolute) mapping is fuzzy regular semi continuous. However, the converse is not true as shown in the following example.

**Example 3.3.** Let $X = \{a, b, c\} = Y$, $\lambda_1, \lambda_2 \in I^X$, $\lambda_3$ and $\lambda_4 \in I^Y$ are defined as

\[
\begin{align*}
\lambda_1(a) &= 0.4, & \lambda_1(b) &= 0.5, & \lambda_1(c) &= 0.6, \\
\lambda_2(a) &= 0.4, & \lambda_2(b) &= 0.5, & \lambda_2(c) &= 0.4, \\
\lambda_3(a) &= 0.5, & \lambda_3(b) &= 0.5, & \lambda_3(c) &= 0.6, \\
\lambda_4(a) &= 0.4, & \lambda_4(b) &= 0.5, & \lambda_4(c) &= 0.6.
\end{align*}
\]

We define smooth topologies $\tau, \eta : I^X \to I$ as follows:

\[
\tau(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\
\frac{1}{2} & \text{if } \lambda = \lambda_1, \\
\frac{1}{3} & \text{if } \lambda = \lambda_2, \\
0 & \text{otherwise},
\end{cases}
\quad \eta(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{\overline{0}, \overline{1}\}, \\
\frac{1}{2} & \text{if } \lambda = \lambda_3, \\
\frac{1}{2} & \text{if } \lambda = \lambda_4, \\
0 & \text{otherwise}.
\end{cases}
\]

For $\tau = \frac{1}{2}$ and $\eta(\lambda_3) \geq r$, there exist an $r$-frso set $\lambda_1 \in I^X$ such that $\lambda_1 \leq \lambda_3 \leq \overline{C}_\tau(\lambda_1, r)$ and $f^{-1}(\lambda_3)$ is an $r$-frso set in $X$, but $f^{-1}(\lambda_3)$ is not $r$-frso set in $X$. Then the identity mapping $f$ is fuzzy regular semi continuous but not fuzzy regular continuous.
Example 3.4. Let \( X = \{a, b, c\} = Y, \lambda_1, \lambda_2 \in I^X, \lambda_3 \) and \( \lambda_4 \in I^Y \) are defined as
\[
\begin{align*}
\lambda_1(a) &= 0.4, & \lambda_1(b) &= 0.5, & \lambda_1(c) &= 0.6, \\
\lambda_2(a) &= 0.4, & \lambda_2(b) &= 0.5, & \lambda_2(c) &= 0.4, \\
\lambda_3(a) &= 0.6, & \lambda_3(b) &= 0.5, & \lambda_3(c) &= 0.4, \\
\lambda_4(a) &= 0.4, & \lambda_4(b) &= 0.5, & \lambda_4(c) &= 0.6.
\end{align*}
\]

We define smooth topologies \( \tau, \eta : I^X \to I \) as follows:
\[
\tau(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{0, 1\}, \\
\frac{1}{2} & \text{if } \lambda = \lambda_1, \\
\frac{1}{2} & \text{if } \lambda = \lambda_2, \\
0 & \text{otherwise},
\end{cases}
\]
\[
\eta(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{0, 1\}, \\
\frac{1}{2} & \text{if } \lambda = \lambda_3, \\
\frac{1}{2} & \text{if } \lambda = \lambda_4, \\
\frac{1}{2} & \text{if } \lambda = \lambda_3 \lor \lambda_4, \\
0 & \text{otherwise},
\end{cases}
\]

For \( r = \frac{1}{2} \) and \( \eta(\lambda_4) \geq r \), there exist an \( r \)-frso set \( \lambda_1 \) such that \( \lambda_1 \leq \lambda_4 \leq C_r(\lambda_1, r) \) and \( f^{-1}(\lambda_4) \) is an \( r \)-frsc set in \( X \). But \( \lambda_3 \) is an \( r \)-frso set in \( Y \), \( f^{-1}(\lambda_3) \) is not an \( r \)-frsc set in \( X \). Then the identity mapping \( f \) is fuzzy regular semi continuous but not fuzzy regular semi irresolute.

Theorem 3.5. Let \((X, \tau)\) and \((Y, \eta)\) be smooth topological spaces. Let \( f : X \to Y \) be a mapping. The following statements are equivalent:

1. \( f \) is fuzzy regular semi irresolute.
2. For each \( r \)-frsc set \( \mu \in I^Y \), \( f^{-1}(\mu) \) is an \( r \)-frsc set in \( X \).
3. \( f(RSC_\tau(\lambda, r)) \leq RSC_\eta(f(\lambda), r) \), for each \( \lambda \in I^X \) and \( r \in I_0 \).
4. \( RSC_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSC_\eta(\mu, r)) \), for each \( \mu \in I^Y \) and \( r \in I_0 \).
5. \( f^{-1}(RSI_\eta(\mu, r)) \leq RSI_\tau(f^{-1}(\mu), r) \), for each \( \mu \in I^Y \) and \( r \in I_0 \).

Proof. (1)\(\Rightarrow\)(2) Obvious.
(2)\(\Rightarrow\)(3) Let (2) holds and let
\[
f(RSC_\tau(\lambda, r)) \not\leq RSC_\eta(f(\lambda), r),
\]
for some \( \lambda \in I^X \) and \( r \in I_0 \). So there exists \( y \in Y \) and \( t \in (0, 1] \) such that:
\[
f(RSC_\tau(\lambda, r))(y) > t > RSC_\eta(f(\lambda), r)(y).
\]
If \( f^{-1}(y) = \emptyset \), it is a contradiction, because
\[
f(RSC_\tau(\lambda, r))(y) = \emptyset.
\]
So, if \( f^{-1}(y) \neq \emptyset \) and there exists \( x \in f^{-1}(y) \) such that
\[
(3.1) \quad f(RSC_\tau(\lambda, r))(y) \geq RSC_\tau(\lambda, r)(x) > t > RSC_\eta(f(\lambda), r)(f(x)).
\]
Also, $RSC_{\eta}(f(\lambda), r)(f(x)) < t$, implies that there exists an $r$-frsc set $\mu$ with $f(\lambda) = \mu$ such that

$$RSC_{\eta}(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t.$$  

Moreover, $f(\lambda) \leq \mu$ implies $\lambda \leq f^{-1}(\mu)$. From (2), $f^{-1}(\mu)$ is $r$-frsc. Thus $RSC_{\tau}(\lambda, r) \leq f^{-1}(\mu)$ and this implies:

$$RSC_{\tau}(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t. \quad (3.2)$$

From relations (3.1) and (3.2) we see that:

$$RSC_{\tau}(f^{-1}(\mu), r) \leq f^{-1}(RSC_{\eta}(\mu, r)). \quad (4)$$

(4) $\Rightarrow$ (5) It follows immediately by taking the complement of (4).

(5) $\Rightarrow$ (1) Let (5) holds and let $\mu$ be $r$-frso in $Y$. By (5)

$$f^{-1}(RI_{\eta}(\mu, r)) \leq RSI_{\tau}(f^{-1}(\mu), r).$$

Then,

$$f^{-1}(\mu) \leq RSI_{\tau}(f^{-1}(\mu), r).$$

But

$$RSI_{\tau}(f^{-1}(\mu), r) \leq f^{-1}(\mu).$$

Hence

$$RSI_{\tau}(f^{-1}(\mu), r) = f^{-1}(\mu).$$

So, $f^{-1}(\mu)$ is $r$-frso and there by $f$ is fuzzy regular semi irresolute. □

**Theorem 3.6.** Let $(X, \tau)$ and $(Y, \eta)$ be smooth topological spaces. Let $f : X \rightarrow Y$ be a mapping. The following statements are equivalent:

1. $f$ is fuzzy regular semi continuous.
2. $f(RSC_{\tau}(\lambda, r)) \leq RC_{\eta}(f(\lambda), r)$, for each $\lambda \in I^{X}$.
3. $RSC_{\tau}(f^{-1}(\mu), r) \leq f^{-1}(RC_{\eta}(\mu, r))$ for each $\mu \in I^{Y}$.
4. $f^{-1}(RI_{\eta}(\mu, r)) \leq RSI_{\tau}(f^{-1}(\mu), r)$, for each $\mu \in I^{Y}$.

**Proof.** (1) $\Rightarrow$ (2) Let (1) holds and let $\lambda \in I^{X}$. Suppose that there exists $\lambda \in I^{X}$ and $r \in I_{0}$ such that

$$f(RSC_{\tau}(\lambda, r)) \not\leq RC_{\eta}(f(\lambda), r).$$

So there exists $y \in Y$ and $t \in (0, 1]$ such that

$$f(RSC_{\tau}(\lambda, r))(y) > t > RC_{\eta}(f(\lambda), r)(y).$$

(3) $\Rightarrow$ (4) It follows immediately by taking the complement of (3).
If \( f^{-1}(y) = \emptyset \), then \( f(RSC_\tau(\lambda, r))(y) = \emptyset \). So, there exists \( x \in f^{-1}(y) \) such that:

\[
(3.3) \quad f(RSC_\tau(\lambda, r))(y) > RSC_\tau(\lambda, r))(x) > t > RC_\eta(f(\lambda), r)(y).
\]

Since \( RC_\eta(f(\lambda), r)(y) < t \), there exists an \( r \)-frso set \( \mu \) such that \( f(\lambda) = \mu \). Then, \( RC_\eta(f(\lambda), r)(y) \leq \mu \) and this implies

\[
RC_\eta(f(\lambda), r)(f(x)) \leq \mu(f(x)) < t.
\]

Moreover, \( f(\lambda) \leq \mu \) implies \( \lambda \leq f^{-1}(\mu) \). So, \( \overline{\lambda} - \overline{\lambda} \geq f^{-1}(\mu) \).

Since \( f^{-1}(\overline{\lambda} - \overline{\lambda}) \) is \( r \)-frso, then from (1),

\[
f^{-1}(\overline{\lambda} - \overline{\lambda}) \leq RSI_\tau(f^{-1}(\lambda), r),
\]

or

\[
\overline{\lambda} - RSI_\tau(f^{-1}(\lambda), r) \leq \overline{\lambda} - f^{-1}(\overline{\lambda} - \overline{\lambda}).
\]

Then, \( RSC_\tau(\lambda, r) \leq f^{-1}(\mu) \). This gives

\[
RSC_\tau(\lambda, r)(x) \leq f^{-1}(\mu)(x) = \mu(f(x)) < t.
\]

But the last relation contradicts the relation \( (3.3) \) above. Hence, the result in (2) is true.

(2)\(\Rightarrow\)(3) Take \( \lambda = f^{-1}(\mu)(\mu \in I^Y) \) and apply (2) we have the required result.

(3)\(\Rightarrow\)(4) Taking the complement of (3), we have the result.

(4)\(\Rightarrow\)(1) Let (4) holds and let \( \mu \in I^Y \) be an \( r \)-frso set, since every \( r \)-frso set is \( r \)-fuzzy open. So \( \eta(\mu) \geq r \). By (4),

\[
f^{-1}(RI_\eta(\mu, r)) \leq RSI_\tau(f^{-1}(\mu), r),
\]

or

\[
f^{-1}(\mu) \leq RSI_\tau(f^{-1}(\mu), r),
\]

(Since \( \eta(\mu) \geq r \)). But \( RSI_\tau(f^{-1}(\mu), r) \leq f^{-1}(\mu) \). Then, \( f^{-1}(\mu) = RSI_\tau(f^{-1}(\mu), r) \) and this means that \( f^{-1}(\mu) \) is \( r \)-frso. Hence, \( f \) is fuzzy regular semi continuous.

\(\square\)

**Theorem 3.7.** Let \((X, \tau)\) and \((Y, \eta)\) be smooth topological spaces. Let \( f : X \rightarrow Y \) be a mapping. The following statements are equivalent:

1. A map \( f \) is fuzzy regular semi irresolute open.
2. \( f(RSI_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r), \) for each \( \lambda \in I^X \) and \( r \in I_0 \).
3. \( RSI_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSI_\eta(\mu, r)), \) for each \( \mu \in I^Y \) and \( r \in I_0 \).
4. For any \( \lambda \in I^Y \) and any \( r \)-frso set \( \mu \in I^X \) such that \( \mu \geq f^{-1}(\lambda)\), there exists an \( r \)-frso set \( \rho \in I^Y \) with \( \lambda \leq \rho \) such that \( f^{-1}(\rho) \leq \lambda \).
Proof. (1)⇒(2) Since $RSI_\tau(\lambda, r) \leq \lambda$, then $RSI_\tau(\lambda, r) \leq f(\lambda)$. By (1),
\[ f(RSI_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r). \]

(2)⇒(3) Let (2) holds. Take $\lambda = f^{-1}(\mu)$, $\lambda \in \mathcal{I}^Y$ and apply part (2).

(3)⇒(4) Let $\mu \in \mathcal{I}^X$ be an $r$-frsc set. Since $f^{-1}(\lambda) \leq \mu$, then
\[ f^{-1}(\lambda) = f^{-1}(\lambda - \mu) = f^{-1}(\lambda - \mu). \]
It follows,
\[ RSI_\tau(f^{-1}(\lambda - \mu), r) \leq f^{-1}(\lambda - \mu) \leq RSI_\eta(\lambda - \mu, r). \]
By (3),
\[ f^{-1}(\lambda - \mu) \leq RSI_\tau(f^{-1}(\lambda - \mu), r) \leq f^{-1}(RSI_\eta(\lambda - \mu, r)). \]
It implies
\[ \mu \geq f^{-1}(RSI_\eta(\lambda - \mu, r)) = f^{-1}(\lambda - RSI_\eta(\lambda - \mu, r)) = f^{-1}(RSI_\eta(\lambda, r)). \]
So, $\mu \geq f^{-1}(RSC_\tau(\lambda, r))$. Take $\rho = RSC_\tau(\lambda, r)$. Then $\rho$ is $r$-frsc such that $\mu \geq f^{-1}(\rho)$ and $\rho \geq \lambda$. Hence the result.

(4)⇒(1) Let $\omega$ be $r$-frso in $X$. Put $\lambda = f^{-1}(\omega)$ and $\mu = f^{-1}(\omega)$. It is easy to see that $\mu \geq f^{-1}(\lambda)$. By part (4), there exists an $r$-frsc set $\rho \in \mathcal{I}^Y$ such that $\rho \geq \lambda$ and $\mu \geq f^{-1}(\rho)$ or $\lambda - \omega \geq f^{-1}(\rho)$. It implies
\[ \omega \leq f^{-1}(\rho) \leq f^{-1}(\lambda - \rho). \]
Thus $f(\omega) \leq f(f^{-1}(\lambda - \rho)) \leq f(\lambda) - \rho$. On the other hand, $\lambda \geq \rho$, $f(\omega) = f^{-1}(\lambda - \omega)$ implies $f(\omega) = f^{-1}(\lambda - \rho) \geq f^{-1}(\rho)$. Finally, we have $f(\omega) = f^{-1}(\rho)$ and therefore $f(\omega)$ is $r$-frso. Hence $f$ is fuzzy regular semi irresolute open. \[ \square \]

**Theorem 3.8.** Let $(X, \tau)$ and $(Y, \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:

1. $f$ is fuzzy regular semi open.
2. $f(I_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r)$, for each $\lambda \in \mathcal{I}^X$.
3. $I_\tau(f^{-1}(\mu), r) \leq f^{-1}(RSI_\eta(\mu, r))$, for each $\mu \in \mathcal{I}^Y$.
4. For any $\lambda \in \mathcal{I}^Y$ and any $\mu \in \mathcal{I}^X$ such that $\tau(\lambda - \mu) \geq r$, $\mu \geq f^{-1}(\lambda)$, there exists an $r$-frsc set $\rho \in \mathcal{I}^Y$ with $\lambda \leq \rho$ and $\mu \geq f^{-1}(\rho)$.

**Proof.** (1)⇒(2) Since $I_\tau(\lambda, r) \leq \lambda$, $(\lambda \in \mathcal{I}^X)$, then $f(I_\tau(\lambda, r)) \leq f(\lambda)$.
But $f(I_\tau(\lambda, r))$ is $r$-frso (by (1)), then $f(I_\tau(\lambda, r)) \leq RSI_\eta(f(\lambda), r)$. 

(2)⇒(3) Let (2) holds. Take $\lambda = f^{-1}(\mu)$, $\lambda \in \mathcal{I}^Y$ and apply part (2).
(2)⇒(3) Let (2) holds. Put $\lambda = f^{-1}(\mu)$, $\mu \in I^Y$ and apply (2), we have

$$f(I_r(f^{-1}(\mu), r)) \leq RSI_\eta(f f^{-1}(\mu), r),$$

$$\leq RSI_\eta(\mu, r).$$

Hence, $I_r(f^{-1}(\mu), r) \leq f^{-1}(RSI_\eta(\mu, r))$.

(3)⇒(4) Let (3) holds and let $\lambda \in I^Y$ and $\mu \in I^X$ such that $\tau(\bar{1} - \mu) \geq r$ and $\mu = f^{-1}(\lambda)$. Since $\bar{1} - \mu \leq \bar{1} - f^{-1}(\lambda) = f^{-1}(\bar{1} - \lambda)$, it implies

$$I_r(\bar{1} - \mu) = \bar{1} - \mu \leq I_r(f^{-1}(\bar{1} - \lambda), r),$$

or

$$\bar{1} - \mu \leq f^{-1}(RSI_\eta(\bar{1} - \lambda, r)).$$

Then,

$$\mu \geq \bar{1} - f^{-1}(RSI_\eta(\bar{1} - \lambda, r))$$

$$= f^{-1}(RSC_\eta(\lambda, r)).$$

Take $\rho = RSC_\eta(\lambda, r)$. Hence $\rho$ is r-frsc and $\rho \geq \lambda$ such that $\mu \geq f^{-1}(\rho)$.

(4)⇒(1) Let $\omega$ be a fuzzy set such that $\omega \in I^X$ and $\tau(\omega) \geq r$. Put $\mu = \bar{1} - \omega$ and $\lambda = \bar{1} - f(\omega)$. Then $\mu = f^{-1}(\lambda)$. So there exists an r-frsc set $\rho$ such that $\rho \geq \lambda$ and $\mu \geq f^{-1}(\rho)$. Then $\bar{1} - \omega \geq f^{-1}(\rho)$ implies $\omega \leq f^{-1}(\bar{1} - \rho)$. Thus $f(\omega) \leq \bar{1} - \rho$. Also, $\lambda \leq \rho$, $f(\omega) = \bar{1} - \lambda \geq \bar{1} - \rho$. Hence $f(\omega) = \bar{1} - \rho$ and therefore $f(\omega)$ is r-frso. So $f$ is fuzzy regular semi open.

\[\Box\]

**Theorem 3.9.** Let $(X, \tau)$ and $(Y, \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:

1. $f$ is fuzzy regular semi irresolute closed.
2. $f(RSC_\tau(\lambda, r)) \geq RSC_\eta(f(\lambda), r)$, for each $\lambda \in I^X$.

**Proof.** (1)⇒(2) Let $\lambda \in I^X$. Since $\lambda \leq RSC_\tau(\lambda, r)$, then

$$f(\lambda) \leq f(RSC_\tau(\lambda, r)).$$

It implies $RSC_\eta(f(\lambda), r) \leq f(RSC_\tau(\lambda, r))$.

(2)⇒(1) Let (2) holds and $\lambda \in I^X$ such that $\lambda$ is r-frsc. Then

$$RSC_\eta(f(\lambda), r) \leq f(\lambda).$$

But $f(\lambda) \leq RSC_\eta(f(\lambda), r)$. Hence $f(\lambda)$ is r-frsc and therefore $f$ is fuzzy regular semi irresolute closed.

\[\Box\]

**Theorem 3.10.** Let $(X, \tau)$ and $(Y, \eta)$ be smooth topological spaces. Let $f : X \to Y$ be a mapping. The following statements are equivalent:
(1) \( f \) is fuzzy regular semi closed.
(2) \( \text{RSC}_\eta(f(\lambda), r) \leq f(\text{RC}_\tau(\lambda, r)), \) for each \( \lambda \in I^X \).

**Proof.** (1)\( \Rightarrow \) (2) Let (1) holds and let \( \lambda \in I^X \). Since \( \lambda \leq \text{RC}_\tau(\lambda, r), \) then \( f(\lambda) \leq f(\text{RC}_\tau(\lambda, r)) \) and therefore, \( \text{RSC}_\eta(f(\lambda), r) \leq f(\text{RC}_\tau(\lambda, r)) \).

(2)\( \Rightarrow \) (1) Let (2) holds and let \( \lambda \in I^X \) be an \( r \)-frsc set.
Then \( \text{RSC}_\eta(f(\lambda), r) \leq f(\text{RC}_\tau(\lambda, r)) \). It implies \( \text{RSC}_\eta(f(\lambda), r) \leq f(\lambda) \). But \( \text{RSC}_\eta(f(\lambda), r) \geq f(\lambda) \). Then, \( f(\lambda) = \text{RSC}_\eta(f(\lambda), r) \) and it follows that \( f(\lambda) \) is \( r \)-frsc. Hence, \( f \) is fuzzy regular semi closed.

\( \square \)

**Theorem 3.11.** Let \((X, \tau)\) and \((Y, \eta)\) be smooth topological spaces. Let \( f : X \to Y \) be bijective. Then:

(1) \( f \) is fuzzy regular semi irresolute closed iff
\[
\text{RSC}_\tau(f^{-1}(\text{RS}C_\eta(\mu, r)),r) \leq \text{RSC}_\eta(f^{-1}(\mu), r),
\]
for each \( \mu \in I^Y \) and \( r \in I_0 \).

(2) \( f \) is fuzzy regular semi irresolute closed iff \( f \) is fuzzy regular semi irresolute open.

**Proof.** (1)\( \Rightarrow \): Let \( f \) be fuzzy regular semi irresolute closed. From Theorem 3.9, we have:
\[
f(\text{RSC}_\tau(\lambda, r)) \geq \text{RSC}_\eta(f(\lambda), r), \quad \lambda \in I^X.
\]
Let \( \mu \in I^Y \) and put \( \lambda = f^{-1}(\mu) \), we have:
\[
f(\text{RSC}_\tau(f^{-1}(\mu), r)) \geq \text{RSC}_\eta(f^{-1}(\mu), r)
= \text{RSC}_\eta(\mu, r).
\]
It implies \( \text{RSC}_\tau(f^{-1}(\mu), r) \geq f^{-1}(\text{RSC}_\eta(\mu, r)) \).

\( \Leftarrow \): On the other hand let the condition is satisfied and let \( \mu \in I^X \) such that \( \mu \) is \( r \)-frsc. Then \( f(\mu) \in I^Y \). Apply the condition we have:
\[
\text{RSC}_\tau(f^{-1}f(\mu), r) \geq f^{-1}(\text{RSC}_\eta(f(\mu), r))
\]
It implies that
\[
\text{RSC}_\tau(\mu, r) \geq f^{-1}(\text{RSC}_\eta(\mu, r)).
\]
Then, \( f(\text{RSC}_\tau(\mu, r)) \geq \text{RSC}_\eta(f(\mu), r) \). So by Theorem 3.9, \( f \) is fuzzy regular semi irresolute closed.

(2) Apply Theorem 3.9 and taking the complement we have the required result.

\( \square \)

From Theorems 3.5, 3.7, 3.9, 3.11, we obtain the following Theorem.

**Theorem 3.12.** Let \((X, \tau)\) and \((Y, \eta)\) be smooth topological spaces. Let \( f : X \to Y \) be a mapping. The following statements are equivalent:
(1) $f$ is fuzzy regular semi irresolute homeomorphism.
(2) $f$ is fuzzy regular semi irresolute and fuzzy regular semi irresolute open.
(3) $f$ is fuzzy regular semi irresolute and fuzzy regular semi irresolute closed.
(4) $f(RSI_r(\lambda, r)) = RSI_r(f(\lambda), r)$, for each $\lambda \in I^X$, $r \in I_0$.
(5) $f(RSC_r(\lambda, r)) = RSC_r(f(\lambda), r)$, for each $\lambda \in I^X$, $r \in I_0$.
(6) $RSC_r(f^{-1}(\mu), r) = f^{-1}(RSI_r(\mu, r))$.
(6) $RSC_r(f^{-1}(\mu), r) = f^{-1}(RSC_r(\mu, r))$, $\mu \in I^Y$, $r \in I_0$.

Note that the composition of two fuzzy regular semi irresolute mappings is fuzzy regular semi irresolute. In general, the composition of two fuzzy regular semi continuous mappings is not fuzzy regular semi continuous.

**Example 3.13.** Let $X = \{a, b, c\} = Y = Z$, $\lambda_1, \lambda_2 \in I^X$, $\lambda_3, \lambda_4 \in I^Y$ and $\lambda_5 \in I^Z$ are defined as

$$
\begin{align*}
\lambda_1(a) &= 0.4, & \lambda_1(b) &= 0.5, & \lambda_1(c) &= 0.6, \\
\lambda_2(a) &= 0.4, & \lambda_2(b) &= 0.5, & \lambda_2(c) &= 0.4, \\
\lambda_3(a) &= 0.6, & \lambda_3(b) &= 0.5, & \lambda_3(c) &= 0.4, \\
\lambda_4(a) &= 0.4, & \lambda_4(b) &= 0.5, & \lambda_4(c) &= 0.6, \\
\lambda_5(a) &= 0.6, & \lambda_5(b) &= 0.5, & \lambda_5(c) &= 0.4.
\end{align*}
$$

We define smooth topologies $\tau, \sigma, \eta : I^X \rightarrow I$ as follows:

$$
\tau(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{\bar{0}, \bar{1}\}, \\
\frac{1}{2} & \text{if } \lambda = \lambda_1, \\
\frac{1}{2} & \text{if } \lambda = \lambda_2, \\
0 & \text{otherwise,}
\end{cases}
\quad \sigma(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{\bar{0}, \bar{1}\}, \\
\frac{1}{2} & \text{if } \lambda = \lambda_3, \\
\frac{1}{2} & \text{if } \lambda = \lambda_4, \\
\frac{1}{2} & \text{if } \lambda = \lambda_3 \lor \lambda_4, \\
0 & \text{otherwise},
\end{cases}
\quad \eta(\lambda) = \begin{cases} 
1 & \text{if } \lambda \in \{\bar{0}, \bar{1}\}, \\
\frac{1}{2} & \text{if } \lambda = \lambda_5, \\
0 & \text{otherwise.}
\end{cases}
$$

For $r = \frac{1}{2}$ and $\eta(\lambda_5) \geq r$, there exist an $r$-fro set $\lambda_3 \in I^Y$ such that $\lambda_3 \leq \lambda_5 \leq C_\sigma(\lambda_3, r)$ and $f^{-1}(\lambda_5)$ is an $r$-frso set in $Y$. $\sigma(\lambda_4) \geq r$, there exist an $r$-frso set $\lambda_1 \in I^X$ such that $\lambda_1 \leq \lambda_4 \leq C_\tau(\lambda_1, r)$ and $f^{-1}(\lambda_4)$ is an $r$-frso set in $X$. But $\eta(\lambda_5) \geq r$, $f^{-1}(\lambda_5)$ is not $r$-frso set in $X$. The identity mapping $id : (X, \tau) \rightarrow (Y, \sigma)$ and $id : (Y, \sigma) \rightarrow (Z, \eta)$ are fuzzy regular semi continuous, but $id : (X, \tau) \rightarrow (Z, \eta)$ is not fuzzy regular semi continuous.
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