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ON ISOMORPHISM OF TWO BASES IN MORREY-LEBESGUE TYPE SPACES

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Abstract. Double system of exponents with complex-valued coefficients is considered. Under some conditions on the coefficients, we prove that if this system forms a basis for the Morrey-Lebesgue type space on $[-\pi, \pi]$, then it is isomorphic to the classical system of exponents in this space.

1. Introduction

Consider the double system of exponents

$$\left\{A\left(t\right)e^{int};\, B\left(t\right)e^{-ikt}\right\}_{n\in Z_+,\, k\in N}, \tag{1.1}$$

with complex-valued coefficients $A\left(t\right) = \left|A\left(t\right)\right| e^{i\alpha(t)}$, $B\left(t\right) = \left|B\left(t\right)\right| e^{i\beta(t)}$ on the interval $[-\pi,\, \pi]$, where N is the set of natural numbers and $Z_+ = \{0\}\bigcup N$. The system (1.1) is a generalization of the following double sine and cosine system

$$1\bigcup\left\{\cos\left(nt+\gamma\left(t\right)\right);\, \sin\left(nt+\gamma\left(t\right)\right)\right\}_{n\in Z_+}, \tag{1.2}$$

where $\gamma : [-\pi,\, \pi] \to C$ is a complex-valued function in general. The study of basis properties (such as completeness, minimality, basicity) of the systems of type (1.1) and (1.2) in the space $L_p\left(-\pi,\, \pi\right)$, $1 \leq p < +\infty$ ($L_\infty\left(-\pi,\, \pi\right) \equiv C\left[-\pi,\, \pi\right]$), dates back to the classical works by Paley-Wiener [31] and N. Levinson [22] who considered the perturbed systems of exponents. A lot of works have appeared in this field since then. A

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well known "Kadets-1/4" theorem [19] also refers to this range of issues. Criterion for the basicity of a system of exponents

$$\left\{e^{i(n+\alpha \mathrm{sign} n)t}\right\}_{n\in Z}, \tag{1.3}$$

for $L_p(-\pi, \pi)$ $1 < p < +\infty$, where Z is the set of all integers, was first found by A.M. Sedletsky [38]. Similar result was obtained by E.I. Moiseyev [24] who used the method of boundary value problems.

Note that the single versions of these systems are the cosine system of

$$1\bigcup\{\cos(n+\alpha)t\}_{n\in N}, \tag{1.4}$$

and the sine system of

$$\{\sin(n+\alpha)t\}_{n\in N}, \tag{1.5}$$

that arise when solving equations of mixed type by the Fourier method (see, e.g., [25–27, 33, 34]). The basis properties of the systems (1.4) and (1.5) in $L_p(0, \pi)$ were fully studied by E.I. Moiseyev [24,28] in case where $a \in R$ is a real parameter. G.G. Devdariani [14,15] extended these results to the case of complex parameter. Riesz basicity in $L_2(-\pi, \pi)$ for the system (1.2), when $\gamma : [-\pi, \pi] \to C$ is a Hölder function, was studied by A.N. Barmenkov [1]. One of the most effective methods for treating basis properties of systems like (1.1)-(1.5) is the method of boundary value problems of the theory of analytic functions which dates back to A.V. Bitsadze [11]. This method was successfully used by many authors [1–4, 11, 14, 15, 24–28, 33, 34]. B.T. Bilalov [2–5] considered the most general case, namely, the systems of the form (1.1), and using the results concerning basis properties of (1.1), found a basicity criterion for the completeness and minimality of the sine system of the form

$$\{\sin(nt+\gamma(t))\}_{n\in N}, \tag{1.6}$$

in $L_p(0, \pi)$, $1 < p < +\infty$, where $\gamma : [0, \pi] \to C$ is a piecewise continuous function. Similar results for the system (1.6) were obtained earlier in [13].

The study of basis properties of systems (1.1)-(1.6) in different spaces of functions is still ongoing. The weighted case of the L_p space was considered in [6, 29, 35], while [30, 36, 39] treated the case of Sobolev spaces. In [8, 9], the basicity of the system (1.3) was studied for generalized Lebesgue spaces.

It should be noted that in recent years interest in the study of various problems of analysis in Morrey-type spaces increased to a great extent. There is obviously a need to study the approximation properties of systems like (1.1)-(1.6) in Morrey-type spaces. Some questions of the approximation theory were studied in [17, 18, 21]. In [10], the basicity of classical exponential systems in Morrey-Lebesgue type spaces was treated.

In this paper, we consider a system of exponents (1.1) and prove that if it forms a basis for Morrey-Lebesgue type space $M_{p,\alpha} = M_{p,\alpha}(-\pi, \pi)$, then it is isomorphic to the classical system of exponents in this space. To do so, we use the method of boundary value problems. A similar result was obtained earlier in [7].

2. Necessary Information

We will need some facts about the theory of Morrey-type spaces. Let Γ be a rectifiable Jordan curve in the complex plane C. By $|M|_\Gamma$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

The expression $f(x) \sim g(x)$, $x \in M$, means

$$\exists \delta > 0 : \delta \leq \left| \frac{f(x)}{g(x)} \right| \leq \delta^{-1}, \quad \forall x \in M.$$

Similar meaning is intended by the expression $f(x) \sim g(x)$, $x \to a$.

By Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 \leq \alpha \leq 1$, $p \geq 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm $\|\cdot\|_{L^{p,\alpha}(\Gamma)}$:

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_B \left(\left| B \bigcap \Gamma \right|_\Gamma^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p \, |d\xi| \right)^{1/p} < +\infty.$$

$L^{p,\alpha}(\Gamma)$ is a Banach space with $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_\infty(\Gamma)$. The inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 \leq \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in [0, 1]$, $\forall p \geq 1$. For $\Gamma = [-\pi, \pi]$ we will use the notation $L^{p,\alpha}(-\pi, \pi) = L^{p,\alpha}$.

More details on Morrey-type spaces can be found in [12, 20, 23, 32, 37, 40].

Let $\omega = \{z \in C : |z| <\}$ be the unit circle on C and $\partial\omega = \gamma$ be its circumference.

Define the Morrey-Hardy space $H_+^{p,\alpha}$ of analytic functions $f(z)$ inside ω equipped with the following norm

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0<r<1} \|f(re^{n})\|_{L^{p,\alpha}}.$$

In [10], the following theorem was proved.

Theorem 2.1. *The function $f(\cdot)$ belongs to $H_+^{p,\alpha}$ only when the nontangential boundary values f^+ belong to $L^{p,\alpha}$, $1<p<+\infty$, $0<\alpha\le 1$, and the Cauchy formula*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f^+(\tau)\,d\tau}{\tau - z},$$

holds.

Denote by $\tilde{L}^{p,\alpha}$ the linear subspace of $L^{p,\alpha}$ functions, whose shifts are continuous in $L^{p,\alpha}$, i.e. $\|f(\cdot+\delta) - f(\cdot)\|_{L^{p,\alpha}} \to 0$ as $\delta \to 0$. Consider the closure of $\tilde{L}^{p,\alpha}$ in $L^{p,\alpha}$ and denote it by $M^{p,\alpha}$. The following theorem was also proved in [10].

Theorem 2.2. *Functions infinitely differentiable on $[0, 2\pi]$ are dense in the space $M^{p,\alpha}$, $1\le p<+\infty$, $0<\alpha\le 1$.*

In the sequel, we will extensively use the following lemma.

Lemma 2.3. *If $f \in L_\infty$, $g \in M^{p,\alpha}$, $1<p<+\infty$, $0<\alpha\le 1$, then $fg \in M^{p,\alpha}$.*

Consider the following singular operator

$$(Sf)(\tau) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - \tau}, \quad \tau\in\gamma.$$

Using the results of [18,20,37], it is easy to prove the following.

Theorem 2.4. *Singular operator S acts boundedly in $M^{p,\alpha}(\gamma)$, $1<p<+\infty$, $0<\alpha\le 1$.*

Consider the space $H_+^{p,\alpha}$. Denote by $L_+^{p,\alpha}$ the subspace of $L^{p,\alpha}$, generated by the restrictions of the functions from $H_+^{p,\alpha}$ to γ. From the results mentioned above it immediately follows that the spaces $H_+^{p,\alpha}$ and $L_+^{p,\alpha}$ are isomorphic and $f^+(\cdot) = (Jf)(\cdot)$, where $f\in H_+^{p,\alpha}$, f^+ are nontangential boundary values of f on γ, and J performs the corresponding isomorphism. Let $M_+^{p,\alpha} = M^{p,\alpha}\bigcap L_+^{p,\alpha}$. It is clear that $M_+^{p,\alpha}$ is a subspace of $M^{p,\alpha}$ with respect to the norm $\|\cdot\|_{L^{p,\alpha}}$. Let $MH_+^{p,\alpha} = J^{-1}\left(M_+^{p\,\alpha}\right)$. The latter is a subspace of $H_+^{p,\alpha}$. Let $f\in H_+^{p,\alpha}$ and f^+ be its boundary values. It is absolutely clear that the norm in $\|f\|_{H_+^{p,\alpha}}$ can be defined also as $\|f\|_{H_+^{p,\alpha}} = \|f^+\|_{L^{p,\alpha}}$.

Similar to the classical case, we define the Morrey-Hardy class outside ω. Let $D^- = C\backslash\bar{\omega}$ $(\bar{\omega} = \omega\bigcup\gamma)$. We say the analytic function f in D^-

has finite order k at infinity, if its Laurent series in the neighborhood of the point at infinity has the following form

$$f\left(z\right)=\sum_{n=-\infty}^{k}a_{n}z^{n},\quad k<+\infty,a_{k}\neq 0. \tag{2.1}$$

Thus, for $k>0$ the function $f\left(z\right)$ has a pole of order k; for $k=0$ it is bounded; and in case of $k<0$ it has a zero of order $\left(-k\right)$. Let $f\left(z\right)=f_{0}\left(z\right)+f_{1}\left(z\right)$, where $f_{0}\left(z\right)$ is the principal, and $f_{1}\left(z\right)$ is the regular part of decomposition (2.1) of the function $f\left(z\right)$. Consequently, if $k\leq 0$, then $f_{0}\left(z\right)=0$. If $k>0$, then $f_{0}\left(z\right)$ is a polynomial of degree k. We say that the function $f\left(z\right)$ belongs to the class ${}_{m}H_{-}^{p,\,\alpha}$, if it has the order at infinity less or equal to m, i.e. $k\leq m$ and $f_{1}\left(\frac{1}{z}\right)\in H_{+}^{p,\,\alpha}$.

The class ${}_{m}MH_{-}^{p,\,\alpha}$ is defined in a very similar manner to the case of $MH_{+}^{p,\,\alpha}$. In other words, ${}_{m}MH_{-}^{p,\,\alpha}$ is the subspace of ${}_{m}H_{-}^{p,\,\alpha}$-functions whose shifts are continuous on the unit circle's circumference with respect to the norm $\left\|\cdot\right\|_{L^{p,\,\alpha}(\gamma)}$.

We will also use the following result of [10].

Theorem 2.5 ([10]). *The system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for $M^{p,\,\alpha}$, $1<p<+\infty$, $0<\alpha\leq 1$.*

We will need the Sokhotski-Plemelj formula for the boundary values of a Cauchy type integral

$$\Phi\left(z\right)=\frac{1}{2\pi i}\int_{\gamma}\frac{f\left(\tau\right)d\tau}{\tau-z}, \tag{2.2}$$

where $f\left(e^{it}\right)\in L_{1}\left(-\pi,\,\pi\right)$. Then the boundary values $\Phi^{\pm}\left(\tau\right)$, $\tau\in\gamma$, satisfy the following Sokhotski-Plemelj expression

$$\Phi^{\pm}\left(\tau\right)=\pm\frac{1}{2}f\left(\tau\right)+\left(Sf\right)\left(\tau\right),\quad \text{a.e..}\quad \tau\in\gamma, \tag{2.3}$$

where $S\left(\cdot\right)$ is a singular integral

$$\left(Sf\right)\left(\tau\right)=\frac{1}{2\pi i}\int_{\gamma}\frac{f\left(\xi\right)d\xi}{\xi-\tau},\quad \tau\in\gamma.$$

Let's show the validity of the following direct decomposition

$$L^{p,\,\alpha}=H_{+}^{p,\,\alpha}\underset{-1}{\dot{+}}H_{-}^{p,\,\alpha}, \tag{2.4}$$

with $1<p<+\infty,0<\alpha<1$.

In fact, let $f \in \mathrm{L}^{p,\alpha}$. Then it is clear that $f \in L_p$. Consider the Cauchy type integral (2.2). Then, by Theorem 2.1, we have $\Phi(z) \in H_+^{p,\alpha}$ for $|z| < 1$, and $\Phi(z) \in_{-1} H_-^{p,\alpha}$ for $|z| > 1$. From the Sokhotski-Plemelj formulas (2.3) it follows that

$$f(\tau) = \Phi^+(\tau) - \Phi^-(\tau), \quad \text{a.e.} \quad \tau \in \gamma. \tag{2.5}$$

Denote by ${}_{-1}L_-^{p,\alpha}$ the subspace of $L^{p,\alpha}$, generated by the restrictions of the functions from ${}_{-1}H_-^{p,\alpha}$ to γ. It is not difficult to see that $L_+^{p,\alpha} \bigcap_{-1} L_-^{p,\alpha} = \{0\}$. Then from (2.4) we immediately derive the direct decomposition

$$L^{p,\alpha} = L_+^{p,\alpha} \underset{-1}{\dot{+}} L_-^{p,\alpha}. \tag{2.6}$$

Identifying $H_+^{p,\alpha} \leftrightarrow L_+^{p,\alpha}$ and ${}_{-1}L_-^{p,\alpha} \leftrightarrow_{-1} H_-^{p,\alpha}$, we obtain the decomposition (2.4).

Applying Theorem 2.4 to the expression (2.5), we obtain in a similar way that the direct decomposition $M^{p,\alpha} = MH_+^{p,\alpha} \underset{-1}{\dot{+}} MH_-^{p,\alpha}$ also holds.

Let $P^{\pm}$ be projectors

$$P^+ : M^{p,\alpha} \to M_+^{p,\alpha}, \qquad P^- : M^{p,\alpha} \to_{-1} M_-^{p,\alpha},$$

generated by the decomposition (2.4) (or (2.6). Denote by $T^{\pm} : M^{p,\alpha} \to M^{p,\alpha}$ the multiplication operators defined as follows

$$T^+ f = Af, \qquad T^- f = Bf, \quad \forall f \in L^{p,\alpha}.$$

Let $A^{\pm 1}; B^{\pm 1} \in L_\infty(-\pi, \pi)$. Suppose that the system (1.1) forms a basis for $M^{p,\alpha}$. Take $\forall g \in M^{p,\alpha}$ and expand it in terms of this basis

$$g(t) = A(t) \sum_{n=0}^{\infty} g_n e^{int} + B(t) \sum_{n=1}^{\infty} g_{-n} e^{-int}.$$

As $A^{\pm} \in L_\infty$, $B^{\pm} \in L_\infty$, it follows from Lemma 2.3 that the series

$$f^+(t) = \sum_{n=0}^{\infty} g_n e^{int},$$

and

$$f^-(t) = \sum_{n=0}^{\infty} g_{-n} e^{-int},$$

represent some functions from $M^{p,\alpha}$. Suppose

$$f(t) = \sum_{n=-\infty}^{+\infty} g_n e^{int}, \quad t \in [-\pi, \pi].$$

It's clear that $f \in M^{p,\alpha}$. Let us show that the inclusions

$$f^+ \in M_+^{p,\alpha}, \qquad f^- \in_{-1} M_-^{p,\alpha},$$

hold. In fact, we have

$$\begin{aligned}\int_\gamma f^+(\arg\xi)\,\xi^n d\xi &= i\int_{-\pi}^{\pi} f^+(t)\, e^{i(n+1)t} dt \\ &= i\sum_{n=0}^{\infty} g_k \int_{-\pi}^{\pi} e^{i(k+n+1)t} dt \\ &= 0, \quad \forall n \in Z_+.\end{aligned}$$

Then from the theorem of Privalov [16] it follows that $f^+(t)$ are the boundary values of the function $F^+ \in H_1^+$ with

$$F^+(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\arg\xi)}{\xi - z} d\xi, \quad |z| < 1. \tag{2.7}$$

We have

$$\begin{aligned}F^+(z) &= \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} g_n \frac{e^{int}}{e^{it} - z} de^{it} \\ &= \frac{1}{2\pi i}\sum_{n=0}^{\infty} g_n \int_\gamma \frac{\xi^n d\xi}{\xi - z} \\ &= \sum_{n=0}^{\infty} g_n z^n, \quad |z| < 1.\end{aligned}$$

As $f \in M^{p,\alpha}$, it follows from (2.7) that $F^+ \in MH_+^{p,\alpha}$.

In a similar way we can prove that $F^- \in_{-1} MH_-^{p,\alpha}$, where

$$F^-(z) = \sum_{n=1}^{\infty} g_{-n} z^{-n}, \quad |z| > 1.$$

Consider the operator $T = T^+P^+ + T^-P^-$. We have

$$\begin{aligned}Tf &= T^+P^+f + T^-P^-f \\ &= T^+f^+ + T^-f^- \\ &= A(\cdot)\, f^+ + B(\cdot)\, f^- \\ &= g.\end{aligned}$$

Then, the equation

$$Tf = g, \quad g \in M^{p,\alpha}, \tag{2.8}$$

has a solution in $M^{p,\alpha}$ for all $g \in M^{p,\alpha}$, i.e. $R_T = M^{p,\alpha}$, where R_T is the range of the operator T. Let $f \in KerT$. Expand f in the basis $\left\{e^{int}\right\}_{n\in Z}$:

$$f(t) = \sum_{n=-\infty}^{+\infty} f_n e^{int}.$$

We have

$$0 = Tf = A(t) \sum_{n=0}^{\infty} f_n e^{int} + B(t) \sum_{n=1}^{\infty} f_{-n} e^{-int}.$$

As the system (1.1) forms a basis for $M^{p,\alpha}$, this gives us $f_n = 0$, $\forall n \in Z$, i.e. $KerT = \{0\}$. By Banach theorem, from $T \in L(M^{p,\alpha})$ it follows that $T^{-1} \in L(M^{p,\alpha})$, and this in turn implies the correct solvability of the equation (2.8).

Now, on the contrary, let the equation (2.8) be correctly solvable in $M^{p,\alpha}$. Take an arbitrary $g \in M^{p,\alpha}$ and let $f = T^{-1}g$. Expand f in the basis $\left\{e^{int}\right\}_{n\in Z}$ in $M^{p,\alpha}$:

$$f(t) = \sum_{n=-\infty}^{+\infty} f_n e^{int}, \quad t \in [-\pi, \pi].$$

We have

$$P^+ f = \sum_{n=0}^{\infty} f_n e^{int}; \qquad P^- f = \sum_{n=1}^{\infty} f_{-n} e^{-int},$$

and therefore

$$Tf = A(t) \sum_{n=0}^{\infty} f_n e^{int} + B(t) \sum_{n=1}^{\infty} f_{-n} e^{-int} = g(t),$$

i.e. every element of $M^{p,\alpha}$ can be expanded with respect to the system (1.1) in $M^{p,\alpha}$. Let's show that this expansion is unique. Let

$$A(t) \sum_{n=0}^{\infty} f_n e^{int} + B(t) \sum_{n=1}^{\infty} f_{-n} e^{-int} = 0.$$

Suppose

$$f(t) = \sum_{n=-\infty}^{+\infty} f_n e^{int}.$$

It's clear that $f \in M^{p,\,\alpha}$. We have

$$\begin{aligned} &Tf = A(t) \sum_{n=0}^{\infty} f_n e^{int} + B(t) \sum_{n=0}^{\infty} f_{-n} e^{-int} = 0 \\ &\Rightarrow f = T^{-1} 0 = 0 \\ &\Rightarrow f_n = 0, \quad \forall n \in Z. \end{aligned}$$

Thus, we have proved the following theorem.

Theorem 2.6. *Let $A^{\pm 1}$, $B^{\pm 1} \in L_\infty(-\pi,\, \pi)$. The system (1.1) forms a basis for $M^{p,\,\alpha}$ only when the equation (2.8) is correctly solvable in $M^{p,\,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$.*

Now let's prove the following main theorem.

Theorem 2.7. *Let $A^{\pm 1}$, $B^{\pm 1} \in L_\infty(-\pi,\, \pi)$. If the system (1.1) forms a basis for $M^{p,\,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$, then it is isomorphic to the classical system of exponents $\left\{e^{int}\right\}_{n \in Z}$ in $M^{p,\,\alpha}$, with the isomorphism given by means of the operator T_0:*

$$(T_0 f)(t) = A(t) \sum_{n=0}^{\infty} \left(f;\, e^{inx}\right) e^{int} + B(t) \sum_{n=1}^{\infty} \left(f;\, e^{-inx}\right) e^{-int}, \tag{2.9}$$

where

$$(f;\, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, \overline{g(t)} dt.$$

Proof. Let the system (1.1) forms a basis for $M^{p,\,\alpha}$. Take an arbitrary $f \in M^{p,\,\alpha}$. As the system $\left\{e^{int}\right\}_{n \in Z}$ forms a basis for $M^{p,\,\alpha}$, it is clear that the series

$$f^+(t) = \sum_{n=0}^{\infty} \left(f;\, e^{inx}\right) e^{int},$$

and

$$f^-(t) = \sum_{n=1}^{\infty} \left(f;\, e^{-inx}\right) e^{-int},$$

converge in $M^{p,\,\alpha}$. Moreover, the following inequality holds

$$\left\|f^{\pm}\right\|_{p,\,\alpha} \le c \left\|f\right\|_{p,\,\alpha}.$$

Then from (2.9) it directly follows that $T_0 \in L\left(M^{p,\alpha}\right)$. Let's show that $KerT_0 = \{0\}$. Let $f \in KerT_0$, i.e.

$$T_0 f = A\left(t\right) \sum_{n=0}^{\infty} \left(f;\, e^{inx}\right) e^{int} + B\left(t\right) \sum_{n=1}^{\infty} \left(f;\, e^{-inx}\right) e^{-int} = 0.$$

From the basicity of the system (1.1) for $M^{p,\alpha}$ it follows that $\left(f;\, e^{inx}\right) = 0$, $\forall n \in Z \Rightarrow f = 0$, because the system $\left\{e^{inx}\right\}_{n \in Z}$ forms a basis for $M^{p,\alpha}$. Consequently, $KerT_0 = \{0\}$. Now let's show that $R_{T_0} = M^{p,\alpha}$. Let $g \in M^{p,\alpha}$ be an arbitrary element. By Theorem 2.1, $\exists f \in M^{p,\alpha}$: $Tf = g$. On the other hand, it is not difficult to see that $T_0 = T$, and as a result, $R_T = M^{p,\alpha}$. Then it follows from the Banach theorem that T_0 is an automorphism in $M^{p,\alpha}$. It's clear that $T_0\left[e^{inx}\right] = A\left(t\right) e^{int}$, $\forall n \in Z_+$ and $T_0\left[e^{-inx}\right] = B\left(t\right) e^{-int}$, $\forall n \in N$. The theorem is proved. □

This theorem has the following immediate corollary.

Corollary 2.8. *If the perturbed system of exponents*

$$\left\{e^{i(n+\alpha \mathrm{sign} n)t}\right\}_{n \in Z},$$

forms a basis for the space $M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$, then it is isomorphic to the classical system of exponents $\left\{e^{int}\right\}_{n \in Z}$ in this space.

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