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ON EXPONENTIABLE SOFT TOPOLOGICAL SPACES

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Abstract. An object X of a category $\mathbf{C}$ with finite limits is called exponentiable if the functor $-\times X : \mathbf{C} \to \mathbf{C}$ has a right adjoint. There are many characterizations of the exponentiable spaces in the category **Top** of topological spaces. Here, we study the exponentiable objects in the category **STop** of soft topological spaces which is a generalization of the category **Top**. We investigate the exponentiability problem and give a characterization of exponentiable soft spaces. Also we give the definition of exponential topology on the lattice of soft open sets of a soft space and present some characterizations of it.

1. Introduction and preliminaries

Most of problems in engineering, medical science, economics, environments, etc have various uncertainties. In 1999, Molodtsov [17] introduced the notion of soft set theory which is a completely new approach for modeling vagueness and uncertainty. Applications of soft set theory in other disciplines and real-life problems are now catching momentum. Molodtsov [17–20] successfully applied the soft set theory into several directions, such as smoothness of functions, game theory, Riemann integration, Perron integration, theory of measurement, and so on. Research works on soft set theory and its applications are progressing rapidly in various fields, including topology [3, 6, 10, 11, 13, 16, 22, 24], algebra [2, 9, 14, 23], decision making [4, 5, 15], information systems [21, 25], and so on.

Shabir and Naz [22] first introduced the concept of soft topological spaces. They defined basic notions of soft topological spaces such as soft open, soft closure, soft subspace and soft separation axioms. Consequently, Hussian and Ahmad [13], Cagman and Karatas [6], continued

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introducing new concepts and investigated their properties. The aim of this paper is to study the exponentiable objects in the category **STop** of soft topological spaces which is a generalization of the category **Top** of topological spaces.

Let **C** be a category with finite limits. Recall that an object X of **C** is called exponentiable if the functor $- \times X : \mathbf{C} \to \mathbf{C}$ has a right adjoint [1]. It is known that a topological space is exponentiable in the category **Top** if and only if it is core-compact, in the sense that any given open neighborhood V of a point x contains an open neighborhood U of x with the property that every open cover of V has a finite subcover of U [7, 8]. Also a topological space X is exponentiable if and only if the lattice OX of open subsets of X is continuous, in the sense of Scott [12].

The paper is organized as follows. In section 2 we formulate the exponentiability problem and give a characterization of exponentiable soft spaces. In section 3 we give the definition of exponential topology on the lattice of soft open sets of a soft space and investigate some properties of it. In section 4 we give the definition of the Alexandroff and Scott soft topologies on the lattice of soft open sets and show that a soft space has an exponential topology on its lattice if and only if the Scott soft topology is approximating in the sense of Scott, i.e. it is a continuous lattice. Finally, in section 5 we give the definition of core compact soft spaces and show that if a soft space is exponentiable, then it is core compact.

Definition 1.1 ([17])**.** Let X be an initial universe set and A a set of parameters, and $P(X)$ the power set of X. A pair (F, A) (briefly F), where F is a map from A to P(X), is called a soft set over X.

In what follows by $S(X, A)$ we denote the family of all soft sets over X.

Definition 1.2 ([17])**.** Let F, $G \in S(X, A)$. We say that F is a soft subset of G, written $F \leq G$, if $F(a) \subseteq G(a)$, for every $a \in A$.

Definition 1.3 ([17])**.** Let I be an arbitrary index set and $\{F_i \mid i \in I\} \subseteq S(X, A)$. The soft union of these soft sets is the soft set $\bigvee_{i\in I} F_i$ defined by $\bigvee_{i\in I} F_i(a) = \bigcup_{i\in I} F_i(a)$, for every $a \in A$. Also the soft intersection of these soft sets is the soft set $\bigwedge_{i\in I} F_i$ defined by $\bigwedge_{i\in I} F_i(a) = \bigcap_{i\in I} F_i(a)$, for every $a \in A$.

Definition 1.4 ([17])**.** The soft set $F \in S(X, A)$, where $F(a) = \emptyset$, for every $a \in A$ is called the A-null soft set of $S(X, A)$ and denoted by 0_A. The soft set $F \in S(X, A)$, where $F(a) = X$, for every $a \in A$ is called the A-absolute soft set of $S(X, A)$ and denoted by 1_A.

Definition 1.5 ([10])**.** Let X and Y be two initial universe sets, A and B two sets of parameters, $f : X \to Y$ and $e : A \to B$ two maps. Then,

1) The map $\varphi_{fe} : S(X, A) \to S(Y, B)$ defined as follows:

$$\varphi_{fe}(F)(b) = \bigcup_{a \in e^{-1}(b)} f(F(a)), \quad \text{for every} \quad b \in B.$$

2) The map $\psi_{fe} : S(Y, B) \to S(X, A)$ defined as follows:

$$\psi_{fe}(G)(a) = f^{-1}(G(e(a))), \quad \text{for every} \quad a \in A.$$

Notice that if $e^{-1}(b) = \emptyset$, then $\varphi_{fe}(F)(b) = \emptyset$.

Proposition 1.6 ([10])**.** *The following statements hold:*

1) $(S(X, A), \vee, \wedge, 0_A, 1_A)$ *is a complete lattice.*
2) *The maps* φ_{fe} *and* ψ_{fe} *are order preserving.*
3) $\varphi_{fe} \circ \psi_{fe} \le id$ *and* $id \le \psi_{fe} \circ \varphi_{fe}$*, where id is the identity map.*

Thus the pair $(\varphi_{fe}, \psi_{fe})$ is a Galois connection (see [1]), and hence φ_{fe} preserves joins and ψ_{fe} preserves meets.

Definition 1.7 ([22])**.** Let X be an initial universe set, A a set of parameters, and $\tau \subseteq S(X, A)$. We say that the family τ defines a soft topology on X if the following axioms are true:

1) $0_A, 1_A \in \tau$.
2) If $F, G \in \tau$, then $F \wedge G \in \tau$.
3) If $F_i \in \tau$ for every $i \in I$, then $\bigvee_{i \in I} F_i \in \tau$.

The triplet (X, τ, A) (or for simply the pair (X, A)) is called a soft topological space or soft space. The members of τ are called soft open sets in X.

Definition 1.8 ([10])**.** Let (X, A) be a soft space, $a \in A$, and $x \in X$. We say that a soft open set F is an a-soft open neighborhood of x if $x \in F(a)$.

Definition 1.9 ([10])**.** Let (X, A) and (Y, B) be two soft spaces, $x \in X$, and $e : A \to B$. A map $f : X \to Y$ is called soft e-continuous at the point x if for every $a \in A$ and every $e(a)$-soft open neighborhood G of $f(x)$ in (Y, B) there exists an a-soft open neighborhood F of x in (X, A) such that $\varphi_{fe}(F) \le G$.

If the map f is soft e-continuous at any point $x \in X$, then we say that the map f is soft e-continuous or the pair (f, e) is soft continuous.

Proposition 1.10 ([10])**.** *Let* (X, A) *and* (Y, B) *be two soft spaces,* $e : A \to B$ *and* $f : X \to Y$ *two maps. Then* f *is soft* e*-continuous if and only if* $\psi_{fe}(G) \in \tau_X$*, for every* $G \in \tau_Y$.

Example 1.11. Let (X, τ) be a topological space, $1 = \{1\}$ a one point set, and $\tau_s = \{F_U \mid U \in \tau\}$, where the map $F_U : 1 \to P(X)$ defined by $F_U(1) = U$. Then the triplet $(X, \tau_s, 1)$ is a soft space. If $f : (X, \tau) \to (Y, \tau')$ be a map between two topological spaces, then f is continuous if and only if (f, id) is soft continuous, where $id : 1 \to 1$ is the identity map.

Let **STop** be the category whose objects are all soft spaces (X, A) and whose morphisms are all pairs of maps $(f, e) : (X, A) \to (Y, B)$ such that f is soft e-continuous. Then the functor $E :$ **Top** $\to$ **STop** defined by $E(f : (X, \tau) \to (Y, \tau')) = (f, id) : (X, \tau_s, 1) \to (Y, \tau'_s, 1)$ is a full embedding. Hence **Top** is a fully embeddable category into **STop** (see [1]).

2. Exponentiable soft spaces

For soft spaces (X, A) and (Y, B), we denote by:

1) $SC(X, Y)$ the set of all maps $f : X \to Y$ such that f is soft e-continuous for some $e : A \to B$.
2) B^A the set of all maps $e : A \to B$ which there is a map $f : X \to Y$ such that f is soft e-continuous.

Let $F \in S(X, A)$ and $G \in S(Y, B)$. The cartesian product of F and G is the soft set $F \times G : A \times B \to P(X \times Y)$ defined by $F \times G(a, b) = F(a) \times G(b)$, for every $(a, b) \in A \times B$. Also the product of two soft spaces (X, A) and (Y, B) in **STop** is the soft space $(X \times Y, \tau, A \times B)$, where τ is the collection of all soft unions of elements of $\{F \times G \mid F \in \tau_X, G \in \tau_Y\}$.

Notation. Let $f : X \times Y \to Z$ be a map and $x \in X$. Then by f_x we denote the map of Y into Z defined by $f_x(y) = f(x, y)$, for every $y \in Y$.

Lemma 2.1 ([11])**.** *Let $(g, e) : (Z \times X, C \times A) \to (Y, B)$ be a soft continuous map, $z \in Z$ and $c \in C$. Then the soft map $(g_z, e_c) : (X, A) \to (Y, B)$ is continuous.*

Definition 2.2. The soft transpose $(\bar{g}, \bar{e}) : (Z, C) \to \big(SC(X, Y), B^A\big)$ of a soft continuous map $(g, e) : (Z \times X, C \times A) \to (Y, B)$ is defined by $\bar{g}(z) = g_z$, for every $z \in Z$ and $\bar{e}(c) = e_c$, for every $c \in C$.

Definition 2.3. Let (X, A) and (Y, B) be two soft spaces. A soft topology on the pair $\big(SC(X, Y), B^A\big)$ is called:

1) weak if soft continuity of $(g, e) : (Z \times X, C \times A) \to (Y, B)$ implies that of $(\bar{g}, \bar{e}) : (Z, C) \to \big(SC(X, Y), B^A\big)$,
2) strong if soft continuity of $(\bar{g}, \bar{e}) : (Z, C) \to \big(SC(X, Y), B^A\big)$ implies that of $(g, e) : (Z \times X, C \times A) \to (Y, B)$,
3) exponential if it is both weak and strong.

Thus a soft topology on $\big(SC(X,Y),B^A\big)$ is exponential if and only if it makes the transposition operation $(g,e)\mapsto(\bar{g},\bar{e})$ into a bijection from the set $STop(Z\times X, C\times A; Y,B)$ to the set $STop\big(Z,C;SC(X,Y),B^A\big)$.

Definition 2.4. We say that a soft space (X,A) is exponentiable if the pair $\big(SC(X,Y),B^A\big)$ admits an exponential topology for every soft space (Y,B). In this case, the pair $\big(SC(X,Y),B^A\big)$ endowed with the exponential soft topology is denoted by (Y^X,B^A).

Theorem 2.5. *A soft topology on $\big(SC(X,Y),B^A\big)$ is strong if and only if the soft evaluation map $(E_X,E_A):\big(SC(X,Y)\times X, B^A\times A\big)\to(Y,B)$ defined by $E_X(f,x)=f(x)$ and $E_A(e,a)=e(a)$ is soft continuous.*

Proof. The transpose of (E_X,E_A) is the soft identity map which is soft continuous. Thus by the definition of strong soft topology, (E_X,E_A) is soft continuous. Conversely, assume that the soft evaluation map is continuous for a given soft topology on $\big(SC(X,Y),B^A\big)$ and let $(g,e):(Z\times X,C\times A)\to(Y,B)$ be a soft map with a soft continuous $(\bar{g},\bar{e})$. Then (g,e) is also soft continuous because it is a composition $(E_X,E_A)\circ(id\times\bar{g}, id\times\bar{e})$ of soft continuous maps, where id is the identity map. □

Corollary 2.6. *If a soft space (X,A) is exponentiable, then the exponential soft topology on $\big(SC(X,Y),B^A\big)$ is uniquely determined.*

Proof. By Theorem 2.5, the result follows. □

Lemma 2.7. *Let (X,A) be an exponentiable soft space and $(g,e):(Y,B)\to(Z,C)$ a soft continuous map. Then the soft map $(g^X,e^A):(Y^X,B^A)\to(Z^X,C^A)$ defined by $g^X(f)=g\circ f$ and $e^A(e')=e\circ e'$ is soft continuous.*

Proof. Since the soft evaluation map (E_X,E_A) is continuous, so the composition $(g,e)\circ(E_X,E_A)$ is also soft continuous and hence its transpose $(\overline{g\circ E_X},\overline{e\circ E_A})$ is soft continuous. But we have that $(g^X,e^A)=(\overline{g\circ E_X},\overline{e\circ E_A})$. □

Theorem 2.8. *A soft space (X,A) is exponentiable if and only if the product functor $-\times(X,A):\mathbf{STop}\to\mathbf{STop}$ has a right adjoint.*

Proof. By Lemma 2.7, the functor $(-)^{(X,A)}:\mathbf{STop}\to\mathbf{STop}$ defined by $(Y,B)\mapsto(Y^X,B^A)$ and $(g,e)\mapsto(g^X,e^A)$ is well defined. Thus $(-)^{(X,A)}$ is a right adjoint to the functor $-\times(X,A)$. Conversely, let $R:\mathbf{STop}\to\mathbf{STop}$ be a right adjoint to $-\times(X,A)$ and (Y,B) a soft space. Then we have the following bijection:

$$\mathrm{STop}(1\times X,1\times A;Y,B)\cong\mathrm{STop}\left(1,1;R(Y,B)\right),$$

where $(1,1)$ is the one point soft space. Let $(Y', B') = R(Y, B)$. Then we have a bijection soft map $(h, e) : \big(SC(X, Y), B^A\big) \to (Y', B')$. Now suppose that T is the soft topology on $\big(SC(X, Y), B^A\big)$ induced by the soft map (h, e). Then for every soft space (Z, C) we have the following bijections:

$$\begin{aligned} \mathrm{STop}(Z \times X, C \times A; Y, B) &\cong \mathrm{STop}(Z, C; Y', B') \\ &\cong \mathrm{STop}\left(Z, C; SC(X, Y), B^A\right). \end{aligned}$$

This shows that (X, A) is exponentiable. □

3. Topologies on lattices of soft open sets

In this section we give the definition of exponential topology on lattice $O(X, A)$ of soft open sets of a soft space (X, A) and investigate some properties of it.

Let (X, A) and (Y, B) be tow soft spaces and $e \in B^A$. We denote by $SC_e(X, Y)$ the set of all soft e-continuous maps $f : X \to Y$.

Definition 3.1. Let (X, A) and (Y, B) be two soft spaces and $e \in B^A$. A topology on the set $SC_e(X, Y)$ is called:

1) weak if for every topological space Z, soft continuity of $(g, e) : (Z \times X, A) \to (Y, B)$ implies the continuity of $\bar{g} : Z \to SC_e(X, Y)$,
2) strong if for every topological space Z, continuity of $\bar{g} : Z \to SC_e(X, Y)$ implies the soft continuity of $(g, e) : (Z \times X, A) \to (Y, B)$,
3) exponential if it is both weak and strong.

Let A be an arbitrary set. Then the Sierpinski A-soft space is the soft space $\left(\prod_{a \in A} S_a, \tau_S, A\right)$ with $S_a = \{0, 1\}$ for every $a \in A$, and $\tau_S = \{0_A, 1_A, G_A\}$, where the soft open set G_A is defined by $G_A(a) = \pi_a^{-1}\{1\}$ for every $a \in A$.

Lemma 3.2. *Let (X, A) be a soft space and $\left(\prod_{a \in A} S_a, A\right)$ the Sierpinski A-soft space. Then the set $O(X, A)$ is one-to-one correspondence to the set $SC_{id}\left(X, \prod_{a \in A} S_a\right)$, where $id : A \to A$ is the identity map.*

Proof. We define the map $H : SC_{id}\left(X, \prod_{a \in A} S_a\right) \to O(X, A)$ by $H(f) = \psi_{fid}(G_A)$ and show that it is a bijection. Let $H(f) = H(g)$. Then $\psi_{fid}(G_A) = \psi_{gid}(G_A)$, so $f^{-1}(\pi_a^{-1}\{1\}) = g^{-1}(\pi_a^{-1}\{1\})$ for all $a \in A$, and hence $\pi_a \circ f = \pi_a \circ g$. Since the product is a mono source, we have $f = g$, which shows that H is injective. Now let $F \in O(X, A)$. Then by the property of product there is a unique map $f : X \to \prod_{a \in A} S_a$, such that $\pi_a \circ f = \chi_{F(a)}$ for every $a \in A$, where $\chi_{F(a)}$ is the characteristic function. It is easy to see that $\psi_{fid}(G_A) = F$. Thus $H(f) = F$, which shows that H is surjective. □

Definition 3.3. Let (X, A) be a soft space. A topology on $O(X, A)$ is called exponential if it is induced by an exponential topology on the set $SC_{id}(X, \prod_{a\in A} S_a)$ via the bijection defined in Lemma 3.2.

Thus a topology on $O(X, A)$ is weak if for every topological space Z, soft continuity of $(g, e) : (Z \times X, A) \to (Y, B)$ implies the continuity of $\bar{g} : Z \to O(X, A)$, where $\bar{g}$ defined by $\bar{g}(z) = \psi_{g_z id}(G_A)$, and it is strong if for every topological space Z, continuity of $\bar{g} : Z \to O(X, A)$ implies the soft continuity of $(g, e) : (Z \times X, A) \to (Y, B)$.

Example 3.4. If X is a topological space, then a topology on $O(X) \cong O(X, 1)$ is exponential if and only if it is induced by an exponential topology on the set $SC_{id}(X, S) = C(X, S)$, where S is the Sierpinski topological space. Thus $O(X)$ has an exponential topology if and only if X is exponentiable in the category **Top**.

Theorem 3.5. *A topology on $O(X, A)$ is strong if and only if the soft graph ε_X defined by $\varepsilon_X(a) = \{(F, x) \in O(X, A) \times X \mid x \in F(a)\}$ for every $a \in A$, is soft open.*

Proof. Similar to the proof of Theorem 2.5, we have that a topology on $O(X, A)$ is strong if and only if the soft evaluation map $(E_X, id) : (O(X, A) \times X, A) \to \left(\prod_{a\in A} S_a, A\right)$ defined by $E_X(F, x) = f(x)$ is continuous, where f is the map defined in Lemma 3.2. But it is easy to see that $\psi_{E_X id}(G_A) = \varepsilon_X$ and hence the result follows. □

Let (X, A) and (Y, B) be two soft spaces and $e \in B^A$. If T is a topology on $O(X, A)$, the topology on $SC_e(X, Y)$ generated by the subbasic open sets

$$T(O, V) = \{f \in SC_e(XY) \mid \psi_{fe}(V) \in O\},$$

where O ranges over T and V ranges over $O(Y, B)$, is referred to as the topology induced by T.

Lemma 3.6. *Let (X, A) be a soft space and T be a topology on $O(X, A)$. Then the topology T is weak if and only if it induces a weak topology on $SC_e(X, Y)$, for every soft space (Y, B) and every $e \in B^A$.*

Proof. Let T be a weak topology on $O(X, A)$ and (Y, B) a soft space, $e \in B^A$. To show that $\bar{g} : Z \to SC_e(X, Y)$ is continuous for $SC_e(X, Y)$ endowed with the topology induced by T, it is enough to show that $\bar{g}^{-1}(T(O, V))$ is open for $O \in T$ and $V \in O(Y, B)$. Since $\psi_{ge}(V) \in O(Z \times X, A)$, so by Lemma 3.2, there is an id-soft continuous map $f : Z \times X \to \prod_{a\in A} S_a$ such that $\psi_{ge}(V) = \psi_{fid}(G_A)$. By assumption the transpose $\bar{f} : Z \to O(X, A)$ defined by $\bar{f}(z) = \psi_{f_z id}(G_A)$ is continuous. Now we show that $\bar{g}^{-1}(T(O, V)) = \bar{f}^{-1}(O)$. But for every $a \in A$, we have that $\psi_{ge}(V)(a) = \psi_{fid}(G_A)(a)$, this implies $g^{-1}(V(e(a))) =$

$f^{-1}\pi_a^{-1}(\{1\})$. Thus $g_z^{-1}(V(e(a))) = f_z^{-1}\pi_a^{-1}(\{1\})$ for every $z \in Z$, and hence $\bar{f}(z) = \psi_{f_z id}(G_A) = \psi_{g_z e}(V)$ for every $z \in Z$. Therefore the chain of equivalences $z \in \bar{g}^{-1}(T(O,V)) \Leftrightarrow g_z \in T(O,V) \Leftrightarrow \psi_{g_z e}(V) \in O \Leftrightarrow \bar{f}(z) \in O \Leftrightarrow z \in \bar{f}^{-1}(O)$ concludes the proof. Conversely, let (Y,B) be the Sierpinski A-soft space. Then the set $SC_{id}\left(X, \prod_{a\in A} S_a\right)$ endowed with the topology induced by T is homeomorphic to $O(X,A)$ endowed with T, and hence the result holds. □

Lemma 3.7. *Let (X,A) be a soft space and T be a topology on $O(X,A)$. Then the topology T is strong if and only if it induces a strong topology on $SC_e(X,Y)$, for every soft space (Y,B) and every $e \in B^A$.*

Proof. Let T be a strong topology on $O(X,A)$, and (Y,B) a soft space, $e \in B^A$. Also let $g : Z \times X \to Y$ be a map such that its transpose $\bar{g} : Z \to SC_e(X,Y)$ is continuous, where Z is a topological space. We first show that the soft evaluation map $E_X : SC_e(X,Y) \times X \to Y$ is e-soft continuous. Let $V \in O(Y,B)$. Since the soft graph ε_X is soft open, so $\varepsilon_X = \bigvee_{i\in I} O_i \times F_i$, where $O_i \in T$ and $F_i \in O(X,A)$, for all $i \in I$. Now we prove that $\psi_{E_X e}(V) = \bigvee_{i\in I} T(O_i,V) \times F_i$. Let $(f,x) \in T(O_i,V) \times F_i(a)$, for some $i \in I$. Then $\psi_{fe}(V) \in O_i$ and $x \in \psi_{fe}(V)(a)$, hence $(f,x) \in \psi_{E_X e}(V)(a)$. Conversely, let $(f,x) \in \psi_{E_X e}(V)(a)$. Then $f(x) \in V(e(a))$, this implies $x \in \psi_{fe}(V)(a)$. Thus $(\psi_{fe}(V),x) \in \varepsilon_X(a)$ and hence $(f,x) \in T(O_i,V) \times F_i(a)$ for some $i \in I$. Finally, we have that $(g,e) = (E_X,e) \circ (id \times \bar{g}, id)$, and hence it is soft continuous.
Conversely, suppose that (Y,B) is the Sierpinski A-soft space. Then the set $SC_{id}\left(X, \prod_{a\in A} S_a\right)$ endowed with the topology induced by T is homeomorphic to $O(X,A)$ endowed with T, and hence the result holds. □

Thus, we have a characterization of exponential topology on the lattice of soft open sets.

Theorem 3.8. *A soft space (X,A) has an exponential topology on $O(X,A)$ if and only if it induces an exponential topology on $SC_e(X,Y)$, for every soft space (Y,B) and every $e \in B^A$. In this case, the exponential topology of $SC_e(X,Y)$ is the topology induced by the exponential topology of $O(X,A)$.*

Definition 3.9. Let (X,A) be a soft space, Y and B nonempty subsets of X and A respectively. Then the soft subspace (Y,B) defined as follows: $G \in \tau_Y$ if and only if there exists $F \in \tau_X$ such that $G = F \wedge 1_B$, where $F \wedge 1_B(b) = F(b) \cap Y$, for every $b \in B$.

The following Lemma is an immediate consequence of the definition of soft continuity.

Lemma 3.10. *Let (Y_0, B_0) be a soft subspace of (Y, B). Then the following statements hold.*

1) *The soft inclusion map $(i, i) : (Y_0, B_0) \to (Y, B)$ is soft continuous.*
2) *If $(f, e) : (X, A) \to (Y, B)$ is a soft continuous map such that $f(X) \subseteq Y_0$ and $e(A) \subseteq B_0$, then $(f, e) : (X, A) \to (Y_0, B_0)$ is soft continuous.*

Theorem 3.11. *If (X, A) is an exponentiable soft space, then $O(X, A)$ has an exponential topology.*

Proof. Let $\left(\prod_{a\in A} S_a, A\right)$ be the Sierpinski A-soft space. By assumption, the pair $\left(SC(X, \prod_{a\in A} S_a), A^A\right)$ has an exponential topology. Then we show that the soft subspace topology on $\left(SC_{id}(X, \prod_{a\in A} S_a), \{id_A\}\right)$ is exponential. Suppose that $(g, id) : (Z \times X, A) \to \left(\prod_{a\in A} S_a, A\right)$ is a soft continuous map, where Z is a topological space. By assumption, the transpose $(\bar{g}, \bar{id}) : (Z, 1) \to \left(SC(X, \prod_{a\in A} S_a), A^A\right)$ is soft continuous. Since $\bar{g}(Z) \subseteq SC_{id}\left(X, \prod_{a\in A} S_a\right)$ and $\bar{id}(1) = id$, so by Lemma 3.10, $\bar{g} : Z \to SC_{id}\left(X, \prod_{a\in A} S_a\right)$ is $\bar{id}$-soft continuous, and hence it is continuous. Conversely, let $\bar{g} : Z \to SC_{id}\left(X, \prod_{a\in A} S_a\right)$ be continuous, i.e. it is soft $\bar{id}$-continuous. Again, by Lemma 3.10, the map $(\bar{g}, \bar{id}) \circ (i, i)$ is soft continuous. Thus $(g, id) : (Z \times X, A) \to \left(\prod_{a\in A} S_a, A\right)$ is soft continuous by the exponentiability of (X, A). □

4. Alexandroff and Scott soft topologies

In this section we give the definition of the Alexandroff and the Scott soft topologies on the lattice of soft open sets and show that a soft space has an exponential topology on its lattice if and only if the Scott soft topology is approximating in the sense of Scott, i.e. it is a continuous lattice.

Definition 4.1. Let (X, A) be a soft space.

1) A set $O \in O(X, A)$ is called Alexandroff soft open if conditions $F \in O$ and $F \leq G \in O(X, A)$ together imply that $G \in O$.
2) An Alexandroff soft open set $O \in O(X, A)$ is called Scott soft open if every open cover of a member of O has a finite subcover of a member of O.

It is immediate that the Alexandroff soft open sets (Scott soft open sets) form a topology.

Lemma 4.2. *The Alexandroff soft topology is strong.*

Proof. For every soft open set F, it is clear that the filter set $\uparrow F = \{G \in O(X, A) \mid F \leq G\}$ is an Alexandroff soft open set. Let $a \in A$

and $(F,x) \in \varepsilon_X(a)$. Then $(F,x) \in \uparrow F \times F(a)$. Conversely, if $(G,x) \in \uparrow F \times F(a)$, then $x \in G(a)$, and hence $(G,x) \in \varepsilon_X(a)$. Thus we have that $\varepsilon_X = \bigvee\{\uparrow F \times F \mid F \in O(X,A)\}$. This shows that ε_X is a soft open set in $O(X,A) \times X$. Thus by Theorem 3.5, the result holds. □

Lemma 4.3. *The Scott soft topology is weak.*

Proof. Suppose that Z is a topological space and $(g, id) : (Z \times X, A) \to \left(\prod_{a\in A} S_a, A\right)$ a soft continuous map. We show that the transpose $\bar{g} : Z \to O(X,A)$ is continuous. Let O be a Scott soft open set and $z \in \bar{g}^{-1}(O)$. Then $\psi_{g_z id}(G_A) = \bar{g}(z) \in O$. On the other hand, since $\psi_{gid}(G_A)$ is soft open, so $\psi_{gid}(G_A) = \bigvee_{i\in I} U_i \times F_i$, where $U_i \in O(Z)$ and $F_i \in O(X,A)$, for all $i \in I$. This implies that $\psi_{g_z id}(G_A) \leq \bigvee_{i\in I} F_i$. Since O is Scott soft open, so there is a finite subset J of I such that $\bigvee_{i\in J} F_i \in O$. Suppose that $U = \bigcap_{i\in J} U_i$, then U is an open neighborhood of z. To conclude the proof, we show that $\bar{g}(w) \in O$ for every $w \in U$. For this, we show that $F = \bigvee_{i\in J} F_i \leq \bar{g}(w)$, for every $w \in U$. Let $a \in A$ and $x \in F(a)$. Then $x \in F_i(a)$ for some $i \in J$. Hence $(w,x) \in U_i \times F_i(a) \subseteq \psi_{gid}(G_A)(a)$. Thus $x \in \psi_{g_w id}(G_A)(a) = \bar{g}(w)(a)$, which shows that $F \leq \bar{g}(w)$. □

Let T be a topology on $O(X,A)$ and $F, G \in O(X,A)$. We write $F \prec_T G$ to mean that G belongs to the interior of $\uparrow F$ in the topology T.

Definition 4.4. Let (X,A) be a soft space and T a topology on $O(X,A)$. We say that T is approximating if for every a-soft open neighborhood G of a point x of X, there is an a-soft open neighborhood $F \prec_T G$ of x.

The following Lemma is an immediate consequence of the definition of interior.

Lemma 4.5. *Let (X,A) be a soft space and T a topology on $O(X,A)$. Then the following statements hold.*

1) *The relation $F \prec_T G$ holds if and only if $G \in O$ for some $O \in T$ with $G \leq H$ for all $H \in O$.*
2) *$F \prec_T G$ implies $F \leq G$.*
3) *$F' \leq F \prec_T G$ implies $F' \prec_T G$.*
4) *$F \prec_T G \leq G'$ implies $F \prec_T G'$ if T is weaker than the Alexandroff soft topology.*
5) *$0_A \prec_T F$, and $F \prec_T H$ and $G \prec_T H$ together imply $F \bigvee G \prec_T H$.*

Lemma 4.6. *A topology T on $O(X,A)$ is strong if and only if it is approximating.*

Proof. Let T be strong and G an a-soft open neighborhood of a point x of X. Since the soft graph ε_X is soft open with respect to T and $(G, x) \in \varepsilon_X(a)$, so there are $O \in T$ and $F \in O(X, A)$, such that $(G, x) \in O \times F(a)$ and $O \times F \leq \varepsilon_X$. If $(H, y) \in O \times F(a')$ for every $a' \in A$, then $y \in H(a')$. Thus $F \leq H$ for all $H \in O$, which shows that $F \prec_T G$ and $x \in F(a)$. Conversely, assume that T is approximating and $(G, x) \in \varepsilon_X(a)$. Then $x \in G(a)$ and there is an $F \prec_T G$ with $x \in F(a)$. Let $O \in T$ with $G \in O$ and $F \leq H$ for all $H \in O$. Then $(G, x) \in O \times F(a)$ and $O \times F \leq \varepsilon_X$, which shows that ε_X is soft open. Thus T is strong. □

Recall that a frame is a complete lattice in which finite meets distribute over arbitrary joins. Since the lattice of soft open sets is a frame, thus we have the following Lemma.

Lemma 4.7 ([12])**.** *The Scott soft topology is the intersection of all approximating topologies.*

By the previous Lemmas, we have a characterization of the exponential topology on the lattice of soft open sets.

Theorem 4.8. *A soft space (X, A) has an exponential topology on $O(X, A)$ if and only if the Scott soft topology is approximating, i.e. $O(X, A)$ is a continuous lattice. Moreover, the exponential topology is the Scott soft topology.*

5. Core compact soft spaces

In this section we give the definition of core compact soft spaces and show that a soft space has an exponential topology on its lattice if and only if it is core compact.

For open soft sets F and G of a soft space (X, A) we write $F \ll G$, and say that F is way below G, to mean that every soft open cover of G has a finite subcover of F.

Definition 5.1. A soft space (X, A) is called core-compact if every a-soft open neighborhood G of a point x of X contains an a-soft open neighborhood $F \ll G$ of x. This is equivalent to saying that every soft open G is the union of the soft opens $F \ll G$.

Similar to the results in lattice theory [12] we have the following Lemmas.

Lemma 5.2. *Let F and G be open soft sets of a soft space (X, A). Then the following statements hold.*

1) *The relation $F \prec_{Scoot} G$ implies $F \ll G$.*
2) *If the Scott soft topology of $O(X, A)$ is approximating then the relation $F \ll G$ implies $F \prec_{Scoot} G$.*

Lemma 5.3. *Let (X, A) be a core compact soft space. Then the following statements hold.*

1) *If $F \ll G$ in $O(X, A)$, then $F \ll H \ll G$ for some $H \in O(X, A)$.*
2) *The set $\Uparrow F = \{G \in O(X, A) \mid F \ll G\}$ is Scott soft open.*
3) *If $O \subseteq O(X, A)$ is Scott soft open and $G \in O$, then $F \ll G$ for some $F \in O$.*
4) *The sets $\Uparrow F$ for $F \in O(X, A)$ form a base of the Scott soft topology of $O(X, A)$.*
5) *If $F \ll G$ then $F \prec_{Scott} G$.*

Thus, we have an alternative characterization of the exponential topology on the lattice of soft open sets.

Theorem 5.4. *A soft space (X, A) is core compact if and only if $O(X, A)$ has an exponential topology. Moreover, if (X, A) is a core compact soft space and (Y, B) is any soft space and $e \in B^A$, then the topology of the exponential $SC_e(X, Y)$ is generated by the sets*

$$\{f \in SC_e(X, Y) \mid F \ll \psi_{fe}(G)\},$$

where F and G range over $O(X, A)$ and $O(Y, B)$ respectively.

Proof. Let $O(X, A)$ has an exponential topology. Then (X, A) is soft core compact by Theorem 4.8 and by Lemma 5.2. Conversely, if (X, A) is soft core compact, then $O(X, A)$ has an exponential topology by Theorem 4.8 and by Lemma 5.3 (5). For the second part, it is easy to see that if T is a topology on $O(X, A)$ with a base β, then the topology on $SC_e(X, Y)$ induced by T has the sets $T(O, V)$ as subbase for $O \in \beta$. The result then follows from the fact that the sets $\Uparrow F$ for $F \in O(X, A)$ form a base of the Scott soft topology of $O(X, A)$ if (X, A) is soft core compact, and from the fact that the exponential topology is induced by the Scott soft topology. □

Finally, by Theorems 3.11 and 5.4, we have the following result.

Theorem 5.5. *If a soft space is exponentiable, then it is core compact.*

It is well-known that a topological space X is exponentiable in the category **Top** if and only if it is core compact.

Question. Does the converse of Theorem 5.5 hold?

6. Conclusion

In this paper, we have studied the exponentiable objects in the category **STop** of soft topological spaces which is a generalization of the category **Top** of topological spaces. We have showed that a soft space

(X, A) is exponentiable if and only if the product functor $- \times (X, A)$: **STop** $\to$ **STop** has a right adjoint. In the sequel, we have given the definition of exponential topology on the lattice of soft open sets of a soft space and investigated some properties of it. Also we have given the definition of the Alexandroff and Scott soft topologies on the lattice of soft open sets and showed that a soft space has an exponential topology on its lattice if and only if it is a continuous lattice. Finally, we have introduced the definition of core compact soft spaces and showed that if a soft space is exponentiable, then it is core compact.

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