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Compare and contrast between duals of fusion and discrete frames

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Abstract. Fusion frames are valuable generalizations of discrete frames. Most concepts of fusion frames are shared by discrete frames. However, the dual setting is so complicated. In particular, unlike discrete frames, two fusion frames are not dual of each other in general. In this paper, we investigate the structure of the duals of fusion frames and discuss the relation between the duals of fusion frames with their associated discrete frames.

1. Introduction and preliminaries

Fusion frames, as a generalization of frames, are valuable tools to subdividing a frame system into smaller subsystems and combine locally data vectors. The theory of fusion frames was systematically introduced in [7, 8]. Since then, many useful results about the theory and application of fusion frames have been obtained rapidly [5, 6, 14, 16].

In the context of signal transmission, fusion frames and their alternative duals have important roles in reconstructing signals in terms of the frame elements. The duals of fusion frames for experimental data transmission are investigated in [15]. But the problem that occur, is that the duality properties of fusion frames are not like discrete frames, such as, the duality property of fusion frames is not alternative and fusion Riesz bases have more than one dual. This paper deals with investigating such problems, which help us to obtain alternative dual fusion frames.

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Let $\mathcal{H}$ be a separable Hilbert space. A frame for $\mathcal{H}$ is a sequence $\{f_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ such that there are constants $0 < A \leq B < \infty$ satisfying

$$A\|f\|^2 \leq \sum_{i=1}^{\infty} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.1}$$

The constants A and B are called frame bounds. If $A = B$, we call $\{f_i\}_{i=1}^{\infty}$ a tight frame. If the right-hand side of (1.1) holds, we say that $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence. Given a frame $\{f_i\}_{i=1}^{\infty}$, the frame operator is defined by

$$Sf = \sum_{i=1}^{\infty} \langle f, f_i\rangle f_i.$$

A direct calculation yields

$$\langle Sf, f\rangle = \sum_{i\in I} |\langle f, f_i\rangle|^2.$$

Hence, the series defining Sf converges unconditionally for all $f \in \mathcal{H}$ and S is a bounded, invertible, and self-adjoint operator. Hence, we obtain

$$f = S^{-1}Sf = \sum_{i=1}^{\infty} \langle f, S^{-1}f_i\rangle f_i, \quad f \in \mathcal{H}. \tag{1.2}$$

The possibility of representing every $f \in \mathcal{H}$ in this way is the main feature of a frame. A sequence $\{f_i\}_{i=1}^{\infty}$ is a Bessel sequence if and only if the operator $T : \ell^2 \to \mathcal{H}; \{c_i\} \mapsto \sum_{i=1}^{\infty} c_i f_i$, which is called the synthesis operator, is well-defined and bounded. When $\{f_i\}_{i=1}^{\infty}$ is a frame, the synthesis operator T is well-defined, bounded and onto. A sequence $\{g_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ is called a dual for Bessel sequence $\{f_i\}_{i=1}^{\infty}$ if

$$f = \sum_{i=1}^{\infty} \langle f, g_i\rangle f_i, \quad f \in \mathcal{H}. \tag{1.3}$$

Every frame at least has a dual. In fact, if $\{f_i\}_{i=1}^{\infty}$ is a frame, then (1.2) implies that $\{S^{-1}f_i\}_{i=1}^{\infty}$, which is a frame with bounds B^{-1} and A^{-1}, is a dual for $\{f_i\}_{i=1}^{\infty}$; it is called the canonical dual. To see a general text in frame theory see [9].

Let $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ be Bessel sequences with synthesis operators T and U, respectively. Then, from (1.3) follows immediately that $\{f_i\}_{i=1}^{\infty}$ and $\{g_i\}_{i=1}^{\infty}$ are dual of each other if and only if $UT^* = I_{\mathcal{H}}$; in particular, they are frames. For more studies on frames and the duality properties of frames we refer to [2–4, 9–12, 18, 19].

The following proposition describes a characterization of alternate dual frames.

Proposition 1.1 ([4, 9]). (i) *The dual frames of* $\{f_i\}_{i=1}^{\infty}$ *are precisely as* $\{\Phi\delta_i\}_{i=1}^{\infty}$, *where* $\Phi : \ell^2 \to \mathcal{H}$ *is a bounded left inverse of* T^* *and* $\{\delta_i\}_{i=1}^{\infty}$ *is the canonical orthonormal basis of* ℓ^2.

(ii) *There is a one to one correspondence between dual frames of* $\{f_i\}_{i=1}^{\infty}$ *and operators* $\Psi \in B(\mathcal{H}, \ell^2)$ *such that* $T\Psi = 0$.

We now review preliminary results about fusion frames. Throughout this paper, I denotes a countable index set and π_V is the orthogonal projection onto a closed subspace V of $\mathcal{H}$.

Definition 1.2. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ be a family of weights, i.e. $\omega_i > 0$, $i \in I$. Then $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{i\in I} \omega_i^2 \|\pi_{W_i} f\|^2 \leq B\|f\|^2, \quad f \in \mathcal{H}. \tag{1.4}$$

The constants A and B are called the fusion frame bounds. If we only have the upper bound in (1.4) we call $\{(W_i, \omega_i)\}_{i\in I}$ a Bessel fusion sequence. A fusion frame is called tight, if A and B can be chosen to be equal, and Parseval if $A = B = 1$. If $\omega_i = \omega$ for all $i \in I$, the collection $\{(W_i, \omega_i)\}_{i\in I}$ is called ω-uniform. A fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ is said to be an orthonormal fusion basis if $\mathcal{H} = \bigoplus_{i\in I} W_i$, and it is a Riesz decomposition of $\mathcal{H}$ if for every $f \in \mathcal{H}$ there is a unique choice of $f_i \in W_i$ so that $f = \sum_{i\in I} f_i$.

Recall that for each sequence $\{W_i\}_{i\in I}$ of closed subspaces in $\mathcal{H}$, the space

$$\left(\sum_{i\in I} \oplus W_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in I} : f_i \in W_i, \sum_{i\in I} \|f_i\|^2 < \infty\right\},$$

with the inner product

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle,$$

is a Hilbert space.

For a Bessel fusion sequence $\{(W_i, \omega_i)\}_{i\in I}$ for $\mathcal{H}$, the synthesis operator $T_W : \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2} \to \mathcal{H}$ is defined by

$$T_W\left(\{f_i\}_{i\in I}\right) = \sum_{i\in I} \omega_i f_i, \qquad \{f_i\} \in \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2}.$$

Its adjoint operator $T_W^* : \mathcal{H} \to \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2}$ which is called the analysis operator, is given by

$$T_W^*(f) = \{\omega_i \pi_{W_i}(f)\}, \quad f \in \mathcal{H}.$$

Recall that $\{(W_i,\omega_i)\}_{i\in I}$ is a fusion frame if and only if the bounded operator T_W is onto [7] and its adjoint operator T_W^* is (possibly into) isomorphism. If $\{(W_i,\omega_i)\}_{i\in I}$ is a fusion frame, the fusion frame operator $S_W:\mathcal{H}\to\mathcal{H}$ defined by

$$S_W f = T_W T_W^* f = \sum_{i\in I}\omega_i^2\pi_{W_i} f,$$

is a bounded, invertible and positive operator and we have the following reconstruction formula [7]

$$f = \sum_{i\in I}\omega_i^2 S_W^{-1}\pi_{W_i} f, \quad f\in\mathcal{H}.$$

The family $\left\{(S_W^{-1}W_i,\omega_i)\right\}_{i\in I}$, which is also a fusion frame, is called the canonical dual of $\{(W_i,\omega_i)\}_{i\in I}$ and satisfies the following reconstruction formula [12]

$$f = \sum_{i\in I}\omega_i^2\pi_{S_W^{-1}W_i} S_W^{-1}\pi_{W_i} f, \quad f\in\mathcal{H}.$$

Definition 1.3. Let $\{(W_i,\omega_i)\}_{i\in I}$ be a fusion frame by the frame operator S_W. A Bessel fusion sequence $\{(V_i,\nu_i)\}_{i\in I}$ is called a dual of $\{(W_i,\omega_i)\}_{i\in I}$ if

$$f = \sum_{i\in I}\omega_i\nu_i\pi_{V_i} S_W^{-1}\pi_{W_i} f, \quad f\in\mathcal{H}. \tag{1.5}$$

Definition 1.4. Let $\{W_i\}_{i\in I}$ be a family of closed subspaces of $\mathcal{H}$ and $\{\omega_i\}_{i\in I}$ be a family of weights, i.e. $\omega_i>0$, $i\in I$. We say that $\{(W_i,\omega_i)\}_{i\in I}$ is a fusion Riesz basis for $\mathcal{H}$ if $\overline{span}_{i\in I}\{W_i\}=\mathcal{H}$ and there exist constants $0<C\le D<\infty$ such that for each finite subset $J\subseteq I$

$$C\left(\sum\|f_j\|^2\right)^{\frac{1}{2}} \le \left\|\sum\omega_j f_j\right\| \le D\left(\sum\|f_j\|^2\right)^{\frac{1}{2}}, \quad f_j\in W_j.$$

Some characterizations of fusion Riesz bases are given in the following theorem.

Theorem 1.5 ([7, 17])**.** *Let $\{(W_i,1)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ and $\{e_{i,j}\}_{j\in J_i}$ be a basis for W_i for each $i\in I$. Then the following conditions are equivalent:*
(1) $\{(W_i,1)\}_{i\in I}$ is a Riesz decomposition of $\mathcal{H}$.
(2) The synthesis operator T_W is one-to-one.
(3) The analysis operator T_W^* is onto.
(4) $\{(W_i,1)\}_{i\in I}$ is a fusion Riesz basis for $\mathcal{H}$.
(5) $\{e_{i,j}\}_{i\in I,j\in J_i}$ is a Riesz basis for $\mathcal{H}$.

Lemma 1.6 ([12])**.** *Let $T \in B(\mathcal{H})$ and $V \subseteq \mathcal{H}$ be a closed subspace . Then we have*

$$\pi_V T^* = \pi_V T^* \pi_{\overline{TV}}.$$

This paper is organized as follows: In Section 2, we compare the duality properties of discrete and fusion frames and by presenting examples of fusion frames we show that some well-known results on discrete frames are not valid on fusion frames. Also, we investigate the cases that these properties can satisfy on fusion frames. In Section 3, we investigate the relation between the duals of fusion frames, local frames and the associated discrete frames and we try to characterize the duals of fusion frames.

2. Contrasting of duals of fusion frames

For a fusion frame $\{(W_i, \omega_i)\}_{i\in I}$ and a Bessel fusion sequence $\{(V_i, \nu_i)\}_{i\in I}$, we define

$$\phi : \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2} \to \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2},$$

by

$$\phi\left(\{f_i\}_{i\in I}\right) = \left\{\pi_{V_i} S_W^{-1} f_i\right\}_{i\in I}. \tag{2.1}$$

It is easy to see that ϕ is a linear operator and $\|\phi\| \le \|S_W^{-1}\|$, its adjoint can be given by

$$\phi^*\left(\{g_i\}_{i\in I}\right) = \left\{\pi_{W_i} S_W^{-1} g_i\right\}_{i\in I}, \quad \text{for all} \quad \{g_i\}_{i\in I} \in \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2}.$$

Now, the identity (1.5) can be written in an operator form as follows.

Lemma 2.1. *Let $\{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame. A Bessel fusion frame $\{(V_i, \nu_i)\}_{i\in I}$ is a dual of $\{(W_i, \omega_i)\}_{i\in I}$ if and only if*

$$T_V \phi T_W^* = I_{\mathcal{H}}, \tag{2.2}$$

where T_W and T_V are the synthesis operators of $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$, respectively.

By Lemma 2.1, we deduce that, unlike discrete frames, two fusion frames are not dual of each other in general. Here we present an example which confirms this statement.

Example 2.2. Let $I = \{1, 2, ..., 6\}$. Consider

$$W_1 = \text{span}\{(1,0,0)\}, \quad W_2 = \text{span}\{(0,1,0)\}, \quad W_3 = \text{span}\{(0,0,1)\},$$
$$W_4 = \text{span}\{(0,1,0)\}, \quad W_5 = \text{span}\{(1,0,0)\}, \quad W_6 = \text{span}\{(0,0,1)\},$$

and $\omega_i = 1$, for $i \in I$. Also take

$$\begin{array}{lll} V_1 = \text{span}\{(1,0,0)\}, & V_2 = \text{span}\{(0,1,0)\}, & V_3 = \text{span}\{(0,0,1)\}, \\ V_4 = \text{span}\{(0,0,1)\}, & V_5 = \text{span}\{(0,1,0)\}, & V_6 = \text{span}\{(1,0,0)\}, \end{array}$$

and $\nu_i = 2$, for $i \in I$. Then $\{(W_i, \omega_i)\}_{i\in I}$ and $\{(V_i, \nu_i)\}_{i\in I}$ are fusion frames for $\mathbb{R}^3$ with frame operators $S_W = 2I_{\mathbb{R}^3}$ and $S_V = 8I_{\mathbb{R}^3}$, respectively. The following calculation shows that $\{(V_i, 2)\}_{i\in I}$ is an alternative dual of $\{(W_i, 1)\}_{i\in I}$:

$$\begin{aligned} \sum_{i\in I} \nu_i \omega_i \pi_{V_i} S_W^{-1} \pi_{W_i}(a,b,c) &= 2\left[\frac{1}{2}(a,0,0) + \frac{1}{2}(0,b,0) + \frac{1}{2}(0,0,c)\right] \\ &= (a,b,c), \quad \text{for all} \quad (a,b,c) \in \mathbb{R}^3. \end{aligned}$$

But $\{(W_i, 1)\}_{i\in I}$ is not an alternative dual of $\{(V_i, 2)\}_{i\in I}$. In fact

$$\begin{aligned} \sum_{i\in I} \nu_i \omega_i \pi_{W_i} S_V^{-1} \pi_{V_i}(a,b,c) &= 2\left[\frac{1}{8}(a,0,0) + \frac{1}{8}(0,b,0) + \frac{1}{8}(0,0,c)\right] \\ &\neq (a,b,c). \end{aligned}$$

Now, it is natural to ask when two Bessel fusion frames are dual of each other. To answer this question assume that $\{(W_i, \omega_i)\}_{i\in I}$ is also a dual fusion frame for $\{(V_i, \nu_i)\}_{i\in I}$ or equivalently (by Lemma 2.1)

$$T_W \psi T_V^* = I_{\mathcal{H}}, \tag{2.3}$$

where

$$\psi : \left(\sum_{i\in I} \oplus V_i\right)_{\ell^2} \to \left(\sum_{i\in I} \oplus W_i\right)_{\ell^2}, \qquad \psi\left(\{g_i\}_{i\in I}\right) = \left\{\pi_{W_i} S_V^{-1} g_i\right\}_{i\in I}.$$

Proposition 2.3. *Let $\{W_i\}_{i\in I}$ be a fusion frame with a dual $\{V_i\}_{i\in I}$. Then the fusion frame $\{W_i\}_{i\in I}$ is also a dual of $\{V_i\}_{i\in I}$ if*

$$\phi^* = \psi. \tag{2.4}$$

Moreover the converse is hold if $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ are fusion Riesz bases.

Proof. Let $\{V_i\}_{i\in I}$ be a dual of $\{W_i\}_{i\in I}$, then by using (2.2) and (2.4) we obtain

$$\begin{aligned} \left\langle \sum_{i\in I} \pi_{W_i} S_V^{-1} \pi_{V_i} f, f \right\rangle &= \langle T_W \phi^* T_V^* f, f\rangle \\ &= \langle f, T_V \phi T_W^* f\rangle = \langle f, f\rangle, \quad \text{for all} \quad f \in \mathcal{H}, \end{aligned}$$

i.e. fusion frame $\{W_i\}_{i\in I}$ is also a dual for $\{V_i\}_{i\in I}$. For the proof of moreover part, since $\{W_i\}_{i\in I}$ and $\{V_i\}_{i\in I}$ are fusion Riesz bases, by Theorem 1.5, T_W and T_V^* are invertible. So we deduce the proof by (2.2) and (2.3). $\square$

Corollary 2.4. *Let $\{f_i\}_{i\in I} \subseteq \mathcal{H}$ and $W_i = \overline{span}_{i\in I}\{f_i\}$ for each $i \in I$. Suppose that $\{(W_i, 1)\}_{i\in I}$ is a tight fusion frame for $\mathcal{H}$. Then $\{(W_i, 1)\}_{i\in I}$ is also a dual fusion frame of $\{(S_W^{-1}W_i, 1)\}_{i\in I}$.*

One of the important results in the duality of discrete frames is that every Riesz basis has just a unique dual (canonical dual) and that dual is Riesz basis as well. But the following example shows that this property is not confirmed in fusion Riesz bases.

Example 2.5. Consider

$$\begin{aligned} W_1 &= \mathrm{span}\{(1, 0, 0)\}, \\ W_2 &= \mathrm{span}\{(1, 1, 0)\}, \\ W_3 &= \mathrm{span}\{(0, 0, 1)\}. \end{aligned}$$

Then $\{(W_i, 1)\}_{i=1}^3$ is a fusion frame for $\mathbb{R}^3$ with bounds $1 - \frac{\sqrt{2}}{2}$ and 2, and the frame operator

$$S_W = \begin{bmatrix} 3/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is not difficult to see that $\{(W_i, 1)\}_{i=1}^3$ is a fusion Riesz basis and its canonical dual can be given with

$$\begin{aligned} S_W^{-1}W_1 &= \mathrm{span}\{(1, -1, 0)\}, \\ S_W^{-1}W_2 &= \mathrm{span}\{(0, 1, 0)\}, \\ S_W^{-1}W_3 &= \mathrm{span}\{(0, 0, 1)\}. \end{aligned}$$

To construct an alternate dual consider

$$V_1 = \mathbb{R}^2 \times \{0\}, \qquad V_2 = S_W^{-1}W_2, \qquad V_3 = S_W^{-1}W_3.$$

Then $\{(V_i, 1)\}_{i=1}^3$ is a fusion frame for $\mathbb{R}^3$. Moreover, if $f = (a, b, c)$ then

$$\begin{aligned} \sum_{i=1}^{3} \pi_{V_i} S_W^{-1} \pi_{W_i} f &= \pi_{V_1}(a, -a, 0) + \pi_{V_2}(0, a + b, 0) + \pi_{V_3}(0, 0, c) \\ &= (a, b, c) = f. \end{aligned}$$

Hence, the fusion Riesz basis $\{(W_i, 1)\}_{i=1}^3$ has more than one dual and the second dual is not a fusion Riesz basis.

3. More results on dual construction

Let $\{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$. By considering a frame for each subspace W_i, we can construct a discrete frame for $\mathcal{H}$. We begin with the following key theorem.

Theorem 3.1. *[7] For each $i \in I$ let $\omega_i > 0$ and let $\{f_{i,j}\}_{j\in J_i}$ be a frame sequence in $\mathcal{H}$ with the frame bounds A_i and B_i. Define $W_i = \overline{span}_{j\in J_i}\{f_{i,j}\}$ for all $i \in I$ and assume that*

$$0 < A = \inf_{i\in I} A_i \leq B = \sup_{i\in I} B_i < \infty.$$

Then $\{\omega_i f_{i,j}\}_{i\in I, j\in J_i}$ is a frame for $\mathcal{H}$ if and only if $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$.

In this paper, we call $F_i = \{f_{i,j}\}_{j\in J_i}$, $i \in I$, local frames of W_i and $\{\omega_i f_{i,j}\}_{i\in I,j\in J_i}$, the associated discrete frames of $\mathcal{H}$, which satisfy in above theorem.

Our aim in this section is to study the relation between the duals of fusion frames, local frames and the associated discrete frames of $\mathcal{H}$. In particular, in the following theorem we investigate the relation between the duals of local frames of W_i with the associated discrete frames of $\mathcal{H}$.

Theorem 3.2. *Let $\{(W_i, \omega_i)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ with local frames $\{f_{i,j}\}_{j\in J_i}$ for each $i \in I$. If $\{g_{i,j}\}_{j\in J_i}$ is a dual frame of $\{f_{i,j}\}_{j\in J_i}$, then $\{w_i f_{i,j}\}_{i\in I,j\in J_i}$ is a frame for $\mathcal{H}$ with dual frame $\{w_i S_W^{-1}(g_{i,j})\}_{i\in I,j\in J_i}$.*

Proof. Since $\{g_{i,j}\}_{j\in J_i}$ is the dual frame of $\{f_{i,j}\}_{j\in J_i}$ for W_i for each $i \in I$, we obtain

$$\begin{aligned}\pi_{W_i}(f) &= \sum_{j\in J_i} \langle \pi_{W_i}(f), f_{i,j}\rangle g_{i,j} \\ &= \sum_{j\in J_i} \langle f, f_{i,j}\rangle g_{i,j}, \quad \text{for all} \quad f \in \mathcal{H}, i \in I.\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{i\in I}\sum_{j\in J_i} \langle f, w_i f_{i,j}\rangle S_W^{-1}(w_i g_{i,j}) &= S_W^{-1} \sum_{i\in I} w_i^2 \pi_{W_i} f \\ &= S_W^{-1} S_W f = f.\end{aligned}$$

□

Suppose that $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$ and S_{F_i} is the frame operator of local frames F_i for each $i \in I$. Now the question is whether the canonical dual of each frame F_i is also a frame for the canonical dual of $\{(W_i, \omega_i)\}_{i\in I}$. The following example shows that the answer is not true in general.

Example 3.3. Let $I = \{1, 2, 3, 4\}$. Consider

$$\begin{aligned} W_1 &= \text{span}\{(1, 0, 0)\}, \qquad W_2 = \text{span}\{(1, 1, 0)\}, \\ W_3 &= \text{span}\{(0, 1, 0)\}, \qquad W_4 = \text{span}\{(0, 0, 1)\}, \end{aligned}$$

and $w_1 = w_3 = w_4 = 1, w_2 = \sqrt{2}$. Then by Example 3.1 in [1], $\{(W_i, \omega_i)\}_{i\in I}$ is a fusion frame for $\mathbb{R}^3$. Let $f_{1,1} = (1,0,0)$, $f_{2,1} = (1,1,0)$, $f_{3,1} = (0,1,0)$ and $f_{4,1} = (0,0,1)$. It is clear that $\{f_{i,1}\}$ is a frame for W_i for each $i \in I$. Suppose that S_W is the frame operator of $\{(W_i, \omega_i)\}_{i\in I}$ and S_{F_i} is the frame operator of $\{f_{i,1}\}$ for each $i \in I$. A straightforward calculation shows that

$$S_W = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the subspaces

$$S_W^{-1} W_1 = \text{span}\left\{\left(\tfrac{2}{3}, \tfrac{-1}{3}, 0\right)\right\}, \qquad S_W^{-1} W_2 = \text{span}\left\{\left(\tfrac{1}{3}, \tfrac{1}{3}, 0\right)\right\},$$
$$S_W^{-1} W_3 = \text{span}\left\{\left(\tfrac{-1}{3}, \tfrac{2}{3}, 0\right)\right\}, \qquad S_W^{-1} W_4 = \text{span}\{(0,0,1)\},$$

with the weights $\{\omega_i\}_{i\in I}$ is the canonical dual of $\{(W_i, w_i)\}_{i\in I}$. Moreover, if we take

$$g_{1,1} = (1,0,0), \qquad g_{2,1} = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0\right),$$
$$g_{3,1} = (0,1,0), \qquad g_{4,1} = (0,0,1),$$

then $\{g_{i,1}\}$ is the canonical dual of $\{f_{i,1}\}$ for each $i \in I$. However, $\{g_{i,1}\}$ is not a frame for $S_W^{-1} W_i$ for each $i \in I$.

The following example shows that there is no significant relation between the duals of fusion frames and their associated discrete frames, i.e. if $\{(V_i, \nu_i)\}_{i\in I}$ is a dual of $\{(W_i, \omega_i)\}_{i\in I}$, then it is not necessary that their associated discrete frames be dual of each other.

Example 3.4. Let

$$W_1 = \text{span}\{(1,0,0)\}, \quad W_2 = \text{span}\{(0,1,0)\}, \quad W_3 = \text{span}\{(0,0,1)\},$$
$$W_4 = \text{span}\{(0,1,0)\}, \quad W_5 = \text{span}\{(1,0,0)\}, \quad W_6 = \text{span}\{(0,0,1)\},$$

and

$$V_1 = \text{span}\{(1,0,0)\}, \quad V_2 = \text{span}\{(0,1,0)\}, \quad V_3 = \text{span}\{(0,0,1)\},$$
$$V_4 = \text{span}\{(0,0,1)\}, \quad V_5 = \text{span}\{(0,1,0)\}, \quad V_6 = \text{span}\{(1,0,0)\}.$$

Then $\{(W_i, 1)\}_{i=1}^6$ is a fusion frame for $\mathbb{R}^3$ with an alternate dual $\{(V_i, 2)\}_{i=1}^6$. Consider

$$f_{1,1} = f_{5,1} = (1,0,0), \quad f_{2,1} = f_{4,1} = (0,1,0), \quad f_{3,1} = f_{6,1} = (0,0,1),$$

and

$$g_{1,1} = g_{6,1} = (2,0,0), \quad g_{2,1} = g_{5,1} = (0,2,0), \quad g_{3,1} = g_{4,1} = (0,0,2).$$

Then $\{f_{i,1}\}_{i=1}^6$ and $\{g_{i,1}\}_{i=1}^6$ are frames for $\mathbb{R}^3$, but they are not dual of each other.

In the following proposition we give a necessary condition to elucidate duals of fusion frames.

Proposition 3.5. *Let $\{(W_i,1)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ and $\{(V_i,1)\}_{i\in I}$ be a Parseval fusion frame for $\mathcal{H}$. Suppose that $W_k \perp S_W^{-1}V_i$ for each $i\neq k$. Then $\{(V_i,1)\}_{i\in I}$ is an alternative dual of $\{(W_i,1)\}_{i\in I}$.*

Proof. Since $\{(V_i,1)\}_{i\in I}$ is a Parseval fusion frame, we have

$$\begin{aligned} f &= S_V f \\ &= \sum_{i\in I}\pi_{V_i}(S_W^{-1}S_W f) \\ &= \sum_{i\in I}\pi_{V_i}S_W^{-1}\sum_{k\in I}\pi_{W_k} f \\ &= \sum_{i\in I}\pi_{V_i}S_W^{-1}\pi_{W_i} f + \sum_{i\in I}\pi_{V_i}S_W^{-1}(\sum_{k\in I,k\neq i}\pi_{W_k} f), \quad f\in\mathcal{H}. \end{aligned} \tag{3.1}$$

By Lemma 1.6, we have

$$\sum_{i\in I}\sum_{k\in I,k\neq i}\pi_{V_i}S_W^{-1}\pi_{W_k} f = \sum_{i\in I}\sum_{k\in I,k\neq i}\pi_{V_i}S_W^{-1}\pi_{S_W^{-1}V_i}\pi_{W_k} f = 0. \tag{3.2}$$

So we get the proof by (3.1) and (3.2). □

Proposition 3.6. *Let $\{(W_i,\omega_i)\}_{i\in I}$ and $\{(V_i,\nu_i)\}_{i\in I}$ be fusion frames for $\mathcal{H}$. Suppose that $W_i \perp V_i$ for each $i\in I$. Then $\{(S_W V_i,\nu_i)\}_{i\in I}$ can not be an alternative dual of $\{(W_i,\omega_i)\}_{i\in I}$.*

Proof. By Proposition 1.1 in [7], $\{(S_W V_i,\nu_i)\}_{i\in I}$ is a fusion frame for $\mathcal{H}$. By Lemma 1.6, we have

$$\sum_{i\in I}\omega_i\nu_i\pi_{S_W V_i}S_W^{-1}\pi_{W_i} f = \sum_{i\in I}\omega_i\nu_i\pi_{S_W V_i}S_W^{-1}\pi_{V_i}\pi_{W_i} f = 0, \quad f\in\mathcal{H}.$$

□

In the rest of the paper we try to characterize the duals of fusion frames. We first discuss the Riesz case. Let $\{(W_i,\omega_i)\}_{i\in I}$ be a Riesz decomposition of $\mathcal{H}$ and $\{(V_i,\nu_i)\}_{i\in I}$ be its dual. Associated to the canonical dual $\{(S_W^{-1}W_i,\omega_i)\}_{i\in I}$, we can consider the operator

$$\phi_1 : \left(\sum_{i\in I}\oplus W_i\right)_{\ell^2} \to \left(\sum_{i\in I}\oplus S_W^{-1}W_i\right)_{\ell^2},$$

given by

$$\phi_1(\{f_i\}_{i\in I}) = \left\{\pi_{S_W^{-1}W_i}S_W^{-1}f_i\right\}_{i\in I}.$$

Applying (2.2) and Theorem 1.5, we conclude that $T_V\phi = T_{S_W^{-1}W}\phi_1$, where $T_{S_W^{-1}W}$ is the synthesis operator of $\{(S_W^{-1}W_i, \omega_i)\}_{i\in I}$. It follows easily that

$$\pi_{V_i}S_W^{-1}f_i = S_W^{-1}f_i, \quad i \in I, f_i \in W_i,$$

or equivalently

$$S_W^{-1}W_i \subseteq V_i, \quad i \in I.$$

The following example shows that unfortunately, we can not characterize the duals of fusion frames by the duals of their associated discrete frames and the first part of Proposition 1.1.

Example 3.7. Consider the fusion frame $\{(W_i, \omega_i)\}_{i=1}^4$ introduced in Example 3.3. By Theorem 3.1 the sequence

$$\{\omega_i f_{i,1}\}_{i=1}^4 = \left\{(1,0,0), \sqrt{2}(1,1,0), (0,1,0), (0,0,1)\right\},$$

is a frame for $\mathbb{R}^3$ with the frame operator

$$S_F = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Denote its canonical dual by $\{g_{i,1}\}_{i=1}^4$. Then

$$\{g_{i,1}\}_{i=1}^4 = \left\{\frac{1}{5}(3,-2,0), \frac{\sqrt{2}}{5}(1,1,0), \frac{1}{5}(-2,3,0), (0,0,1)\right\}.$$

Consider

$$\begin{aligned} V_1 &= \text{span}\{(3,-2,0)\}, \qquad & V_2 &= \text{span}\{(1,1,0)\}, \\ V_3 &= \text{span}\{(-2,3,0)\}, \qquad & V_4 &= \text{span}\{(0,0,1)\}, \end{aligned}$$

and $\nu_1 = \nu_3 = \frac{1}{5}$, $\nu_2 = \frac{\sqrt{2}}{5}$, $\nu_4 = 1$, then $\{(V_i, \nu_i)\}_{i=1}^4$ is a fusion frame for $\mathbb{R}^3$. But $\{(V_i, \nu_i)\}_{i=1}^4$ is not an alternative dual of $\{(W_i, \omega_i)\}_{i=1}^4$ and vise-versa.

We give an explicit construction of a dual fusion frame in the following theorem.

Theorem 3.8. *Let $\{(W_i, 1)\}_{i\in I}$ be a fusion frame for $\mathcal{H}$ and $\{h_i\}_{i\in I}$ be a Bessel sequence of normalized vectors such that $h_i \in (S_W^{-1}W_i)^{\perp}$. Take*

$$V_i = S_W^{-1}W_i + Z_i, \quad i \in I,$$

where Z_i is the 1-dimensional subspace generated by h_i. Then $\{(V_i, 1)\}_{i\in I}$ is a dual for $\{(W_i, 1)\}_{i\in I}$.

Proof. First, it is not difficult to see that

$$\pi_{Z_i} f = \langle f, h_i \rangle h_i, \quad i \in I, f \in \mathcal{H}.$$

Now by using 8.12 of [13] and Corollary 2.5 of [12], we conclude that

$$\begin{aligned}
\sum_{i\in I} \|\pi_{V_i} f\|^2 &= \sum_{i\in I} \|\pi_{S_W^{-1}W_i} f + \pi_{Z_i} f\|^2 \\
&\leq \sum_{i\in I} \|\pi_{S_W^{-1}W_i} f\|^2 + \sum_{i\in I} |\langle f, h_i\rangle|^2 \\
&\quad + 2\left(\sum_{i\in I} \|\pi_{S_W^{-1}W_i} f\|^2\right)^{\frac{1}{2}} \left(\sum_{i\in I} |\langle f, h_i\rangle|^2\right)^{\frac{1}{2}} \\
&\leq \left(B\|S_W\|^2\|S_W^{-1}\|^2 + D + 2\sqrt{BD}\|S_W\|\|S_W^{-1}\|\right)\|f\|^2,
\end{aligned}$$

where B and D are the frame bounds of $\{(W_i, 1)\}_{i\in I}$ and $\{h_i\}_{i\in I}$, respectively. Hence, $\{(V_i, 1)\}_{i\in I}$ is a Bessel fusion frame. Moreover, by Lemma 1.6 we have

$$\begin{aligned}
\sum_{i\in I} \pi_{V_i} S_W^{-1} \pi_{W_i} f &= \sum_{i\in I} \pi_{S_W^{-1}W_i} S_W^{-1}\pi_{W_i} f + \sum_{i\in I} \pi_{Z_i} S_W^{-1} \pi_{W_i} f \\
&= f, \quad f \in \mathcal{H}.
\end{aligned}$$

□

Remark 3.9. The above theorem gives us a very simple method to construct duals of finite fusion frames. More precisely, let $\{(W_i, 1)\}_{i\in I}$ be a finite fusion frame for $\mathcal{H}$. Take

$$V_i = \begin{cases} S_W^{-1}W_i & \text{for } \left(S_W^{-1}W_i\right)^{\perp} = \emptyset; \\ S_W^{-1}W_i \oplus Z_i & \text{for } \left(S_W^{-1}W_i\right)^{\perp} \neq \emptyset, \end{cases}$$

where Z_i is a 1-dimensional subspace of $(S_W^{-1}W_i)^{\perp}$. To illustrate this algorithm, let us consider the fusion frame $\{W_i\}_{i\in I}$ in Example 2.5. Clearly

$$\begin{aligned}
\left(S_W^{-1}W_1\right)^{\perp} &= \operatorname{span}\{(1,1,0),(0,0,1)\}, \\
\left(S_W^{-1}W_2\right)^{\perp} &= \operatorname{span}\{(1,0,0),(0,0,1)\}, \\
\left(S_W^{-1}W_3\right)^{\perp} &= \operatorname{span}\{(1,0,0),(0,1,0)\}.
\end{aligned}$$

Therefore, we can introduce some duals:

(i)	$V_1 = \operatorname{span}\{(1,-1,0),(0,0,1)\}$,	$V_2 = \mathbb{R}^2 \times \{0\}$,	$V_3 = \{0\} \times \mathbb{R}^2$.
(ii)	$V_1 = span\{(1,-1,0),(0,0,1)\}$,	$V_2 = \{0\} \times \mathbb{R}^2$,	$V_3 = \{0\} \times \mathbb{R}^2$.
(iii)	$V_1 = span\{(1,-1,0),(1,1,0)\}$,	$V_2 = \mathbb{R}^2 \times \{0\}$,	$V_3 = \{0\} \times \mathbb{R}^2$.
(iv)	$V_1 = span\{(1,-1,0),(1,1,0)\}$,	$V_2 = \{0\} \times \mathbb{R}^2$,	$V_3 = \{0\} \times \mathbb{R}^2$.

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References

1. Z. Amiri, M.A. Dehghan, and E. Rahimi, *Subfusion frames*, Abstr. Appl. Anal., 2012 (2012), pp. 1-12.
2. A. Arefijamaal and S. Ghasemi, *On characterization and stability of alternate dual of g-frames*, Turk. J. Math., 37 (2013), pp. 71-79.
3. A. Arefijamaal, E. Zekaee, *Image processing by alternate dual Gabor frames*, To appear in Bull. Iranian Math. Soc.
4. A. Arefijamaal and E. Zekaee, *Signal processing by alternate dual Gabor frames*, Appl. Comput. Harmon. Anal., 35 (2013), pp. 535-540.
5. R. Calderbank, P.G. Casazza, A. Heinecke, G. Kutyniok, and A. Pezeshki, *Sparse fusion frames: existence and construction*, Adv. Comput. Math., 35 (2011), pp. 1-31.
6. P.G. Casazza and M. Fickus, *Minimizing fusion frame potential*, Acta, Appl. Math., 107 (2009), pp. 7-24.
7. P.G. Casazza and G. Kutyniok, *Frames of subspaces*, Contemp. Math., 345 (2004), pp. 87-113.
8. P.G. Casazza, G. Kutyniok, and S. Li, *Fusion frames and distributed processing*, Appl. Comput. Harmon. Anal., 25 (2008), pp. 114-132.
9. O. Christensen, *Frames and Bases: An Introductory Course*, Birkhäuser, Boston, 2008.
10. O. Christensen and E. Osgooei, *On frame properties for Fourier-like systems*, J. Approx. Theory, 172 (2013), pp. 47-57.
11. M.A. Dehghan and M.A. Hasankhani Fard, *G-dual frames in Hilbert spaces*, U. P. B. Sci. Bull. Series A, 75 (2013), pp. 129-140.
12. P. Găvruţa, *On the duality of fusion frames*, J. Math. Anal. Appl., 333 (2007), pp. 871-879.
13. W.H. Greub, *Linear Algebra*, Springer-Verlag, New York, 1981.
14. S.K. Kaushik, *A generalization of frames in Banach spaces*, J. Contemp. Math. Anal., 44 (2009), pp. 212-218.
15. J. Leng, Q. Guo, and T. Huang, *The duals of fusion frames for experimental data transmission coding of high energy physics*, Adv. High Energy Phys., 2013 (2013), pp. 1-9.
16. P.G. Massey, M.A. Ruiz, and D. Stojanoff, *The structure of minimizers of the frame potential on fusion frames*, J. Fourier Anal. Appl., 16 (2010), pp. 514-543.

17. A. Najati, A. Rahimi, and M.H. Faroughi, *Continuous and discrete frames of subspaces in Hilbert spaces*, Southeast Asian Bull. Math., 32 (2008), pp. 305-324.
18. E. Osgooei and M.H. Faroughi, *Hilbert-Schmidt sequences and dual of g-frames*, Acta Univ. Apulensis, 36 (2013), pp. 1-15.
19. A. Rahimi, *Invariance of Fréchet frames under perturbation*, Sahand Communications in Mathematical Analysis, 1 (2014), pp. 41-51.

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