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LATIN-MAJORIZATION AND ITS LINEAR PRESERVERS

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Abstract. In this paper we study the concept of Latin-majorization. Geometrically this concept is different from other kinds of majorization in some aspects. Since the set of all xs Latin-majorized by a fixed y is not convex, but, consists of union of finitely many convex sets. Next, we hint to linear preservers of Latin-majorization on $\mathbb{R}^n$ and $M_{n,m}$.

1. Introduction

Let $x = (x_1, \ldots, x_n)^t$ and $y = (y_1, \ldots, y_n)^t$ be two column vectors in $\mathbb{R}^n$ and $x^\downarrow = \left(x_1^\downarrow, \ldots, x_n^\downarrow\right)$ be rearrangment of coordinates of x in nonincreasing order. Then x is said to be majorized by y (written $x \prec y$) if for all $k\,(k = 1, 2, \ldots, n)$ we have

$$\sum_{i=1}^{k} x_i^\downarrow \leq \sum_{i=1}^{k} y_i^\downarrow,$$

and

$$\sum_{i=1}^{n} x_i^\downarrow = \sum_{i=1}^{n} y_i^\downarrow,$$

see [8].

A doubly stochastic matrix is a real square matrix with nonnegative elements all of whose row sums and column sums equal to 1. One of the most important properties of doubly stochastic matrices is as follows.

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Theorem 1.1 (Birkhoff's theorem)**.** *The set of $n \times n$ doubly stochastic matrices is a convex set whose extreme points are the $n \times n$ permutation matrices.*

Denote the set of all $n \times n$ permutation matrices by S_n, and an $n \times n$ matrix all of whose entries equal to 1 by J_n. Hardy, Littlewood, and Polya proved that for all $x, y \in \mathbb{R}^n$, $x \prec y$ if and only if $x = Dy$ for some doubly stochastic matrix $D \in M_n$, see [3].

2. Latin-majorization

By using different types of doubly stochastic matrices, differnt types of majorizations have been introduced and investigated. In the following we investigate Latin-majorization.

Definition 2.1. A Latin square of order n is an $n \times n$ matrix, each row and each column of which is a permutation of the set of letters $I_n = \{1, 2, \dots, n\}$.

Each Latin square is equivalent to n-tuple $(P_1, P_2, \dots, P_n)$, where $P_1, P_2, \dots, P_n \in S_n$ and $P_1 + P_2 + \cdots + P_n = J_n$.

Actually one can consider the set of n different elements instead of $I_n = \{1, 2, \dots, n\}$. Here we omit the condition that the elements must be different, so we have the followings.

Let

$$\Omega_n = \{\{P_1, P_2, \dots, P_n\} \,|\, P_1, P_2, \dots, P_n \in S_n \, and \, P_1 + P_2 + \cdots + P_n = J_n\}.$$

Definition 2.2. An $n \times n$ matrix A is called Latin doubly stochastic matrix if there exists $\{P_1, P_2, \dots, P_n\} \in \Omega_n$ such that

$$A \in conv\,\{P_1, P_2, \dots, P_n\}.$$

Definition 2.3. Let x, y be two column vectors in $\mathbb{R}^n$. Then x is said to Latin-majorized by y if there exists a Latin doubly stochastic matrix D such that $x = Dy$ (denoted by $x \prec_L y$).

Note that with our definitions a Latin doubly stochastic matrix is not necessarily a Latin square matrix.

The number of $n \times n$ Latin square matrices (denoted by L_n) has been found in [10].

Theorem 2.4 ([10])**.** *Let B_n be the set of $n \times n$ $(0,1)$ matrices, $\sigma_0(A)$ be the number of zero elements of the matrix A, per A be the permanent of A, then we have*

$$L_n = n! \sum_{A \in B_n} (-1)^{\sigma_0(A)} \binom{per\ A}{n}.$$

Since the number of Ω_n is $L_n/n!$, the set $\{x|x\prec_L y\}$ for fixed y (with distinct coordinates) consists of union of

$$\sum_{A\in B_n}(-1)^{\sigma_0(A)}\binom{per\ A}{n},$$

convex sets. For example Figure 1 shows the set $\{x|x\prec_L y\}$ in $\mathbb{R}^3$.

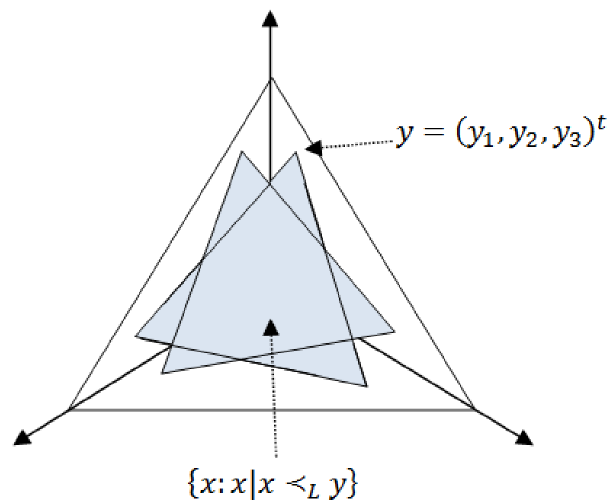

FIGURE 1. $\{x|x \prec_{\mathrm{L}} y\} \in \mathbb{R}^3$.

If $x\prec y$ and $y\prec x$ we write $x \sim y$. Similarly, if $x\prec_L y$ and $y\prec_L x$ we write $x \sim_L y$.

Theorem 2.5. *Let $x, y \in \mathbb{R}^n$. Then $x \sim_L y$ if and only if there exists $P \in S_n$ such that $x = Py$.*

Proof. Let $x \sim_L y$ then $x\prec_L y$ and $y\prec_L x$. Hence $x\prec y$ and $y\prec x$. This implies that $x \sim y$ and $x = Py$ for some $P \in S_n$. The converse is trivial. □

3. Linear preservers of Latin-majorization on $\mathbb{R}^n$

Suppose that R is a relation on $\mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator. Then T is a linear preserver of R if for all $x, y \in \mathbb{R}^n$, xRy implies $TxRTy$. Similarly T is said to be a strong linear preserver of R if for all $x, y \in \mathbb{R}^n$,

$$xRy \quad \Leftrightarrow \quad TxRTy.$$

The linear preservers on $M_{m,n}$ can be defined similarly.

Let $A \in M_{m,n}$ and $x \in \mathbb{R}^n$. We can represent Ax in the form

$$Ax = \left(\left\langle x, a_{(1)}\right\rangle, \ldots, \left\langle x, a_{(m)}\right\rangle\right)^t,$$

where $a^t_{(i)}$ is the ith row of A and $<,>$ is the standard inner product on $\mathbb{R}^n$. Similar to [1] we have this lemma.

Lemma 3.1. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator such that $Ax \sim_L Ay$ whenever $x \sim_L y$. Suppose that $||a_{(j)}|| \geq ||a_{(i)}||$ for every i $(1 \leq i \leq n)$, where $||.||$ is the Euclidean norm and $a^t_{(i)}$ is the ith row of A. Then for any permutation Π, there exists k such that $\Pi a_{(j)} = a_{(k)}$.*

The following theorem characterizes linear preservers of Latin-majorization on $\mathbb{R}^n$.

Theorem 3.2. *Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator. The following conditions are equivalent:*

(i) *A is a linear preserver of $\prec_L$.*
(ii) *$Ax \sim_L Ay$ whenever $x \sim_L y$.*
(iii) *A has one of the following forms:*
  (a) *$Ax = tr(x)a$ for some $a \in \mathbb{R}^n$.*
  (b) *$Ax = \alpha\Pi x + \beta tr(x)e$ for some $\alpha, \beta \in \mathbb{R}$ and permutation Π.*

Proof. It is trivial that 1 implies 2. If A has form (a) and $x = Dy$ for some Latin doubly stochastic matrix D, then, $Ax = Ay$ and hence, $Ax \prec_L Ay$. If A has form (b) then,

$$\begin{aligned} Ax &= \alpha\Pi x + \beta tr(x)e \\ &= \alpha\Pi Dy + \beta J_n Dy \\ &= \alpha\Pi Dy + \beta J_n y \\ &= \alpha\Pi D\Pi^{-1}\Pi y + \beta J_n y \\ &= \alpha D'\Pi y + \beta D' J_n y \\ &= D'\left(\alpha\Pi y + \beta J_n y\right) \\ &= D'Ay. \end{aligned}$$

Hence,

$$Ax \prec_L Ay.$$

So, 3 implies 1. Now, we prove that 2 implies 3. Take j such that

$$\left\|a_{(j)}\right\| = \max\left\{\left\|a_{(i)}\right\|; 1 \leq i \leq n\right\}.$$

By Lemma 3.1, two cases can occure:

Case 1. All coordinates of $a_{(j)}$ are the same. Since $x \sim_L y$, there exists Π such that $x = \Pi y$. So, $< x, a_{(j)} >=< y, a_{(j)} >$ and $A_1 x = QA_1 y$ for some permutation Q, where A_1 is obtaind from A by omitting jth row of A. Therefore $A_1x \sim_L A_1y$ whenever $x \sim_L y$. Suppose the norm of $a_{(k)}$ is greater than or equal to the norm of other rows of A_1. By Lemma 3.1 all coordinates of $a_{(k)}$ are the same. Continuing this method we conclude that A has form (a).

Case 2. Only one coordinate of $a_{(j)}$ is different from others. Then A has form (b). □

4. (Strong) Linear preservers of Latin-majorization on $M_{n,m}$

We begin with definition of Latin-majorization on $M_{n,m}$.

Definition 4.1. Let $A, B \in M_{n,m}$, we say that A is Latin-majorized by B, if there exists a Latin doubly stochastic matrix D such that $A = DB$ (denoted by $A \prec_L B$).

In the sequel we find the structure of (strong) linear preservers of this majorization.

In proof of [Theorem 2, [7]], the authors deduced from $x = Py$ for some $x, y \in \mathbb{R}^n$ and permutation P that $x \prec y$ but we can deduce from this fact that $x \sim y$ as well. So with minor variation in that proof we have:

Theorem 4.2. *Let $T : M_{n,m} \to M_{n,m}$ be a linear operator. Then the following are equivalent:*

(i) *T is a linear preserver of majorization.*
(ii) *$Tx \sim Ty$ whenever $x \sim y$.*
(iii) *T has one of the following forms:*
  (a) *There exist $A_1, \ldots, A_m \in M_{n,m}$ such that*

$$TX = \sum_{j=1}^{m} (trx_j)\, A_j.$$

  (b) *There exist $R, S \in M_m$ and $P \in S_n$ such that*

$$TX = PXR + JXS.$$

Since $x \sim y$ if and only if $x = Py$ for some permutation P if and only if $x \sim_L y$ we have

Theorem 4.3. *Let $T : M_{n,m} \to M_{n,m}$ be a linear operator. Then the following are equivalent:*

(i) *T is a linear preserver of Latin majorization.*
(ii) *$Tx \sim_L Ty$ whenever $x \sim_L y$.*
(iii) *$Tx \sim Ty$ whenever $x \sim y$.*
(iv) *T has one of the following forms:*
  (a) *There exist $A_1, \ldots, A_m \in M_{n,m}$ such that*

$$TX = \sum_{j=1}^{m} (trx_j)\, A_j.$$

(b) *There exist $R, S \in M_m$ and $P \in S_n$ such that*

$$TX = PXR + JXS.$$

Let $X, Y \in M_{n,m}$. It is said that X is multivariate majorized by Y (written $X \prec_m Y$) if $X = DY$ for some doubly stochastic matrix D. The strong linear preservers of $\prec_m$ on $M_{n,m}$ has been characterized in [5]. We state it here. The definitions and theorems hold for Latin-majorization instead of multivariate majorization.

Theorem 4.4 ([5])**.** *Let $T : M_{nm} \to M_{nm}$ be a linear operator. The following assertions are equivalent:*

(a) *T is invertible and preserves multivariate majorization $\prec_m$.*
(b) *T has the form*

$$X \mapsto PXR + JXS \qquad X \in M_{nm},$$

where $P \in \rho(n)$ and $R, S \in M_m$ are such that $R(R + nS)$ is invertible.
(c) *T strongly preserves multivariate majorization $\prec_m$.*

(With our notations M_{nm} is $M_{n,m}$ and $\rho(n)$ is S_n.)

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