Sahand Communications in Mathematical Analysis (SCMA) Vol. 4 No. 1 (2016), 43-55
http://scma.maragheh.ac.ir

CONSTRUCTION OF CONTINUOUS g-FRAMES AND CONTINUOUS FUSION FRAMES

MAHDIYEH KHAYYAMI[1] AND AKBAR NAZARI[2] *

Abstract. A generalization of the known results in fusion frames and g-frames theory to continuous fusion frames which defined by M. H. Faroughi and R. Ahmadi, is presented in this study. Continuous resolution of the identity (CRI) is introduced, a new family of CRI is constructed, and a number of reconstruction formulas are obtained. Also, new results are given on the duality of continuous fusion frames in Hilbert spaces.

1. Introduction

Frames for Hilbert spaces were formally defined by Duffin and Schaeffer [7] in 1952 for studying some problems in non-harmonic Fourier series. Wenchang Sun [14] introduced a generalization of frames, showed that it included more other cases of generalizations of frame concept, and proved that many basic properties can be derived within this more general context.

Continuous frames were proposed by G. Kaiser [12] and independently by Ali, Antoine, and Gazeau [2] to a family indexed by some locally compact space endowed by a Radon measure. Gabardo and Han [10] called these frames as the ones associated with measurable spaces. Abdollahpour and Faroughi [1] introduced the concept of continuous g-frames as a generalization of discrete g-frames. They characterized the continuous g-frames and showed that, under some conditions, an element of a continuous g-frame could be removed such that the remaining set could be also a continuous g-frame. In this paper we extend some known results in fusion frames and g-frame theory to continuous fusion frames which

2010 *Mathematics Subject Classification.* Primary: 42C15; Secondary: 42C99.
Key words and phrases. Fusion frame, Continuous fusion frame, Continuous g-frame, Continuous resolution.
Received: 23 December 2015, Accepted: 24 April 2016.
* Corresponding author.

defined in [8] and [13]. We extend Proposition 2.1, Theorem 2.2, Proposition 2.5 and Proposition 2.9 of [13] and also we extend Lemma 3.4 of [8].

Throughout this paper, (Ω, μ) is a measure space, H and K are two Hilbert spaces, and $\{K_\omega\}_{\omega\in\Omega}$ is a sequence of closed Hilbert subspaces of K. For each $\omega \in \Omega$, $\mathcal{B}(H, K_\omega)$ is the collection of all bounded linear operators from H to K_ω. We also denote

$$\bigoplus_{\omega\in\Omega} K_\omega = \left\{ \{g_\omega\}_{\omega\in\Omega} : g_\omega \in K_\omega \quad \text{and} \quad \int_\Omega \|g_\omega\|^2 d\mu(\omega) < \infty \right\}.$$

Definition 1.1 ([1])**.** A sequence $\mathbf{\Lambda} := \{\Lambda_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ is called a continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$, if

(i) for each $f \in H$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is strongly measurable,
(ii) there are two constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 d\mu(\omega) \le B\|f\|^2; \quad (f \in H). \tag{1.1}$$

We call A and B the lower and upper continuous g-frame bounds, respectively. In this case we say that $\mathbf{\Lambda}$ is a $(A-B)$-continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$. If only the right-hand inequality of (1.1) is satisfied, we call $\mathbf{\Lambda}$ the continuous g-Bessel sequence for H with respect to $\{K_\omega\}_{\omega\in\Omega}$ with continuous g-Bessel bound B. If $A = B = \lambda$, we call $\mathbf{\Lambda}$ the λ-tight continuous frame. Moreover, if $\lambda = 1$, $\mathbf{\Lambda}$ is called the Parseval continuous frame.

For any $\{f_\omega\}_{\omega\in\Omega}, \{g_\omega\}_{\omega\in\Omega} \in \bigoplus_{\omega\in\Omega} K_\omega$, if the inner product is defined by $\langle f, g\rangle = \int_\Omega \langle f_\omega, g_\omega\rangle d\mu(\omega)$ and the norm is defined by $\|f\| = \langle f, f\rangle^{\frac{1}{2}}$, then $\bigoplus_{\omega\in\Omega} K_\omega$ is a Hilbert space. We define synthesis operator for a continuous g-Bessel sequence $\mathbf{\Lambda}$ as follows:

$$\langle T_\Lambda \{f_\omega\}_{\omega\in\Omega}, g\rangle = \int_\Omega \langle f_\omega, \Lambda_\omega g\rangle d\mu(\omega), \qquad \left(\{f_\omega\}_{\omega\in\Omega} \in \bigoplus_{\omega\in\Omega} K_\omega, g \in H \right).$$

The operator T_Λ is well-defined and bounded so the operator T_Λ^* is defined by

$$T_\Lambda^* : H \longrightarrow \bigoplus_{\omega\in\Omega} K_\omega, \qquad T_\Lambda^*(f) = \{\Lambda_\omega f\}_{\omega\in\Omega},$$

which is the adjoint of T_Λ and is called the analysis operator. The bounded linear operator S_Λ defined by

$$S_\Lambda : H \longrightarrow H, \qquad \langle S_\Lambda f, g\rangle = \int_\Omega \langle f, \Lambda_\omega^* \Lambda_\omega g\rangle d\mu(\omega),$$

is called the continuous g-frame operator of $\mathbf{\Lambda}$. Based on their definitions, if the measure space is set to $\Omega := \mathbb{N}$ and μ is the counting measure, then the continuous g-frame will be a discrete g-frame.

In this paper, the definitions of fusion frames and continuous fusion frames are stated in Hilbert spaces and new continuous g-frames and continuous fusion frames are constructed by considering their components. Continuous resolution of identity (simply CRI) is also introduced. Moreover, alternate dual is defined for two Bessel continuous fusion sequences and the alternate dual of continuous fusion frame is demonstrated as a continuous fusion frame. Also a new operator is then defined for two Bessel continuous fusion sequences and accordingly, a number of reconstruction formulas and a family of CRI are obtained.

2. Continuous g-frames and continuous fusion frames

In this section, the concept of continuous fusion frames is reviewed and extended the concepts from fusion frames to continuous fusion frames. Then, some results about g-frames and fusion frames are generalized to continuous g-frames and continuous fusion frames, respectively. First, the definition of a fusion frame is stated.

Definition 2.1 ([13])**.** Let J be a countable index set, $\{W_j\}_{j\in J}$ be a family of closed subspaces in H, and $\{\nu_j\}_{j\in J}$ be a family of weights, i.e., $\nu_j > 0$ for all $j \in J$. Then $\{(W_j, \nu_j)\}_{j\in J}$ is a fusion frame, if there exist constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \sum_{j\in J} \nu_j^2 \|\pi_{W_j}(f)\|^2 \leq D\|f\|^2 \quad (f \in H), \tag{2.1}$$

where π_{W_j} is the orthogonal projection onto the subspace W_j . We call C and D the fusion frame bounds. The family $\{(W_j, \nu_j)\}_{j\in J}$ is called a C-tight fusion frame, if in (2.1) the constants C and D can be chosen so that $C = D$, a Parseval fusion frame provided that $C = D = 1$.

This definition leads to the introduction of the following definition:

Definition 2.2 ([9])**.** Let $\{K_\omega\}_{\omega\in\Omega}$ be a family of closed subspaces of a Hilbert space H and (Ω, μ) be a measure space with positive measure μ and $\mathrm{m} : \Omega \longrightarrow \mathbb{R}^+$. Then $\mathcal{K} = \{(K_\omega, \mathrm{m}(\omega))\}_{\omega\in\Omega}$ is called a continuous fusion frame with respect to (Ω, μ) and m, if

(i) for each $f \in H$, $\{\Pi_{K_\omega} f\}_{\omega\in\Omega}$ is strongly measurable and m is a measurable function from Ω to $\mathbb{R}^+$;
(ii) there are two constants $0 < C \leq D < \infty$ such that

$$C\|f\|^2 \leq \int_\Omega \mathrm{m}^2(\omega)\|\Pi_{K_\omega} f\|^2 d\mu(\omega) \leq D\|f\|^2 \quad (f \in H), \tag{2.2}$$

where Π_{K_ω} is the orthogonal projection onto the subspace K_ω. We call C and D the lower and upper continuous fusion frame bounds, respectively. In this case we say that $\mathcal{K}$ is a $(C-D)$-continuous fusion frame for H.

If only the right-hand inequality of (2.2) is satisfied, we call $\mathcal{K}$ the D-continuous fusion Bessel sequence. If $C = D = \lambda$, we call $\mathcal{K}$ the λ-tight continuous fusion frame. Moreover, if $\lambda = 1$, $\mathcal{K}$ is called the Parseval continuous fusion frame.

Synthesis operator for a Bessel continuous fusion sequence $\mathcal{K}$ is defined as follows:

$$T_{\mathcal{K}} : \bigoplus_{\omega\in\Omega} K_\omega \longrightarrow H,$$

is weakly defined by:

$$\langle T_{\mathcal{K}}\{f_\omega\}_{\omega\in\Omega}, g\rangle = \int_\Omega \langle f_\omega, \mathrm{m}(\omega)\Pi_{K_\omega} g\rangle d\mu(\omega), (\{f_\omega\}_{\omega\in\Omega} \in \bigoplus_{\omega\in\Omega} K_\omega, g \in H).$$

Operator $T_{\mathcal{K}}$ is well-defined and bounded. It can be easily shown that the analysis operator $T_{\mathcal{K}}^*$, which is defined to be the adjoint operator is given by:

$$T_{\mathcal{K}}^* : H \longrightarrow \bigoplus_{\omega\in\Omega} K_\omega,$$

by

$$T_{\mathcal{K}}^*(f) = \{\mathrm{m}(\omega)\Pi_{K_\omega} f\}_{\omega\in\Omega}.$$

The bounded linear operator $S_{\mathcal{K}}$ defined by:

$$S_{\mathcal{K}} : H \longrightarrow H, \qquad \langle S_{\mathcal{K}} f, g\rangle = \int_\Omega \langle f, \mathrm{m}^2(\omega)\Pi_{K_\omega} g\rangle d\mu(\omega),$$

is called the continuous fusion frame operator of $\mathcal{K}$. Let Υ_ω be a subspace of Ω for all $\omega \in \Omega$ and consider a (C_ω, D_ω)-continuous fusion frame $\{(W_{\omega\nu}, \mathrm{m}(\omega,\nu))\}_{\nu\in\Upsilon_\omega}$ for each K_ω such that:

$$0 < C = \inf C_\omega \le \sup D_\omega = D < \infty.$$

In this case, $\{(W_{\omega\nu}, \mathrm{m}(\omega,\nu))\}_{\nu\in\Upsilon_\omega}$ is (C, D)-bounded for all $\omega \in \Omega$.

Definition 2.3. Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ be a family of bounded operators from H to K_ω. If $\{f : \Lambda_\omega f = 0, \quad \text{for a.e. } [\mu] \text{ on } \Omega\} = \{0\}$, then we say that $\{\Lambda_\omega : \omega \in \Omega\}$ is almost every where g-complete.

Proposition 2.4. *Let $\{\Lambda_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ be an $(A - B)$-continuous g-frame and $\{\Gamma_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ be an almost every where g-complete family of bounded operators. If $\varphi : H \longrightarrow H$ defined by*

$$\langle \varphi f, g\rangle = \int_\Omega \langle f, (\Gamma_\omega^*\Gamma_\omega - \Lambda_\omega^*\Lambda_\omega)g\rangle d\mu(\omega), \quad (f, g \in H),$$

is a compact operator, then $\{\Gamma_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ is a continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$.

Proof. Let $T : H \longrightarrow H$ be an operator defined by $T = S_\Lambda + \varphi$. By the same proof as the on of Proposition 2.1 in [13], T is a bounded, linear and self-adjoint operator and

$$\|Tf\| \leq (B + \|\varphi\|)\|f\|.$$

Hence

$$\int_\Omega \|\Gamma_\omega f\|^2 d\mu(\omega) = \langle Tf, f\rangle \leq (B + \|\varphi\|)\|f\|^2. \tag{2.3}$$

On the other hand since φ is a compact operator, φS_Λ^{-1} is also a compact operator on H. Therefore as we see in [3] Theorem 2.8, T has closed range. Now we show that T is injective. Let f be an element of H such that $Tf = 0$, then

$$\int_\Omega \|\Gamma_\omega f\|^2 d\mu(\omega) = \langle Tf, f\rangle = 0.$$

Hence $\Gamma_\omega f = 0$ a.e. $[\mu]$ on Ω. Since $\{\Gamma_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ is almost every where g-complete, we have $f = 0$. Furthermore, we have

$$\text{Rang}T = (\ker T^*)^\perp = (\ker T)^\perp = H.$$

Hence T is onto and therefore invertible on H. Moreover, by using Cauchy-Schwartz inequality and (2.3), we have

$$\begin{aligned}
\|Tf\|^4 = (\langle Tf, Tf\rangle)^2 &= \left(\int_\Omega \langle \Gamma_\omega f, \Gamma_\omega Tf\rangle d\mu(\omega)\right)^2 \\
&\leq \left(\int_\Omega \|\Gamma_\omega f\|^2 d\mu(\omega)\right)\left(\int_\Omega \|\Gamma_\omega Tf\|^2 d\mu(\omega)\right) \\
&\leq (B + \|\varphi\|)\|Tf\|^2 \left(\int_\Omega \|\Gamma_\omega f\|^2 d\mu(\omega)\right).
\end{aligned}$$

Thus $\|Tf\|^2 \leq (B + \|\varphi\|)\left(\int_\Omega \|\Gamma_\omega f\|^2 d\mu(\omega)\right)$, which implies that

$$\int_\Omega \|\Gamma_\omega f\|^2 d\mu(\omega) \geq (B + \|\varphi\|)^{-1}\|Tf\|^2 \geq (B + \|\varphi\|)^{-1}\|T^{-1}\|^{-2}\|f\|^2.$$

□

Theorem 2.5. *Let $\{\Lambda_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ be a family of bounded operators and $\{\Gamma_{\omega\nu} \in \mathcal{B}(K_\omega, W_{\omega\nu}) : \nu \in \Upsilon_\omega\}$ be a $(C_\omega - D_\omega)$-continuous g-frame for K_ω with respect to $\{W_{\omega\nu}\}_{\nu\in\Upsilon_\omega}$ and suppose that they are (C, D)-bounded. Then the following conditions are equivalent.*

(i) *$\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$;*
(ii) *$\{\Gamma_{\omega\nu}\Lambda_\omega\}_{\omega\in\Omega,\nu\in\Upsilon_\omega}$ is a continuous g-frame for H with respect to $\{W_{\omega\nu}\}_{\omega\in\Omega,\nu\in\Upsilon_\omega}$.*

Proof. (1) $\Rightarrow$ (2) Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$ with continuous g-frame bounds A and B. Then for all $f \in H$, we have

$$\begin{aligned}\int_\Omega \int_{\Upsilon_\omega} \|\Gamma_{\omega\nu}\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) &\le \int_\Omega D_\omega \|\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\omega)\\ &\le DB\|f\|^2.\end{aligned}$$

Similarly, we have

$$\int_\Omega \int_{\Upsilon_\omega} \|\Gamma_{\omega\nu}\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) \ge CA\|f\|^2,$$

so $\{\Gamma_{\omega\nu}\Lambda_\omega\}_{\omega\in\Omega,\nu\in\Upsilon_\omega}$ is continuous g-frame for H with respect to

$$\{W_{\omega\nu}\}_{\omega\in\Omega,\nu\in\Upsilon_\omega},$$

with continuous g-frame bounds AC and BD.

(2) $\Rightarrow$ (1) Let $\{\Gamma_{\omega\nu}\Lambda_\omega\}_{\omega\in\Omega,\nu\in\Upsilon_\omega}$ is a continuous g-frame for H with respect to $\{W_{\omega\nu}\}_{\omega\in\Omega,\nu\in\Upsilon_\omega}$ with continuous g-frame bounds A' and B'. Since $\Lambda_\omega f \in K_\omega$, we have

$$\begin{aligned}C\int_\Omega \|\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\omega) &\le \int_\Omega C_\omega \|\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\omega)\\ &\le \int_\Omega \int_{\Upsilon_\omega} \|\Gamma_{\omega\nu}\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega)\\ &\le B'\|f\|^2.\end{aligned}$$

Therefore

$$\int_\Omega \|\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\omega) \le \frac{B'}{C}\|f\|^2.$$

Similarly, we have

$$\int_\Omega \|\Lambda_\omega f\|^2 \ \mathrm{d}\mu(\omega) \ge \frac{A'}{D}\|f\|^2.$$

So $\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$ with continuous g-frame bounds $\frac{B'}{C}$ and $\frac{A'}{D}$. $\square$

Now, consider continuous fusion frames, instead of continuous g-frames, and the next corollary can be obtained.

Corollary 2.6. *Let $\{\Lambda_\omega \in \mathcal{B}(H, K_\omega) : \omega \in \Omega\}$ be a family of bounded operators and $\big\{\big(W_{\omega\nu}, \mathrm{m}(\omega,\nu)\big)\big\}_{\nu\in\Upsilon_\omega}$ be a $(C_\omega - D_\omega)$-continuous fusion frame for each K_ω with respect to $\{W_{\omega\nu}\}_{\nu\in\Upsilon_\omega}$ and suppose that they are (C, D)-bounded. Then the following conditions are equivalent.*

(i) *$\{\Lambda_\omega\}_{\omega\in\Omega}$ is a continuous g-frame for H with respect to $\{K_\omega\}_{\omega\in\Omega}$.*
(ii) *$\{\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega\}_{\omega\in\omega,\nu\in\Upsilon_\omega}$ is a continuous g-frame for H with respect to $\{W_{\omega\nu}\}_{\omega\in\omega,\nu\in\Upsilon_\omega}$.*

Proposition 2.7. *Let $\{\Lambda_\omega\}_{\omega\in\Omega}$ is an $(A-B)$-continuous g-frame with the continuous g-frame operator S_Λ and the canonical dual continuous g-frame $\{\tilde{\Lambda}_\omega\}_{\omega\in\Omega}$. Suppose that $\big\{\big(W_{\omega\nu}, \mathrm{m}(\omega,\nu)\big)\big\}_{\nu\in\Upsilon_\omega}$ be a Parseval continuous fusion frame for each K_ω with respect to $\{W_{\omega\nu}\}_{\nu\in\Upsilon_\omega}$. Then we have $\mathcal{S} = S_\Lambda$, where $\mathcal{S}$ is the continuous g-frame operator for*

$$\{\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega\}_{\omega\in\omega,\nu\in\Upsilon_\omega}.$$

Also $\{\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega\}_{\omega\in\omega,\nu\in\Upsilon_\omega}$ and $\{\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega\}_{\omega\in\omega,\nu\in\Upsilon_\omega}$ are canonical dual continuous g-frames with respect to each other.

Proof. Let $f, g \in H$, then we have

$$\begin{aligned}
\langle S_\Lambda f, g\rangle &= \int_\Omega \langle \Lambda_\omega f, \Lambda_\omega g\rangle d\mu(\omega) \\
&= \int_\Omega \int_{\Upsilon_\omega} \langle \Lambda_\omega f, \mathrm{m}^2(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega g\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) \\
&= \int_\Omega \int_{\Upsilon_\omega} \langle f, \mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega)^*\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega g\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) \\
&= \langle \mathcal{S} f, g\rangle.
\end{aligned}$$

So $\mathcal{S} = S_\Lambda$. Now by the hypothesis we have

$$\begin{aligned}
&\int_\Omega \int_{\Upsilon_\omega} \big\langle f, \big(\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega\big)^*\big(\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega\big)g\big\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) \\
&\quad= \int_\Omega \int_{\Upsilon_\omega} \big\langle \Lambda_\omega f, \mathrm{m}^2(\omega,\nu)\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega g\big\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) \\
&\quad= \int_\Omega \langle f, \Lambda_\omega^*\tilde{\Lambda}_\omega g\rangle \ \mathrm{d}\mu(\omega) \\
&\quad= \langle f, g\rangle.
\end{aligned}$$

Similarly we have

$$\int_\Omega \int_{\Upsilon_\omega} \big\langle f, \big(\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega\big)^*\big(\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega\big)g\big\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) = \langle f, g\rangle.$$

On the other hand we have $\mathcal{S} = S_\Lambda$. Hence

$$\begin{aligned}
\widetilde{\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}}\Lambda_\omega &= \mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega\mathcal{S}^{-1} \\
&= \mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega.
\end{aligned}$$

□

Definition 2.8. A sequence $\{T_\omega \in \mathcal{B}(H) : \omega \in \Omega\}$ is a continuous resolution of identity (simply CRI) on H if, for each $f, g \in H$:

(i) $\omega \longmapsto \langle f, T_\omega^* g\rangle$ is a measurable functional on Ω.
(ii) $\langle f, g\rangle = \int_\Omega \langle f, T_\omega^* g\rangle \ \mathrm{d}\mu(\omega)$.

Corollary 2.9. *Let* $\mathcal{K}_\omega = \{\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\}_{\nu\in\Upsilon_\omega}$ *be a Parseval continuous fusion frame for* K_ω *for all* $\omega \in \Omega$. *Then the family*

$$\{\mathrm{m}^2(\omega,\nu)\Lambda_\omega^*\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega\}_{\omega\in\Omega,\nu\in\Upsilon_\omega}$$

of bounded operators is a CRI on H.

Proof. For all $f, g \in H$ by using the above proposition we have

$$\begin{aligned}\langle f,g\rangle &= \int_\Omega\int_{\Upsilon_\omega} \langle f, \big(\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega\big)^*\big(\mathrm{m}(\omega,\nu)\Pi_{W_{\omega\nu}}\Lambda_\omega\big)g\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega)\\ &= \int_\Omega\int_{\Upsilon_\omega} \langle f, \big(\mathrm{m}^2(\omega,\nu)\tilde{\Lambda}_\omega^*\Pi_{W_{\omega\nu}}\Lambda_\omega\big)g\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega)\\ &= \int_\Omega\int_{\Upsilon_\omega} \langle f, \mathrm{m}^2(\omega,\nu)\big(\Lambda_\omega^*\Pi_{W_{\omega\nu}}\tilde{\Lambda}_\omega\big)^*g\rangle \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega).\end{aligned}$$

□

Proposition 2.10. *Let* $\mathcal{K} = \{(K_\omega, \mathrm{m}(\omega))\}_{\omega\in\Omega}$ *be an* $(A-B)$*-continuous fusion frame for* H. *Let* $\{\Lambda_{\omega\nu} \in \mathcal{B}(K_\omega, W_{\omega\nu}) : \nu \in \Upsilon_\omega\}$ *be a* $(C_\omega - D_\omega)$*-continuous g-frame for each* K_ω *with respect to* $\{W_{\omega\nu}\}_{\nu\in\Upsilon_\omega}$, *which are* $(C-D)$*-bounded. Then* $\{\mathrm{m}(\omega)\Lambda_{\omega\nu}\Pi_{K_\omega} \in \mathcal{B}(H, W_{\omega\nu}) : \omega \in \Omega, \nu \in \Upsilon_\omega\}$ *is an* (AC, BD)*-continuous g-frame for* H *with respect to* $\{W_{\omega\nu}\}_{\nu\in\Upsilon_\omega}$.

Proof. For any $f \in H$ we have

$$\begin{aligned}\int_\Omega\int_{\Upsilon_\omega} \|\mathrm{m}(\omega)\Lambda_{\omega\nu}\Pi_{K_\omega}f\|^2\ \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega) &\le \int_\Omega D_\omega \mathrm{m}^2(\omega)\|\Pi_{K_\omega}f\|^2\ \mathrm{d}\mu(\omega)\\ &\le D\int_\Omega \mathrm{m}^2(\omega)\|\Pi_{K_\omega}f\|^2\ \mathrm{d}\mu(\omega)\\ &\le DB\|f\|^2.\end{aligned}$$

Similarly we have

$$AC\|f\|^2 \le \int_\Omega\int_{\Upsilon_\omega} \|\mathrm{m}(\omega)\Lambda_{\omega\nu}\Pi_{K_\omega}f\|^2\ \mathrm{d}\mu(\nu)\mathrm{d}\mu(\omega).$$

□

3. Continuous g-frame operator for a pair of Bessel continuous fusion sequences

Now, a form of the reconstruction formula is provided the dual continuous fusion frame. Using Lemma 2.3 from [11], following can be given:

$$\Pi_{K_\omega}S_\mathcal{K}^{-1} = \Pi_{K_\omega}S_\mathcal{K}^{-1}\Pi_{S_\mathcal{K}^{-1}K_\omega}.$$

Hence

$$S_\mathcal{K}^{-1}\Pi_{K_\omega} = \Pi_{S_\mathcal{K}^{-1}K_\omega}S_\mathcal{K}^{-1}\Pi_{K_\omega}.$$

Then, the reconstruction formula is in the following form:

$$\begin{aligned}(3.1)\qquad \langle f,g\rangle = \langle S_{\mathcal{K}}S_{\mathcal{K}}^{-1}f,g\rangle &= \int_{\Omega}\mathrm{m}^2(\omega)\langle S_{\mathcal{K}}^{-1}f,\Pi_{K_\omega}g\rangle d\mu(\omega)\\ &= \int_{\Omega}\mathrm{m}^2(\omega)\langle f,S_{\mathcal{K}}^{-1}\Pi_{K_\omega}g\rangle d\mu(\omega)\\ &= \int_{\Omega}\mathrm{m}^2(\omega)\langle f,\Pi_{S_{\mathcal{K}}^{-1}K_\omega}S_{\mathcal{K}}^{-1}\Pi_{K_\omega}g\rangle d\mu(\omega),\end{aligned}$$

which leads to introducing the following definition:

Definition 3.1. Let $\mathcal{K} = \{(K_\omega, \mathrm{m}(\omega))\}_{\omega\in\Omega}$ be a continuous fusion frame and $S_{\mathcal{K}}$ be the continuous fusion frame operator. Also, $\mathcal{G} = \{(G_\omega, \mathrm{n}(\omega))\}_{\omega\in\Omega}$ is a Bessel continuous fusion sequence. $\mathcal{G}$ is an alternate dual of $\mathcal{K}$ if:

$$(3.2)\qquad \langle f,g\rangle = \int_{\Omega}\mathrm{m}(\omega)\mathrm{n}(\omega)\langle f,\Pi_{G_\omega}S_{\mathcal{K}}^{-1}\Pi_{K_\omega}g\rangle d\mu(\omega),$$

for all $f,g\in H$.

Using relation (3.1), the dual continuous fusion frame of $\mathcal{K}$ is an alternate dual continuous frame. Also, the following result can be obtained.

Proposition 3.2. *The alternate dual of a continuous fusion frame is a continuous fusion frame.*

Proof. Let $D_{\mathcal{K}}$ be the upper bound of the continuous fusion frame $\mathcal{K}$, by (3.2) we obtain

$$\begin{aligned}\|f\|^2 = \langle f,f\rangle &= \int_{\Omega}\mathrm{m}(\omega)\mathrm{n}(\omega)\langle f,\Pi_{G_\omega}S_{\mathcal{K}}^{-1}\Pi_{K_\omega}f\rangle d\mu(\omega)\\ &= \int_{\Omega}\mathrm{m}(\omega)\mathrm{n}(\omega)\langle \Pi_{G_\omega}f,S_{\mathcal{K}}^{-1}\Pi_{K_\omega}f\rangle d\mu(\omega)\\ &\le \left(\int_{\Omega}\mathrm{m}^2(\omega)\|S_{\mathcal{K}}^{-1}\Pi_{K_\omega}f\|^2 d\mu(\omega)\right)^{\frac{1}{2}}\\ &\quad\times\left(\int_{\Omega}\mathrm{n}^2(\omega)\|\Pi_{G_\omega}f\|^2 d\mu(\omega)\right)^{\frac{1}{2}}\\ &\le \sqrt{D_{\mathcal{K}}}\|S_{\mathcal{K}}^{-1}\|\|f\|\left(\int_{\Omega}\mathrm{n}^2(\omega)\|\Pi_{G_\omega}f\|^2 d\mu(\omega)\right)^{\frac{1}{2}}.\end{aligned}$$

So we have

$$\frac{1}{D_{\mathcal{K}}}\|S_{\mathcal{K}}^{-1}\|^2\|f\|^2 \le \int_{\Omega}\mathrm{n}^2(\omega)\|\Pi_{G_\omega}f\|^2 d\mu(\omega).$$

□

Below, two Bessel continuous fusion sequences are considered: $\mathcal{K} = \{(K_\omega, \mathrm{m}(\omega))\}_{\omega\in\Omega}$ with Bessel bound $D_\mathcal{K}$ and $\mathcal{G} = \{(G_\omega, \mathrm{n}(\omega))\}_{\omega\in\Omega}$ with Bessel bound $D_\mathcal{G}$. The operator

$$\langle S_{KG} f, g\rangle = \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{K_\omega}\Pi_{G_\omega} g\rangle d\mu(\omega), \quad (f, g \in H), \tag{3.3}$$

is introduced. By Cauchy-Schwartz inequality:

$$|\langle S_{KG} f, g\rangle| \leq \left(\int_\Omega \mathrm{m}^2(\omega)\|\Pi_{K_\omega} f\|^2 d\mu(\omega)\right)^{\frac{1}{2}} \left(\int_\Omega \mathrm{n}^2(\omega)\|\Pi_{G_\omega} g\|^2 d\mu(\omega)\right)^{\frac{1}{2}}. \tag{3.4}$$

It follows that:

$$|\langle S_{KG} f, g\rangle| \leq \sqrt{D_\mathcal{K}}\sqrt{D_\mathcal{G}}\|f\|\|g\|.$$

Hence, S_{KG} is a bounded operator and

$$\|S_{KG}\| \leq \sqrt{D_\mathcal{K}}\sqrt{D_\mathcal{G}}.$$

Using (3.4):

$$\|S_{KG} f\| \leq \sqrt{D_\mathcal{G}}\left(\int_\Omega \mathrm{m}^2(\omega)\|\Pi_{K_\omega} f\|^2 d\mu(\omega)\right)^{\frac{1}{2}}, \tag{3.5}$$

and

$$\|S^*_{KG} g\| \leq \sqrt{D_\mathcal{K}}\left(\int_\Omega \mathrm{n}^2(\omega)\|\Pi_{G_\omega} g\|^2 d\mu(\omega)\right)^{\frac{1}{2}}. \tag{3.6}$$

Moreover, using (3.3):

$$\begin{aligned}\langle S_{KG} f, g\rangle &= \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{K_\omega}\Pi_{G_\omega} g\rangle d\mu(\omega) \\ &= \overline{\int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle g, \Pi_{G_\omega}\Pi_{K_\omega} f\rangle d\mu(\omega)} \\ &= \langle f, S_{GK} g\rangle.\end{aligned}$$

Hence

$$S^*_{KG} = S_{GK}.$$

For this operator, the following result can be obtained.

Proposition 3.3. *Let $\mathcal{K} = \{(K_\omega, \mathrm{m}(\omega))\}_{\omega\in\Omega}$ be a continuous fusion frame with continuous fusion frame bounds C and D and continuous fusion frame operator S_K for a Hilbert space H. Let $\mathcal{G} = \{(G_\omega, \mathrm{n}(\omega))\}_{\omega\in\Omega}$ be an alternate dual continuous fusion frame for $\mathcal{K}$ with required positivity. Then we have*

$$CI_H \leq S_{GK} \leq DI_H,$$

and also S_{GK} is invertible.

Proof. Let f be an arbitrary element of H. Then we have

$$\begin{aligned}\|f\|^2 = \langle f, f\rangle &= \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{G_\omega} S_{\mathcal{K}}^{-1}\Pi_{K_\omega} f\rangle d\mu(\omega)\\ &\leq \frac{1}{C}\int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{G_\omega}\Pi_{K_\omega} f\rangle d\mu(\omega)\\ &= \frac{1}{C}\langle S_{GK} f, f\rangle.\end{aligned}$$

Similarly, $\langle S_{GK} f, f\rangle \leq D\|f\|^2$, hence:

$$CI_H \leq S_{GK} \leq DI_H.$$

By the same argument as in the proof of Proposition 2.9 in [13], S_{GK} is invertible and

$$\frac{1}{D} \leq \|S_{GK}^{-1}\| \leq \frac{1}{C}.$$

□

Remark 3.4. By the Proposition 3.3 we have the following reconstruction formulas:

(i) $\langle f, g\rangle = \langle S_{KG} f, S_{KG}^{-1} g\rangle = \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{K_\omega}\Pi_{G_\omega} S_{KG}^{-1} g\rangle d\mu(\omega)$,

(ii) $\langle f, g\rangle = \langle S_{KG} S_{KG}^{-1} f, g\rangle = \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, S_{KG}^{-1}\Pi_{K_\omega}\Pi_{G_\omega} g\rangle d\mu(\omega)$,

(iii) $\langle f, g\rangle = \langle S_{GK} f, S_{GK}^{-1} g\rangle = \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{G_\omega}\Pi_{K_\omega} S_{GK}^{-1} g\rangle d\mu(\omega)$,

(iv) $\langle f, g\rangle = \langle S_{GK} S_{GK}^{-1} f, g\rangle = \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, S_{GK}^{-1}\Pi_{G_\omega}\Pi_{K_\omega} g\rangle d\mu(\omega)$.

Theorem 3.5. *The following are equivalent:*

(i) S_{KG} *is bounded below;*
(ii) *There exist* $U \in \mathcal{L}(H)$ *such that* $\{T_\omega\}_{\omega\in\Omega}$ *is a CRI, where*

$$T_\omega = \mathrm{m}(\omega)\mathrm{n}(\omega)U\Pi_{G_\omega}\Pi_{K_\omega}, \quad (\omega \in \Omega).$$

If one of the conditions holds, then $\mathcal{G}$ *is a continuous fusion frame.*

Proof. (1) ⇒ (2) If S_{KG} is bounded below, then there exists $U \in \mathcal{L}(H)$ such that $US_{KG} = I_H$. It follows that

$$\begin{aligned}\langle f, g\rangle &= \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, \Pi_{K_\omega}\Pi_{G_\omega} U^* g\rangle d\mu(\omega)\\ &= \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, (U\Pi_{G_\omega}\Pi_{K_\omega})^* g\rangle d\mu(\omega).\end{aligned}$$

(2) ⇒ (1) If (2) holds, then for $f, g \in H$ we have

$$\langle US_{KG}f, g\rangle = \langle S_{KG}f, U^*g\rangle = \int_\Omega \mathrm{m}(\omega)\mathrm{n}(\omega)\langle f, (U\Pi_{G_\omega}\Pi_{K_\omega})^*g\rangle d\mu(\omega)$$
$$= \langle f, g\rangle,$$

hence $US_{KG} = I_H$. It follows that S_{KG} is bounded below.
If S_{KG} is bounded below, from (3.5) it follows that $\mathcal{G}$ is a continuous fusion frame. □

Corollary 3.6. *The following are equivalent:*

(i) S_{KG} *is invertible;*
(ii) *There exists invertible operator* $U \in \mathcal{L}(H)$ *such that* $\{T_\omega\}_{\omega\in\Omega}$ *is a CRI, where*

$$T_\omega = \mathrm{m}(\omega)\mathrm{n}(\omega)U\Pi_{G_\omega}\Pi_{K_\omega}, \quad (\omega \in \Omega).$$

If one of the conditions holds, then $\mathcal{K}$ *and* $\mathcal{G}$ *are continuous fusion frames.*

Corollary 3.7. *Let* $\mathcal{G} = \{(G_\omega, \mathrm{n}(\omega))\}_{\omega\in\Omega}$ *be a Bessel continuous fusion sequence. Then* $\mathcal{G}$ *is a continuous fusion frame if and only if there exists* $\mathcal{K} = \{(K_\omega, \mathrm{m}(\omega))\}_{\omega\in\Omega}$, *a Bessel continuous fusion sequence, such that* S_{KG} *is bounded below.*

Proof. If $\mathcal{G}$ is a continuous fusion frame, for all $\omega \in \Omega$, we take $K_\omega = G_\omega$ and $\mathrm{m}(\omega) = \mathrm{n}(\omega)$. For converse, we use Theorem 3.5. □

4. Conclusions

In this paper, new continuous g-frames and continuous fusion frames were constructed by considering their components. Also, the continuous resolution of identity (simply CRI) was introduced and a family of CRI was obtained by a Parseval continuous fusion frame. In the second part, the alternate dual was defined for two Bessel continuous fusion sequences and that the alternate dual of a continuous fusion frame was shown as a continuous fusion frame. Moreover, a new operator was proposed for two Bessel continuous fusion sequences. Accordingly, a number of reconstruction formulas and a family of CRI were obtained.

References

1. M.R. Abdollahpour and M.H. Faroughi, *Continuous g-frames in Hilbert spaces*, Southeast Asian Bull. Math., 32, 1-19(2008).
2. S.T. Ali, J.P. Antoine, and J.P Gazeau, *Continuous frames in Hilbert spaces*, Annals of Physics. 222, 1-37 (1993).
3. M.S. Asgari and A. Khosravi, *Frames and bases of subspaces in Hilbert spaces*, J. Math. Anal. Appl., 308, 541-553 (2005).

4. O. Christensen, *An introduction to frame and Riesz bases*, Birkhäuser, Boston, 2003.
5. I. Daubechies, A. Grossmann, and Y. Meyer, *Painless nonorthogonal expansions*, J. Math. Phys., 27, 1271-1283 (1986).
6. I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF conference series in applied mathematics, 61, SIAM, Philadelphia, 1992.
7. R. Duffin and A. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., 72, 341-366 (1952).
8. M.H. Faroughi and R. Ahmadi, *Fusion integral*, J. Mathematische Nachrichten. 284, No. 5-6, 681-693 (2011).
9. M.H. Faroughi and R. Ahmadi, *Some properties of C-fusion frames*, Turk J Math., 34, 393-415. (2010).
10. J.P. Gabardo and D. Han, *Frames Associated with Measurable Space*, Adv. Comp. Math., 18, 127-147 (2003).
11. P. Gavruta, *On the duality of fusion frames in Hilbert spaces*, J. Math. Anal. Appl., 333, 871-879 (2007).
12. G. Kaiser, *A Friendly Guide to Wavelets*, Birkhäuser, 1994.
13. A. Khosravi and K. Musazadeh, *Fusion frames and g-frames*, J. Math. Anal. Appl., 342, 1068-1083 (2008).
14. W. Sun, *g-frame and g- Riesz bases*, J. Math. Anal. Appl., 322, 437-452 (2006).

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[1] Department of Mathematics, Science and Research Branch, Islamic Azad University, Kerman, Iran.

E-mail address: `Mahdiyehkhayyami@yahoo.com`

[2] Department of Mathematics, Science and Research Branch, Islamic Azad University, Kerman, Iran.

E-mail address: `Nazari@mail.uk.ac.ir`